

**INTRODUCTION  
TO THE THEORY  
OF QUANTIZED FIELDS**

**THIRD EDITION**

**N. N. BOGOLIUBOV  
D. V. SHIRKOV**

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## PREFACE TO THE THIRD EDITION

In this edition we have rewritten the chapters that discuss the methods of continuous integration and the renormalization group, which are two topics in theory that have become extremely important in recent years. We have also reworked and supplemented the sections on the complete Green functions.

This work was done in an atmosphere of friendly advice and fruitful discussions with our colleagues from the Steklov Mathematical Institute of the USSR Academy of Sciences and of the Laboratory of Theoretical Physics of the Joint Institute of Nuclear Studies, to whom we are extremely grateful.

N. N. BOGOLIUBOV  
D. V. SHIRKOV

*Dubna,  
September 1979*





## PREFACE TO THE SECOND EDITION

In preparing the second edition of this book we concentrated on improving the exposition of the most important parts. Thus, for example, we have rewritten sections on the quantization of fields and on general rules for removal of divergencies. Fairly substantial changes were also made in the two last chapters (renormalization group and dispersion relations) where we have attempted to make the exposition easier to understand by freeing it of many complicating problems (for example, proof of the dispersion relationships at  $t \neq 0$ ), since these complicating side issues are fairly well described in the specialized literature.

We are indebted to many colleagues of the Steklov Mathematical Institute of the USSR Academy of Sciences, the Laboratory of Theoretical Physics of the Joint Institute of Nuclear Studies, and of the Institute of Mathematics of the Siberian Department, USSR Academy of Sciences, for useful comments and advice in preparation of this second edition.

N. N. BOGOLIUBOV  
D. V. SHIRKOV



## PREFACE TO THE FIRST RUSSIAN EDITION

This monograph is an attempt to give a systematic presentation of the modern theory of quantized fields from its foundations right up to its most recent achievements.

In writing this book the authors were guided by their intention to present field theory from a unified point of view combining internal logical consistency and closure with completeness of the material covered. We also aimed, wherever possible, to introduce the maximum degree of clarity into the basic assumptions of the theory employed at the present stage of its development. At the same time, naturally, particular attention was devoted to the mathematical correctness of the arguments, as a result of which the extent of coverage of the applications of the theory to calculations of specific physical phenomena can be claimed to be only methodologically complete.

The authors also wanted to treat sufficiently fully the most promising approaches developed quite recently. We hope that because of this, the book will prove to be useful not only to persons first undertaking the study of quantum field theory, but also to theoreticians working in this domain of physics.

The chapter "Dispersion Relations," which presents the most recent results, was included in the book as a supplement in view of the current interest in this topic.

The authors wish to thank the staff and the post-graduate students of the Division of Theoretical Physics of the V. A. Steklov Mathematical Institute of the Academy of Sciences, U.S.S.R., and of the Chair of Statistical Physics and Mechanics of the Physics Faculty of the M. V. Lomonosov Moscow State University, for their remarks and suggestions made during the preparation of the manuscript. We are particularly grateful to B. V. Medvedev, who contributed a number of valuable comments on different parts of the book.

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*February, 1957*  
*Moscow*



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**INTRODUCTION  
TO THE THEORY OF  
QUANTIZED FIELDS**



# INTRODUCTION

*General Plan.* This book contains a systematic exposition of the fundamentals of the modern theory of quantized fields, making full use of all the necessary mathematical concepts. Most of the book (the first seven chapters) consists of a logical development of the formalism based on the Lagrangian formulation of the theory of free fields and the axiomatic introduction of the scattering matrix for interacting fields.

Chapter 1 consists of a presentation of the apparatus of nonquantized relativistic free fields, based on the Lagrangian formalism and the Noether theorem.

The free fields are then quantized in Chapter 2 on the basis of the correspondence principle and without recourse to the canonical formalism. Chapter 3 discusses the properties of commutation and causal functions, and also certain mathematical problems that arise during multiplication of singular functions.

Chapter 4 gives the general theory of the scattering matrix and a development of the perturbation-theory apparatus. In this we follow the ideas of Heisenberg, Feynman, and, especially, Stueckelberg, and define the scattering matrix without recourse to the Hamiltonian formalism, but taking as our basis the physical requirements of covariance, unitarity, and causality. The condition of causality, which is formulated explicitly with the aid of the variational derivative, plays an important role in this development. In other words, we give a constructive development of the perturbation-theory expansion of the  $S$ -matrix within the framework of the axiomatic approach. Careful analysis of the arbitrariness that arises when singular Green's functions are multiplied together is used to obtain the most general expression for the scattering matrix, and this subsequently forms the basis for the procedure used to remove divergences.



Next, in Chapter 5, we take the lowest-order diagrams of spinor electrodynamics to illustrate the practical evaluation of Feynman integrals and the removal of the simplest ultraviolet divergencies in perturbation theory is then discussed in its fullest form, and a rigorous theory is presented of the renormalization of the  $S$ -matrix. A classification of renormalizability of theories is given within the framework of perturbation theory.

Chapter 6 contains an application of general renormalization theory in any order of perturbation theory to spinor electrodynamics, to the theories of scalar and pseudoscalar meson fields with interaction, and to the theory of pseudoscalar interaction of nucleons and pseudoscalar mesons. Particular attention is devoted to the renormalization structure of complete Green's functions. Schwinger's equations are also obtained for these functions.

In Chapter 7, the  $S$ -matrix is used to investigate the evolution of a system of quantized fields in time, and the Tomonaga-Schwinger equation is derived. The dynamic variables are introduced, and generalizations of free-field operators to the case of interaction are constructed. The Dirac equation is obtained with the radiative corrections included, and an account is given of the theory of the Lamb shift.

The three concluding chapters discuss some of the general methods in the theory of quantized fields that are not organically connected with the perturbation-theory expansion.

Chapter 8 contains an account of the method of functional integration. This method is based on a particular representation of complete Green's functions in terms of functional integrals, and is very general. Its possibilities are demonstrated by considering spinor electrodynamics. A generalization of Ward's identity is obtained, the gauge transformation of the electron Green's function is derived, and the structure of its infrared singularities is determined. Studies of non-Abelian gauge fields have made this method increasingly popular. It has become crucially important as an instrument of theoretical analysis of quantized-field models, not based on perturbation theory.

In Chapter 9, we present the renormalization group method which is based on the group character of multiplicative renormalizations. The formalism of functional and differential equations is developed, and considerable attention is devoted to the analysis of asymptotic behavior in the ultraviolet region, which is of considerable practical importance.

Finally, Chapter 10 is devoted to the method of dispersion relations. Our presentation, based on the axiomatic approach, contains a derivation of the spectral representations of single-particle Green's functions and of the dispersion relations for pion-nucleon scattering. Extensive use is made of variational derivatives of the  $S$ -matrix with respect to the quantized fields (i.e., currents) and the matrix elements of their products. Enough material is presented to enable the reader to achieve an adequate understanding of the theory of functions of many complex variables, as used for the analytic continuation of generalized functions.

During the twenty-year history of the theory of dispersion relations, these methods at first seemed exotic, but eventually found extensive application in quantum field theory and now form the basis for most of its results (connection between spin and statistics, the CPT theorem, and so on).

In its broad sense, the method of dispersion relations, i.e., the method of investigating the basic variables of the theory in terms of their analytic properties, has itself undergone considerable development during this period, and has found varied and physically

meaningful applications. Some of them, for example, the high-energy properties of the scattering amplitude (the Pomeranchuk and Logunov theorems, the Froissart limit) are based on rigorously proved analytic properties. Others use additionally postulated propositions (for example, the double spectral representations of Mandelstam). All these problems are, however, outside the framework of our presentation.

*Some Notation.* We now introduce some of the notation used in this book. All the components of four-vectors are chosen to be real. The metric is defined by the Minkowski tensor, taken with reversed sign:

$$g^{mn} = 0 \text{ for } m \neq n; \quad g^{00} = -g^{11} = -g^{22} = -g^{33} = 1,$$

i.e., the product of two contravariant four-vectors  $a$  and  $b$  with components  $\{a^0, a^1, a^2, a^3\} = \{a^0, \mathbf{a}\}$  and  $\{b^0, b^1, b^2, b^3\} = \{b^0, \mathbf{b}\}$  is defined as follows:

$$ab \equiv \sum_{0 \leq m, n \leq 3} g^{mn} a^m b^n = a^0 b^0 - a^1 b^1 - a^2 b^2 - a^3 b^3 = a^0 b^0 - \mathbf{a} \cdot \mathbf{b}.$$

Bold type is used to represent ordinary three-vectors. The transition from contravariant  $a^i$  to covariant  $a_i$  components (lowering of index) is achieved with the aid of the metric tensor

$$a_m = \sum g^{mn} a^n = g_{mn} a^n; \quad g_{mn} = g^{mn},$$

i.e.,

$$a_0 = a^0, \quad a_\alpha = -a^\alpha, \quad \alpha = 1, 2, 3, \quad a_m = \{a^0, -\mathbf{a}\}.$$

Repeated indices imply summation (the summation symbol is not explicitly indicated). As a rule, indices representing summation over all four components 0, 1, 2, 3 are indicated by Latin letters, and those over the three space components by Greek letters. For example,

$$ab = g^{mn} a_m b_n = a^n b_n = a_m b^m, \\ \mathbf{a} \cdot \mathbf{b} = a_\alpha b_\alpha = a^\alpha b^\alpha.$$

By raising (or lowering) one of the indices of the metric tensor we obtain the Kronecker symbol:

$$g_{nm} g^{mk} = \delta_n^k.$$

Indices referring to groups of internal symmetries (for example, isospin indices) are usually represented by Latin letters at the beginning of the alphabet ( $a, b, \dots$ ).

Generalizing notation is occasionally used for (operator) field functions in addition to the individualized notation introduced at the relevant points in our presentation. Fields with integer spin (Bose fields) are denoted by  $\varphi$ , and fields with half-integer spin (Fermi

fields) are denoted by the symbol  $\psi$ . To emphasize the generality of the discussion, continuous space-time coordinates are occasionally combined with discrete coordinates into a single argument, indicated by a Greek letter, for example,

$$\varphi_a(x) \rightarrow \varphi(\xi), \quad \psi_a(x) \rightarrow \psi(\xi), \quad \xi = (x, a).$$

Integration with respect to  $\xi$  is defined as follows:

$$\int d\xi = \sum \int d^4x.$$

The special notation  $u(\xi)$  is used for the most general analysis, valid for both Bose and Fermi fields.

The symbol  $\hat{a}$  represents the contraction of the components of the four-vector  $a_m$  and the Dirac matrices  $\gamma^m$ :

$$\hat{a} = a^m \gamma_m.$$

The following abbreviated notation is occasionally used for derivatives:

$$\frac{\partial \varphi_a}{\partial x^n} = \partial_n \varphi_a = \varphi_{a, n}; \quad \frac{\partial u}{\partial x_n} = \partial^n u = u^{,n},$$

where it is, of course, understood that

$$\varphi_a^{,n} = g^{nm} \varphi_{a, m}.$$

The d'Alembert operator

$$\square = \Delta - \partial_0^2$$

is represented by

$$\square = -\partial^n \partial_n.$$

Throughout this book we use the system of units in which the velocity of light and Planck's constant divided by  $2\pi$ , are both equal to unity, i.e.,

$$c = \hbar = 1.$$

In this system of units, energy, momentum, and mass have the dimensions of the reciprocal of length, and the time  $x_0 = t$  has the dimensions of length.

The four-dimensional Fourier transformation is usually written in the form

$$f(x) \sim \int e^{-ipx} \tilde{f}(p) dp, \quad \tilde{f}(p) \sim \int e^{ipx} f(x) dx.$$

The sign of the argument of the exponential is chosen so that it is consistent with the quantum-mechanical formula

$$f(x^0, \mathbf{x}) = f(t, \mathbf{x}) \sim \int e^{-iEt} \tilde{f}(E, \mathbf{x}) dE.$$

Accordingly, the three-dimensional Fourier transformation has the form

$$\varphi(\mathbf{x}) \sim \int e^{i\mathbf{p}\mathbf{x}} \tilde{\varphi}(\mathbf{p}) d\mathbf{p}, \quad \tilde{\varphi}(\mathbf{p}) \sim \int e^{-i\mathbf{p}\mathbf{x}} \varphi(\mathbf{x}) d\mathbf{x}.$$

The only exception occurs in the case of formulas for the positive-frequency parts of field functions and the positive-frequency parts of Green's functions. Different normalizing factors (powers of  $2\pi$ ) are used for the Fourier transformations at different places in the book.

Bibliographic references are given in the form of the surname of the author followed by the year of publication in parenthesis. Full bibliographic references can be found at the end of the book.



# Chapter 1

## CLASSICAL THEORY OF FREE FIELDS

### §1. Lagrangian Formalism

*1.1. Fields and Particles.* We begin by considering the classical theory of relativistic wave fields. The procedure used for the quantization of fields (Chapter 2) will lead us to the concept of quantized fields. The *quantized wave field* is a fundamental physical concept, within the framework of which the properties of elementary particles and their interactions are formulated.

The method of quantized fields enables us to describe a many-particle system in terms of a single physical object in ordinary space-time, namely, the quantized field. It also enables us to describe an important property of the world of elementary particles, namely, their ability to transform into each other.

A well-known example of a wave field in classical physics is the electromagnetic field describing the interaction between electrically charged particles. The classical description of the electromagnetic field, based on Maxwell's equations, leads to a purely wave theory of electromagnetism. Occasionally, it is also convenient to consider the continuous system (the electromagnetic field) as a discrete mechanical system with an infinite number of degrees of freedom, or, more precisely, an infinite number of the so-called *field oscillators*. This approach enables us to use the apparatus of classical mechanics in studying the field.

The transition to the corpuscular or, more precisely, the unified wave-corporeal point of view, is accomplished by the so-called quantization procedure in which discrete quanta of energy corresponding to different possible energy states of the field oscillators are associated with the field. The electromagnetic quanta are then said to be the particles

describing the interaction between the electrically charged particles. The quanta of the electromagnetic field have zero rest mass and an intrinsic angular momentum, i.e., spin. This property is reflected in the fact that the electromagnetic field is a four-component field described by the four-vector potential  $A_\mu$ .

The properties of wave fields corresponding to other particles (electrons, nucleons, mesons, and so on) also reflect their spin, charge, and other characteristics. These particles are identified with the quanta of the corresponding fields and describe the interaction between other particles that are the sources of the given wave field. A well-known example of this correspondence is provided by pions which are the quanta of the field describing most of the interaction between nucleons. Nuclear forces may be said to be "carried" largely by pions.

The quantum theory of interacting particles is thus the *theory of interaction between elementary particles*.

We shall begin by considering the classical theory of free wave fields (Chapter 1) and will continue with the quantum theory of free wave fields (Chapter 2). We shall conclude with the theory of interacting quantized fields.

In our account of the theory of classical fields, it will often be useful to employ concepts connected with the characteristics of the corresponding particles (mass, spin, and so on). When this is done, it must be remembered that the complete significance of these concepts emerges only after quantization.

**1.2. Hamiltonian and Lagrangian Formalism.** By treating the field as a mechanical system with an infinite number of degrees of freedom we are led to the possibility of formulating a theory of the field by analogy with the classical mechanics of a particle. In this procedure, the field is described by means of a so-called *field function* which corresponds to an infinite number of degrees of freedom. The equations for the field functions may be obtained from the Lagrangian function of the system by means of a variational principle of stationary action, and the dynamic variables may be obtained by introducing quantities similar to the corresponding expressions in the formalism of classical mechanics.

By analogy with mechanics, two methods of constructing a theory of wave fields turn out to be possible.

In the *canonical* (or *Hamiltonian*) *formalism*, the fundamental quantities are the generalized coordinates and momenta, and also the Hamiltonian function for the system, which is constructed with the aid of the Lagrangian function and the generalized coordinates and momenta. The canonical equations and the dynamic variables are obtained directly from the Hamiltonian function, and this method of formulating the field theory permits a detailed formal analogy with classical mechanics to be carried through.

However, in spite of its straightforwardness, the Hamiltonian formalism has an essential disadvantage: since time is singled out, the presentation loses its relativistic invariance which is particularly inconvenient in the case of quantum field theory.

The second method of presenting field theory, to which we shall refer as the *Lagrangian formalism*, is based on the fact that the introduction of generalized coordinates and momenta as well as of the Hamiltonian function is not unavoidable. Starting with the Lagrangian of the system, one may obtain the equations of motion



by means of a variational principle, while the dynamic variables, such as energy-momentum and charge, are defined as invariants corresponding to various transformations of the system of coordinates and of the field functions. While in the canonical formalism time is the fundamental independent variable and coordinate variables play the role of parameters, all four coordinates appear in the Lagrangian formalism in a perfectly *symmetric* way. The individual features of a detailed analogy with a mechanical system of particles become less sharply emphasized, but the presentation of the theory becomes relativistically covariant from the outset. Naturally, the final results are equivalent in the two cases. We shall develop our presentation on the basis of the Lagrangian formalism.

**1.3. Lagrangian Function and Principle of Stationary Action.** The field equations and their invariants are obtained directly from the Lagrangian function. Let us therefore first formulate the fundamental requirements to be imposed on this function. A prominent place among them is occupied by the requirement of *relativistic invariance*, or the condition of invariance with respect to the full inhomogeneous Lorentz group. In this connection let us recall the definition of the Lorentz group. As is well known, the *full Lorentz group* is the group of homogeneous linear transformations of the coordinates of four-dimensional space-time which leave invariant the quadratic form representing the square of a four-interval:

$$x^2 = x_n x^n \equiv (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2,$$

$$x^i \in \mathbb{R}$$

$$x^0 = ct = t, \text{ da } c = 1$$

and which do not reverse the direction of time.\* This group includes spatial rotations in the three planes  $x^1x^2$ ,  $x^2x^3$ ,  $x^3x^1$ , the Lorentz rotations in the three planes  $x^0x^1$ ,  $x^0x^2$ ,  $x^0x^3$ , the reflections of the three space axes  $x^1$ ,  $x^2$ ,  $x^3$ , and all the products of these transformations. The determinants of the transformations of rotation are equal to +1, while the determinants of the reflections of the axes are equal to -1. Therefore the proper Lorentz group of transformations with the determinant +1 is separated from the full group and includes the six rotations and the reflections of an even number of space axes which are equivalent to rotations.

It is often convenient to consider the full Lorentz group together with transformations of translation along all four coordinate axes. For brevity, we shall refer to this combined set of transformations as the inhomogeneous *Lorentz group* (full Poincaré group).

Invariance under this group will be called *relativistic* or *Lorentz invariance*.

Finally, the inclusion of the time reversal operation (transition from the orthochronous to the nonorthochronous group) leads us to the general Lorentz group (correspondingly, the general Poincaré group).

We return now to the Lagrangian function. It is a function of time, and in mechanics it is expressed as a sum over all the material points of the system. For a continuous system such as a wave field, this sum is expressed by a spatial integral of the Lagrangian density function

\*Translator's note. In American literature the "full Lorentz group" is usually taken to include time reversal  $t \rightarrow -t$ .

$$\Lambda(x^0) = \int dx \mathcal{L}(x^0, x). \quad (1.2)$$

However, the variational principle deals not with the Lagrangian function  $\Lambda$ , but with the action  $\mathcal{A}$  obtained from it by integration over  $x^0$ . Therefore the noncovariant expression (1.2) in the Lagrangian formalism in fact turns out to be only an intermediate step, and it is quite sufficient to consider the Lagrangian density function

$$\mathcal{L}(x^0, x) = \mathcal{L}(x),$$

which depends on all four space-time variables. For brevity, we shall in future refer to  $\mathcal{L}(x)$  as the Lagrangian.

If the Lagrangian  $\mathcal{L}(x)$  depends only on the state of the fields in an infinitely small neighborhood of the point  $x$ , i.e., on the values of  $u_i$  and of a finite number of their partial derivatives evaluated at the point  $x$ , then it is called of the local Lagrangian, and the corresponding theory is said to be a local theory. In the opposite case when, for example,  $\mathcal{L}(x)$  is of the form

$$\int dy F\left(u(x), u(y), \frac{\partial u(x)}{\partial x^k}, \frac{\partial u(y)}{\partial y^m}\right),$$

we obtain the so-called nonlocal theories which are not discussed in this book.

The Lagrangian  $\mathcal{L}$  is usually taken to be a real function (in the quantized case, a Hermitian operator; see §21.3) of the field variables  $u_i(x)$  and their first derivatives  $u_{i;k}(x)$ , but not an explicit function of the coordinates or having the property of relativistic invariance.

The local Lagrangian can therefore be written in the form

$$\mathcal{L}(x) = \mathcal{L}(u_i(x), u_{i;k}(x)).$$

The integral of the Lagrangian over a certain volume in space-time

$$\mathcal{A} = \int \mathcal{L}(x) dx \quad (dx = dx^0 dx) \quad (1.3)$$

is given the name of *action*.

From the variational principle of stationary action

$$\delta \mathcal{A} = 0,$$

together with the assumption that the variations of the field functions vanish at the surface of the four-volume over which the integral is taken, one obtains, using integration by parts, the Lagrange-Euler equations

$$\frac{\delta \mathcal{A}}{\delta u_i(x)} = \frac{\partial \mathcal{L}}{\partial u_i(x)} - \frac{\partial}{\partial x^k} \frac{\partial \mathcal{L}}{\partial u_{i;k}(x)} = 0, \quad (1.4)$$

which are the equations determining the field functions  $u_i(x)$ , i.e., the field equations. In accordance with the foregoing properties of the Lagrangian, the field equations are differential equations of order not higher than two.

Since the physical properties of the system are determined by the action, i.e., by the integral given by (1.3), the Lagrangian  $\mathcal{L}(x)$  is not uniquely defined. The definition is completed by adding the four-divergence of a four-vector:

$$\mathcal{L}(x) \rightarrow \mathcal{L}(x) + \frac{\partial}{\partial x^k} F^k(x).$$

The integral of  $F^k_{;k}$  in (1.3) can be transformed with the aid of the Gauss-Ostrogradskiy theorem to a surface integral of  $F^k$  over the three-dimensional boundary of the four-dimensional integration region.

The above ambiguity in the definition of the Lagrangian is usually employed by considering  $F^k$  to be a certain function of the field variables:

$$F^k(x) = \Phi^k(u(x)).$$

Since variations of the field functions  $\delta u$  vanish on the boundary of the four-dimensional volume, we find that in this case, the term  $F^k_{;k}$  does not contribute to the variation of the action and hence does not effect the dynamic characteristics of the system (see §2, below).

*1.4. Transformation Properties of the Field Functions. Tensors and Spinors.* Before proceeding to the construction of field invariants, we shall examine the transformation properties of the field functions. In other words, we must establish the laws governing the transformation of the field functions describing the wave fields when the coordinates are subjected to transformations belonging to the inhomogeneous Lorentz group

$$\begin{aligned} x \rightarrow x' = Lx, \\ x'^k = \Omega^k_l x^l + a^k = \Omega^k_l x^l + a^k; \quad g_{mn} \quad \Omega^{mk} \quad \Omega^{nl} = g^{kl}. \end{aligned} \quad \begin{aligned} x^l \Omega^k_l = \Omega^k_m g^{lm} x^m = \Omega^k_l x^l \end{aligned} \quad (1.5)$$

The field function  $u(x)$  is understood to be either one function (a single-component field function) or several functions (a many-component field function) of the four coordinates  $x^k$  defined in every coordinate system. A transition from one coordinate system  $x$  to another  $x'$  which is related to  $x$  by the Lorentz transformation (1.5) corresponds to a linear homogeneous transformation of the components of the field function

$$u(x) \rightarrow u'(x') = \Lambda u(x), \quad (1.6)$$

with the matrix of the transformation  $\Lambda$  of the functions being completely determined by the matrix of the Lorentz transformation  $L$ . We emphasize that the above transformation (1.6) is not restricted to a replacement of the argument  $x$  by  $x'$ , and describes a transformation from one coordinate system to another, rather than a displacement from one point of space to another.

Thus to each Lorentz transformation  $L$  there corresponds a linear transformation  $\Lambda_L$ , it being evident that to the identity element of the group  $L$  there corresponds the identity transformation  $\Lambda = 1$ , and that to a product of two elements of the Lorentz group there corresponds the product of two transformations

$$\Lambda_{L_1 L_2} = \Lambda_{L_1} \Lambda_{L_2}.$$

The system of operators  $\Lambda$  possessing such properties is called a *linear representation of the group*. The operators  $\Lambda$  evidently may be represented in the form of matrices whose rank is determined by the number of components of the field function  $u$ . When the number of components of  $u$  is finite, the group of transformations  $\Lambda$  is said to form a *finite-dimensional* representation of the Lorentz group, while in the opposite case we have an *infinite-dimensional* representation of this group. In view of the fact that all the principal physical fields are usually described by functions with a finite number of components, we shall confine our attention to a discussion of only the finite-dimensional representations of the Lorentz group.

Thus we may regard the transformations  $\Lambda$  as operators operating in the finite-dimensional space of the components of the field functions, and represent them by square matrices of finite rank.

Sometimes it happens that the space of the components of the field functions within which the representation  $\Lambda$  operates breaks up into subspaces invariant with respect to all the transformations belonging to a given representation (i.e., into subspaces that transform into themselves under the operation  $\Lambda$ ). Such a representation is said to be *reducible*. In the opposite case, the representation is *irreducible*. If the process of separating out invariant subspaces in the space of a reducible representation is carried out to the end, i.e., if the whole space is divided into invariant subspaces which themselves do not contain invariant subspaces, then it is clear that the initial representation will be broken up into irreducible representations operating in the corresponding invariant subspaces. Thus the study of any reducible representation may be replaced by an equivalent study of the irreducible representations of the given group. Thus all possible types of wave functions and their laws of transformation (1.6) may be obtained by means of investigating the finite-dimensional representations of the Lorentz group.

The investigation referred to above forms a special division of the theory of representations of continuous groups and may be briefly summarized as follows. The finite-dimensional representations of the Lorentz group may be single-valued or double-valued. This is related to the fact that the correspondence  $L \rightarrow \Lambda_L$  need not be necessarily single-valued, since, generally speaking, the field functions are not directly experimentally observable quantities (however, the observable quantities may always be expressed in the form of *bilinear combinations* of the field functions). However, the lack of single-valuedness of the operator  $\Lambda_L$  corresponding to the transformation  $L$  must always be such that the observable quantities transform in a completely single-valued manner under any arbitrary Lorentz transformation  $L$ . Moreover, it is necessary that the operators  $\Lambda_L$  should be continuous functions of the parameters of the transformation  $L$ , i.e., that an infinitesimal transformation of the coordinate system should correspond to an infinitesimal

transformation of the field functions. The combined requirements stated above lead to the representations of the Lorentz group being divided into two categories. The first category is characterized by the *single-valuedness of the correspondence*  $L \rightarrow \Lambda_L$  and contains the single-valued so-called *tensor* and *pseudotensor\* representations*. The field functions which transform in accordance with the tensor representation are called *tensors (pseudotensors)*, and in some cases may be observable themselves (the electromagnetic field). In the second case this correspondence turns out to be two-valued:  $L \rightarrow \pm \Lambda_L$ .

The transformation law for a (pseudo) tensor of the  $N$ th rank

$$T^{i_1, i_2, \dots, i_N}$$

under continuous transformations of coordinates (transformations involving reflections of an odd number of axes will be discussed later) has the form

$$T^{i_1, i_2, \dots, i_N}(x') = \frac{\partial x'^{i_1}}{\partial x^{j_1}} \dots \frac{\partial x'^{i_N}}{\partial x^{j_N}} T^{j_1, \dots, j_N}(x) \quad (1.7)$$

or using the notation of (1.5)

$$T^{i_1, \dots, i_N}(x') = \Omega_{j_1}^{i_1} \dots \Omega_{j_N}^{i_N} T^{j_1, \dots, j_N}(x).$$

The double-valued representations are referred to as *spinor* representations, and the corresponding quantities are called *spinors*. The transformation law for spinor quantities has a more complex structure and is given for the simplest spinors in §6. We merely note that the transformation law for tensor quantities which follows from (1.7)

$$u(x) \rightarrow u'(x') = u(x)$$

in the case of the transformation of translation

$$x'^k = x^k + a^k$$

also holds for spinors.

We shall now give the simplest tensor representations and the quantities which correspond to them. The tensor of zero rank which under any continuous transformation transforms in accordance with the law

$$u'(x') = u(x), \quad (1.8)$$

is an invariant and is called a *scalar (pseudoscalar)*.

\*The distinction between tensors and pseudotensors is related to the transformations of the reflections of space axes and is discussed more fully later.

The tensor of the first rank, which under a rotation of coordinates transforms in accordance with the law

$$u'^k(x') = \Omega^k_n u^n(x) = \Omega^{kn} u_n(x), \quad \Omega^{kn} = -\Omega^{nk} \quad n \neq k \quad (1.9)$$

is called a *contravariant vector* (*pseudovector*). The *covariant vector* associated with it

$$u_k(x) = g_{km} u^m(x)$$

transforms in accordance with the law

$$u'_k(x') = \Omega_k^m u_m(x). \quad (1.10)$$

Corresponding formulas may readily be written down for covariant and contravariant tensors of second and higher ranks.

We have already noted that relations such as (1.7)–(1.10) establish the transformation laws of tensor quantities only for transformations of a continuous type. The laws governing their transformation under reflection of an odd number of space axes are not determined by these expressions and must be formulated separately. Because two consecutive applications of such a reflection must correspond to the identity operator, owing to the single-valuedness of the tensor representation, the transformation laws in question may have only the forms

$$u'(x') = u(x) \quad (1.11)$$

or

$$u'(x') = -u(x). \quad (1.12)$$

Those quantities which change sign on reflection, i.e., which transform according to (1.12) in contrast to quantities transforming in accordance with (1.11), are called *pseudoquantities* (pseudoscalar, pseudovector, pseudotensor, and so on).

The distinction between the transformation laws (1.11) and (1.12) seems at first glance to have a somewhat formal character. However, as we shall see later (§8), the *concept of parity* defined by these relations plays an essential role in determining the possible *forms of interaction* between different fields.

One of the important postulated properties of the Lagrangian is its Lorentz invariance, i.e., the Lagrangian is a scalar. This means that

$$\mathcal{L}'(x') \equiv F(u'_i(x'), u'_{i;k}(x')) = F(u_i(x), u_{i;k}(x)) \equiv \mathcal{L}(x). \quad (1.13)$$

Since the infinitesimal volume element  $dx = dx^0 dx^1 dx^2 dx^3$  is also an invariant, we find that for any finite region of space-time  $\Delta$ , the action is unaffected by the Lorentz transformations, i.e.,

$$\mathcal{A}_\Delta = \int_\Delta \mathcal{L}(x) dx = \int_{\Delta'} \mathcal{L}'(x') dx'.$$

In this expression,  $\Delta$  is the region of integration expressed in terms of the variables  $x$ , and  $\Delta'$  is the same region expressed in terms of the variables  $x'$ . The fact that the Lagrangian is a scalar ensures that action is invariant.

The condition given by (1.13) means that the Lagrangian depends only on invariant combinations of the field functions and their first derivatives.

**1.5. Other Groups of Transformations.** The Lorentz group does not exhaust the list of transformations resulting in physically important conservation laws.

The *gauge transformation of the first kind*, which involves only field functions but not the coordinates, corresponds to the conservation of the electric charge. A *complex* field expressed in terms of two real functions

$$u(x) = u_1(x) + iu_2(x),$$

obviously contains an extra degree of freedom which can be associated with an additional discrete parameter, for example, the electric charge.

The condition that the function must be real (or Hermitian in the quantized case) ensures that both the Lagrangian and the time-independent dynamic variables may be functions of the complex fields  $u$  only through the quadratic forms such as  $\bar{u}u$ , where  $u$  and  $\bar{u}$  are mutually complex conjugate functions or (and) their derivatives. Hence it follows directly that complex wave fields  $u(x)$  may be multiplied by an arbitrary unitary phase factor  $\exp(i\alpha)$  without affecting the quadratic form  $\bar{u}u$ , and hence without producing any observable effects.

Assuming that  $u$  and  $\bar{u}$  are linearly independent functions, we may write the gauge transformation of the first kind in the form

$$u_j \rightarrow u'_j = e^{i\alpha} u_j, \quad \bar{u}_j \rightarrow \bar{u}'_j = e^{-i\alpha} \bar{u}_j. \quad (1.14)$$

Assuming that  $\alpha$  is an infinitesimal quantity, we have

$$u'_j = u_j + i\alpha u_j; \quad \bar{u}'_j = \bar{u}_j - i\alpha \bar{u}_j. \quad (1.15)$$

The transformations given by (1.14) differ from the transformations in the Lorentz group by the fact that they involve only the field functions but not the coordinates. Such transformations, known as transformations corresponding to internal symmetries, play an important role in modern physics. They include isospin transformations, unitary transformations, and certain others.

## §2. Noether's Theorem and Dynamic Invariants

The equations of motion alone are not sufficient for the description of a physical system. One must also be able to express the main physical characteristics of the system in terms of



solutions of these equations. Such time-conserved additive dynamic quantities will be called *dynamic invariants* (or simply *invariants*). The well-known Noether theorem is the instrument enabling us to obtain expressions for the dynamic invariants.

2.1. *Noether's theorem\** states that *to every finite-parameter (depending on  $s$  constant parameters) continuous transformation of field functions and simultaneously coordinates, which ensures that the variation of the action is zero, there correspond  $s$  dynamic invariants, i.e., combinations of field functions and their derivatives that are conserved in time.*

To prove this, consider an infinitesimal transformation of coordinates and field functions:

$$x^k \rightarrow x^{k'} = x^k + \delta x^k, \quad (2.1)$$

$$u_i(x) \rightarrow u'_i(x') = u_i(x) + \delta u_i(x). \quad (2.2)$$

The variations  $\delta x^k$  and  $\delta u_i$  can be expressed in terms of the infinitesimal linearly independent transformation parameters  $\delta \omega_n$  as follows:

$$\delta x^k = \sum_{1 \leq n \leq s} X_{(n)}^k \delta \omega_n, \quad \delta u_i(x) = \sum_{1 \leq n \leq s} \Psi_{i(n)} \delta \omega_n. \quad (2.3)$$

The indices  $i$  and  $n$  of the field functions and the transformation parameters may (or may not) have a simple tensorial significance. We shall not specify it, and will agree to interpret repeated indices as indicating summation.

We note that the transformation law for the derivatives of the field functions

$$u_{i;k}(x) \rightarrow u'_{i;k}(x') = u_{i;k}(x) + \delta u_{i;k}(x)$$

contains the variations  $\delta u_{i;k}$  that are nonderivatives of  $\delta u_i$ . In other words, the operations  $\delta$  and  $\partial/\partial x$  do not commute. The point is that  $\delta u_i$  is the variation of the field function due to both the change in its form and the change in its argument. The variation due to the change in the form of the function is defined by

$$\bar{\delta} u_i(x) = u'_i(x) - u_i(x),$$

which to within second-order terms can be written in the form

$$\bar{\delta} u_i(x) = \delta u_i(x) - u_{i;k} \delta x^k = (\Psi_{i(n)} - u_{i;k} X_{(n)}^k) \delta \omega_n. \quad (2.4)$$

By definition, the operation  $\bar{\delta}$  commutes with  $\partial/\partial x$ .

We now define the variation of the action by

$$\delta \mathcal{A} = \delta \int \mathcal{L}(x) dx = \int \mathcal{L}'(x') dx' - \int \mathcal{L}(x) dx, \quad (2.5)$$

\*This was referred to as the first theorem by Noether (1918). The second Noether theorem involves groups of transformations with parameters dependent on  $x$ .

where

$$\mathcal{L}'(x') = \mathcal{L}(u'_i(x'), u'_{i;k}(x')) = \mathcal{L}(x) + \delta\mathcal{L}(x),$$

and

$$\delta\mathcal{L}(x) = \frac{\partial\mathcal{L}}{\partial u_i} \delta u_i + \frac{\partial\mathcal{L}}{\partial u_{i;k}} \delta u_{i;k} = \bar{\delta}\mathcal{L}(x) + \frac{d\mathcal{L}}{dx^n} \delta x^n.$$

In these expressions,  $\bar{\delta}\mathcal{L}$  is the variation of  $\mathcal{L}$  due to variations in the form of  $u_i$  and  $u_{i;k}$ :

$$\bar{\delta}\mathcal{L}(x) = \frac{\partial\mathcal{L}}{\partial u_i} \bar{\delta}u_i + \frac{\partial\mathcal{L}}{\partial u_{i;k}} \bar{\delta}u_{i;k}, \quad (\prec)$$

and the second term describes the total variation due to variations in the coordinates.

Thus:  $m(1) \quad \mathcal{L}'(x') = \mathcal{L}(x) + \bar{\delta}\mathcal{L}(x) + \frac{d\mathcal{L}}{dx^n} \delta x^n$  where  $\delta x^n$

$$\delta\mathcal{A} = \int \left( \bar{\delta}\mathcal{L}(x) + \frac{d\mathcal{L}}{dx^n} \delta x^n \right) dx + \int \mathcal{L}(x) dx' - \int \mathcal{L}(x) dx.$$

We shall now consider the difference between the last two terms, which describes the variation in the volume of integration.

We have

$$dx' \equiv dx'_0 dx'_1 dx'_2 dx'_3 = \frac{\partial(x'_0, x'_1, x'_2, x'_3)}{\partial(x_0, x_1, x_2, x_3)} dx \approx \left( 1 + \frac{\partial\delta x^k}{\partial x^k} \right) dx.$$

and therefore

$$\int \mathcal{L}(x) dx' - \int \mathcal{L}(x) dx = \int \mathcal{L}(x) \frac{\partial\delta x^k}{\partial x^k} dx$$

and

$$\delta\mathcal{A} = \int \left[ \bar{\delta}\mathcal{L}(x) + \frac{d}{dx^n} (\mathcal{L}(x) \delta x_n) \right] dx. \quad (3)$$

Using the equation of motion

$$\frac{\partial\mathcal{L}}{\partial u_i} = \frac{\partial}{\partial x^k} \left( \frac{\partial\mathcal{L}}{\partial u_{i;k}} \right), \quad \text{where } m(2) =$$

we obtain

$$\bar{\delta}\mathcal{L}(x) = \left\{ \frac{\partial}{\partial x^k} \left( \frac{\partial \mathcal{L}}{\partial u_{i;k}} \right) \bar{\delta}u_i + \frac{\partial \mathcal{L}}{\partial u_{i;k}} \frac{\partial}{\partial x^k} (\bar{\delta}u_i) \right\} = \frac{\partial}{\partial x^k} \left[ \frac{\partial \mathcal{L}}{\partial u_{i;k}} \bar{\delta}u_i \right] \quad \text{where}$$

and

$$\delta \mathcal{A} = \int \frac{d}{dx^n} \left\{ \frac{\partial \mathcal{L}}{\partial u_{i;n}} \bar{\delta}u_i + \mathcal{L}(x) \delta x^n \right\} dx.$$

Taking (2.3) and (2.4) into account, we have

$$\delta \mathcal{A} = - \sum_{1 \leq n \leq s} \int \partial_k [\theta_{(n)}^k(x)] dx \delta \omega_n,$$

where

$$\theta_{(n)}^k(x) = - \frac{\partial \mathcal{L}}{\partial u_{i;k}} (\Psi_{i(n)} - u_{i;l} X_{(n)}^l) - \mathcal{L}(x) X_{(n)}^k. \quad (2.5)$$

Since the first variation of action must vanish, and if we equate to zero the coefficients of the independent transformation parameters  $\delta \omega_n$ , we obtain

$$\frac{\partial \mathcal{A}}{\partial \omega_n} = - \int \partial_k [\theta_{(n)}^k(x)] dx = 0. \quad (2.6)$$

Since the region of integration is arbitrary, we obtain the continuity equation

$$\frac{d}{dx^k} \theta_{(n)}^k(x) = 0. \quad (2.7)$$

Transforming the right-hand side of (2.6) by Gauss' theorem, we obtain the conservation laws for the corresponding surface integrals. If we further suppose that the integral in (2.6) is valuated over a volume that expands without limit in space-like directions, but is bounded in time-like directions by space-like three-dimensional surfaces  $\sigma_1$  and  $\sigma_2$ , we find that if the field is practically zero on the boundaries of the spatial volume,

$$\int_{\sigma_1} d\sigma_k \theta_{(n)}^k - \int_{\sigma_2} d\sigma_k \theta_{(n)}^k = 0.$$

In this expression,  $d\sigma_k$  is the projection of the surface area element  $\sigma$  onto the three-plane perpendicular to the  $x^k$  axis. The above equation shows that the surface integrals

$$C_n(\sigma) = \int_{\sigma} d\sigma_k \theta_{(n)}^k$$

are in fact independent of the surface  $\sigma$ . In the special case where the surfaces are the three-planes  $x^0 = t = \text{const}$ , the integral is evaluated over the three-dimensional configuration space, and the integrals

$$C_{(n)}(x^0) = \int dx \theta_{(n)}^0 = \text{const} \quad (2.8)$$

are independent of time.

We have thus shown that to each continuous  $s$ -parameter transformation of coordinates (2.1) and field functions (2.2), there correspond  $s$  time-independent invariants  $C_n$  ( $n = 1, \dots, s$ ) given by (2.8).

The quantities  $\theta_{(n)}^k$  are not unique. Expressions of the form

$$\frac{\partial}{\partial x^m} f_{(n)}^{km}$$

can be added to them if

$$f_{(n)}^{km} = -f_{(n)}^{mk}.$$

This ambiguity does not, however, affect the value of the conserved integrals (2.8).

We must now consider special cases of  $\theta$  and the associated conservation laws (2.8).

**2.2. Energy-momentum Vector.** For infinitesimal space-time translations

$$x'^k = x^k + \delta x^k,$$

we can take  $\delta x^k$  for the transformation parameters  $\delta \omega^i$  and, if we take the transformation law (1.8) into account, we obtain

$$X_l^k = \delta_l^k, \quad \Psi_{ll} = 0 \quad (l = 0, 1, 2, 3),$$

and  $\theta$  transforms into a tensor of rank 2:

$$T_l^k = \frac{\partial \mathcal{L}}{\partial u_{i;k}(x)} \frac{\partial u_i}{\partial x^l} - \mathcal{L} \delta_l^k \quad (k, l = 0, 1, 2, 3).$$

In the fully contravariant form, this tensor becomes

$$T^{lk} = \frac{\partial \mathcal{L}}{\partial u_{i;k}(x)} \frac{\partial u_i}{\partial x_l} - \mathcal{L} g^{kl}. \quad (2.9)$$

Integrals of  $T^{lk}$  of the form given by (2.8) represent the time-conserved four-vector

$$P^l = \int T^{l0} dx. \quad (2.10)$$

$T^{00} = \text{Energy density}$   
 $T^{l0} = \text{Momentum density}$

In classical mechanics, the zero component of this vector  $P^0$ , is the Hamilton function, i.e., the energy. Covariance considerations therefore show that the four-vector given by (2.10) is the energy-momentum vector, whereas (2.9) gives the *energy-momentum tensor*.

We note that we shall be interested only in integral dynamic quantities similar to the energy-momentum four-vector  $P^j$ . The structure of the tensor  $T^{kl}$ , which in our presentation is not even unique, would be of independent interest only in a theory capable of including the gravitational effects in a systematic fashion. It is known, however, that a unified theory of this kind does not as yet exist, and we shall therefore not consider such problems here.

*2.3. Angular Momentum Tensor and Spin Tensor.* Under infinitesimal four-rotations

$$x'^n = x^n + x_m \delta \Omega^{nm}$$

the antisymmetry property

$$\delta \Omega^{nm} = -\delta \Omega^{mn}$$

ensures that we can take linearly independent transformation parameters

$$\delta \Omega^{nm} = \delta \omega^{nm}, \quad n < m.$$

The indices  $n, m$  define the plane in which rotation with the parameter  $\delta \omega^{nm}$  takes place. We see that the index ( $n$ ) in (2.3) splits into two indices:

$$(n) \rightarrow n, m,$$

and, since  $\omega^{nm}$  is antisymmetric, we have

$$\begin{aligned} \delta x^k &= X_j^k \delta \omega^j = \sum_{n < m} X_{nm}^k \delta \omega^{nm} = x_l \delta \omega^{kl} = x_l \delta \omega^{ml} \delta_m^k = \\ &= \sum_{m \leq l} x_l \delta \omega^{ml} \delta_m^k + \sum_{m > l} x_l \delta \omega^{ml} \delta_m^k = \sum_{m \leq l} \delta \omega^{ml} (x_l \delta_m^k - x_m \delta_l^k), \end{aligned}$$

from which it follows that

$$X_{nm}^k = x_m \delta_n^k - x_n \delta_m^k \quad (n \leq m). \quad (2.11)$$

The total variation of the field function will be written in the form

$$u'_i(x') = u_i(x) + \delta u_i, \quad \delta u_i = \sum_{l, k \leq l} A_{ikl}^i u_j(x) \delta \omega^{kl}.$$

For the scalar field

$$A_{ikl}^i = 0,$$

and for the vector field

$$A_{ikt}^l = g_{ik}\delta_l^t - g_{it}\delta_k^l, \quad k \leq l.$$

Substituting for  $X_{nm}^k$  from (2.11) and

$$\Psi_{inm} = A_{inm}^l u_j(x) = g_{in}u_m(x) - g_{im}u_n(x)$$

in (2.5), we obtain the following expression for the angular momentum tensor:

$$M_{lm}^k = \frac{\partial \mathcal{L}}{\partial (\partial u_i / \partial x^k)} \left\{ \frac{\partial u_l}{\partial x^l} x_m - \frac{\partial u_i}{\partial x^m} x_l \right\} + \mathcal{L} (x_l \delta_m^k - x_m \delta_l^k) - \\ - \frac{\partial \mathcal{L}}{\partial (\partial u_i / \partial x^k)} A_{ilm}^l u_j(x) = (x_m T_l^k - x_l T_m^k) - \frac{\partial \mathcal{L}}{\partial (\partial u_i / \partial x^k)} A_{ilm}^l u_j(x). \quad (2.12)$$

It is clear from (2.12) that there is a relationship between the symmetry properties of the energy-momentum tensor  $T_l$  and the structure of the angular momentum tensor  $M_{lm}$ . In the case of the scalar field, the second term in (2.12) is absent and the relationship between  $M$  and  $T$  assumes a form similar to that encountered in particle mechanics. The term

$$M_0^{ml,k} = x^m T^{lk} - x^l T^{mk} \quad (2.13)$$

must therefore be identified with the intrinsic *orbital* angular momentum of the wave field. In the case of a one-component field, this angular momentum is conserved:

$$\frac{\partial M_0^{ml,k}}{\partial x^k} = 0. \quad (2.14)$$

Substituting (2.13) into this expression, we find that the energy-momentum tensor for the scalar field is symmetric in its indices:

$$T^{kl} = T^{lk}.$$

In the case of a multicomponent (vector or spinor) field, the expression for  $M^{lm,k}$  has the form given by (2.12). The second term in this expression, i.e.,

$$S_{lm}^k = - \frac{\partial \mathcal{L}}{\partial (\partial u_i / \partial x^k)} u_j(x) A_{i,lm}^l, \quad (2.15)$$

characterizes the polarization properties of the field and, as follows from quantum theory (see Chapter 2), corresponds to the spin angular momentum of the particles described by the quantized fields.

The spatial density of the orbital and spin angular momenta is given by

$$M_0^{ml,0} = x^m T^{l0} - x^l T^{m0},$$

$$S_{lm}^0 = - \frac{\partial \mathcal{L}}{\partial (\partial u_i / \partial x^0)} u_j(x) A_{l,lm}^i.$$

Integrating these expressions over configuration space, we obtain the orbital and spin angular momentum tensors in the form

$$M_0^{lm} = \int M_0^{ml,0} d\mathbf{x} = \int d\mathbf{x} (x^m T^{l0} - x^l T^{m0}),$$

$$S_{lm} = \int S_{lm}^0 d\mathbf{x} = - \int d\mathbf{x} \frac{\partial \mathcal{L}}{\partial (\partial u_i / \partial x^0)} u_j(x) A_{l,lm}^i. \quad (2.16)$$

Contracting the space components of the last expression with the antisymmetric tensor of rank three,  $\varepsilon_{\alpha\beta\gamma}$ , we obtain the components of the three-dimensional (pseudo) spin vector

$$S_\alpha = \varepsilon_{\alpha\beta\gamma} S_{\beta\gamma}. \quad (2.17)$$

*2.4. Isotopic Spin, Charge, and Current Vector.* We shall consider rotation in the fictitious three-dimensional isospin space. Since the wave functions do not explicitly depend on the coordinates of this space, and the usual coordinates  $x_k$  are unaffected by isospin transformations, we can write the formulas for an infinitesimal transformation for the wave functions alone:

$$u'_i = u_i + \delta u_i, \quad \delta u_i = R_{ij}^{\alpha\beta} u_j \delta \omega_{\alpha\beta}. \quad (2.18)$$

Here,  $\delta \omega_{\alpha\beta}$  are infinitesimal rotations of isospin space that are antisymmetric in the indices  $\alpha, \beta$  ( $=1, 2, 3$ ).

We now see that the tensor  $\theta$  (2.5) for the present case differs from the angular momentum tensor (2.12) by the absence of the orbital term

$$N_{\alpha\beta}^k = - \frac{\partial \mathcal{L}}{\partial u_{i;k}} R_{ij}^{\alpha\beta} u_j. \quad (2.19)$$

It is completely analogous to the spin angular momentum tensor (2.15) with the exception of the fact that it is a tensor in three-dimensional space in so far as the indices  $\alpha, \beta$  are concerned. The corresponding space integrals

$$I_{\alpha\beta} = \int N_{\alpha\beta}^0 d\mathbf{x}$$

are conserved in time and are the components of the isotopic spin axial three-vector (in isospin space)

$$I_\alpha = \varepsilon_{\alpha\beta\gamma} I_{\beta\gamma}. \quad (2.20)$$

The third component of this vector is connected with the electric charge  $Q$  which is given by

$$Q = I_3 + \frac{B+S}{2} = I_3 + \frac{Y}{2}. \quad (2.21)$$

Here  $B$  is the baryon number,  $S$  is the strangeness, and  $Y = B + S$  is the hypercharge.

When  $Y = 0$  ( $\Sigma$  hyperons or pions),  $Q = I_3$ , and the corresponding space density is the zero component of the current four-vector

$$J^0(x) = N_{12}^0. \quad (2.22)$$

In the more general case, one must take into account the second term on the right-hand side of (2.21). Thus, for protons and neutrons forming the isospin nucleon doublet  $S = 0$ ,  $B = Y = 1$ ,

$$Q = I_3 + 1/2. \quad (2.23)$$

For  $K$ -mesons,  $B = 0$ ,  $S = Y = -1$ , and consequently

$$Q = I_3 - 1/2. \quad (2.24)$$

It is well known that  $K$ -mesons form an isospin doublet ( $I_3 = +1/2, -1/2$ ) consisting of  $K^0$  and  $K^-$ .

Consider the isospin doublet for  $\Xi$  hyperons ( $B = -Y = 1, S = -2$ ). The corresponding field function is a two-component function and is a spinor in the three-dimensional isospin space. It transforms as follows under rotations through the angle  $\beta = \omega_{12}$  in the plane  $(z_1, z_2)$  of this space (for further details of the spinor transformation laws see §6.4):

$$u_\sigma \rightarrow u'_\sigma = e^{-i\frac{\beta}{2}(\tau_3)_{\sigma\varrho}} u_\varrho = \left\{ \delta_{\sigma\varrho} \cos \frac{\beta}{2} - i(\tau_3)_{\sigma\varrho} \sin \frac{\beta}{2} \right\} u_\varrho. \quad (2.25)$$

Here,  $\tau_3$  is the spin matrix which can be taken in the diagonal form

$$\tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (2.26)$$

For an infinitesimal  $\beta$  we have

$$u'_\sigma = u_\sigma + \delta u_\sigma, \quad \delta u_1 = -i\frac{\beta}{2}u_1, \quad \delta u_2 = +i\frac{\beta}{2}u_2. \quad (2.27)$$

We have thus obtained formulas that are practically identical with those for the gauge transformation of §1.5. The latter are therefore a special case of isospin rotation through the angle  $\alpha = \beta/2$  around the  $z_3$  axis of isospin space.



Comparison of (2.18) and (2.25) will show that

$$R_{ij}^{12} = -i(\tau_3)_{ij}.$$

Substituting this in (2.19), and using the explicit form of (2.26), we obtain the current four-vector

$$J^n(x) = i \left( \frac{\partial \mathcal{L}}{\partial u_{1;n}} u_1 - \frac{\partial \mathcal{L}}{\partial u_{2;n}} u_2 \right) \quad (2.28)$$

and the charge

$$Q = \int J^0(x) dx = i \int \left( \frac{\partial \mathcal{L}}{\partial u_{1;0}} u_1 - \frac{\partial \mathcal{L}}{\partial u_{2;0}} u_2 \right) dx. \quad (2.29)$$

We emphasize one further point. The requirement of Lorentz invariance (and the ensuing conservation laws for the four-momentum and the angular momentum) is a necessary attribute of any relativistic field theory. Moreover, invariance under transformations such as gauge transformations of the first kind, or the more general isospin transformations, is an additional requirement reflecting some more or less specific properties of elementary particles and the corresponding wave fields. These requirements are based on experiment and are frequently approximate in character. For example, the strong interactions between elementary particles are invariant under any rotation in isospin space. Interactions with the electromagnetic field violate this invariance, retaining only that part of it that is connected with rotations about the  $z_3$  axis, i.e., with the conservation of electric charge. Consequently, the law of conservation of electric charge may be said to be an exact conservation law in strong and electromagnetic interactions, whereas complete isospin invariance (isospin symmetry of interactions) and, in particular, the conservation laws relating to the other isospin components  $I_1$  and  $I_2$ , are only approximate and are violated by electromagnetic interactions.

These approximate conservation laws, based as they are on approximate invariance or symmetry conditions, have played an increasingly important role in elementary-particle physics. The best known among them are the various schemes involving the so-called unitary symmetries and, especially, the  $SU(3)$  scheme.

We shall not discuss these questions in detail here and will confine ourselves to noting that  $SU(3)$  symmetry is approximate even for strong interactions.

### §3. Scalar Field

We now proceed to the various special cases of wave fields and apply to them the general formalism developed above. In Chapter I, we shall consider the following important free wave fields:

(a) the scalar (pseudoscalar) field corresponding to spinless particles (for example, pseudoscalar pions)

- (b) the vector field
- (c) the electromagnetic field
- (d) the simple spinor field with spin 1/2, corresponding to electrons-positrons, nucleons, muons, and certain hyperons
- (e) the massless spinor field corresponding to the neutrino

We emphasize that it would be <sup>more</sup>correct to say that the free fields “describe” the corresponding particles because the complete description of elementary particles, including all physical characteristics (for example, magnetic moments), can only be provided by a theory of interacting fields. It is therefore more correct to say that the individual free fields correspond to different particles and provide a basis for the description of these particles within the framework of the theory of interacting fields.

As indicated in §2, it is usually assumed that the Lagrangian depends only on the field functions and their partial derivatives of order not higher than one. This leads to the fact that the corresponding field equations are differential equations of order not higher than two.

An important property of the free-field Lagrangians follows from the requirement of linearity and homogeneity of the free-field equations. Only Lagrangians that are quadratic in the field functions and their derivatives lead to equations of this type.

These conditions, taken together with the relativistic invariance and the transformation properties of the field functions, define the Lagrangian to within its coefficients.

We shall now consider the simplest wave field, namely, the scalar field in the following two variants:

- (a) the real (pseudo) scalar field describing neutral spinless mesons, and
- (b) the complex (pseudo) scalar field describing charged spinless mesons.

**3.1. Lagrangian Formalism for a Real Scalar Field.** The simplest wave field is described by a single-component real wave function  $\varphi(x)$  which transforms under Lorentz transformations like a scalar or a pseudoscalar. As was noted in the preceding section, the difference between a scalar and a pseudoscalar lies in their transformation properties under an odd number of reflections of the coordinate axes, and this becomes apparent only in the form of the possible laws of interaction (more precisely in the form of the interaction Lagrangian) with other fields. Therefore, in the theory of free fields we shall not make a distinction between scalars and pseudoscalars, vectors and pseudovectors, etc., and shall in fact discuss them simultaneously.

Let us consider the real scalar field which describes neutral spinless particles (neutral (pseudo) scalar mesons with spin zero).

The Lagrangian of this field

$$\mathcal{L} = \frac{1}{2} \frac{\partial \varphi}{\partial x_k} \frac{\partial \varphi}{\partial x^k} - \frac{m^2}{2} \varphi^2(x) = \frac{1}{2} \partial_\mu \varphi \partial^\mu \varphi - \frac{m^2}{2} \varphi^2 \quad (3.1)$$

is determined by the conditions formulated above up to its coefficients which are chosen in such a way that the formulas (1.4) lead to the Klein-Gordon equation

$$-\frac{\partial^2 \varphi(x)}{\partial x_k \partial x^k} - m^2 \varphi = (\square - m^2) \varphi(x) = 0, \quad (3.2)$$

which is obviously the only invariant equation of the second order. Here  $\square$  is the d'Alembert operator

$$\square \equiv -\frac{\partial^2}{\partial x^k \partial x_k} = \frac{\partial^2}{\partial x^1 \partial x^1} + \frac{\partial^2}{\partial x^2 \partial x^2} + \frac{\partial^2}{\partial x^3 \partial x^3} - \frac{\partial^2}{\partial x^0 \partial x^0}. \quad (3.3)$$

Substituting the Lagrangian (3.1) into (2.9) we obtain the energy-momentum tensor of the real scalar field in the form

$$T_{kl} = \frac{\partial \varphi}{\partial x^k} \frac{\partial \varphi}{\partial x^l} - g_{kl} \mathcal{L}. \quad (3.4)$$

From this we find the energy density

$$T^{00} = \frac{1}{2} \sum_n \left( \frac{\partial \varphi}{\partial x^n} \right)^2 + \frac{m^2}{2} \varphi^2 = \frac{1}{2} \frac{\partial \varphi}{\partial x^k} \frac{\partial \varphi}{\partial x^k} + \frac{m^2}{2} \varphi^2 \quad (3.5)$$

and the density of the momentum vector

$$T^{0\alpha} = -\frac{\partial \varphi}{\partial x^0} \frac{\partial \varphi}{\partial x^\alpha} \quad (\alpha = 1, 2, 3). \quad (3.6)$$

With the aid of (2.12) we obtain the expression for the angular-momentum tensor

$$M_{lm}^k = \frac{\partial \varphi}{\partial x^k} \left( x_m \frac{\partial \varphi}{\partial x^l} - x_l \frac{\partial \varphi}{\partial x^m} \right) + \mathcal{L} (x_l \delta_m^k - x_m \delta_l^k). \quad (3.7)$$

Setting  $k = 0$  in this expression, we obtain the spatial density of the angular-momentum tensor conserved in time:

$$M_{lm}^0 = \frac{\partial \varphi}{\partial x^0} \left( x_m \frac{\partial \varphi}{\partial x^l} - x_l \frac{\partial \varphi}{\partial x^m} \right) + \mathcal{L} (x_l \delta_m^0 - x_m \delta_l^0). \quad (3.8)$$

The spin angular momentum of a scalar field is equal to zero since it has only a single component. Since the field is real, the current four-vector is also equal to zero.

**3.2. Momentum Representation and Positive- and Negative-Frequency Components.** The expressions for the dynamic variables assume a more suggestive form in the momentum representation. For this purpose we write the field function  $\varphi(x)$  in the form of a four-dimensional Fourier integral

$$\varphi(x) = \frac{1}{(2\pi)^{1/2}} \int dk e^{ikx} \tilde{\varphi}(k), \quad (3.9)$$

where

$$dk = dk^0 dk = dk^0 dk^1 dk^2 dk^3,$$

$$kx = k_n x^n = k^0 x^0 - \mathbf{k} \cdot \mathbf{x},$$

while the factor  $2\pi$  has been taken to the power  $-\frac{3}{2}$  for the sake of convenience in a later transition to a three-dimensional integral.

The condition that  $\varphi(x)$  should be real leads to the following property of complex conjugation for  $\tilde{\varphi}$ :

$$\tilde{\varphi}^*(k) = \tilde{\varphi}(-k). \quad (3.10)$$

Substituting the expansion (3.9) into the field equation (3.2), we find that the function  $\tilde{\varphi}$  satisfies the equation

$$(k^2 - m^2) \tilde{\varphi}(k) = 0$$

and may therefore be represented in the form

$$\tilde{\varphi}(k) = \delta(k^2 - m^2) \varphi(k). \quad (3.11)$$

The factor  $\delta(k^2 - m^2)$  establishes the relation between the “energy” variable  $k^0$ , the “momentum” variable  $\mathbf{k}$ , and the term  $m^2$  which therefore represents the square of the mass:

$$k^2 - m^2 = (k^0)^2 - \mathbf{k}^2 - m^2 = 0.$$

If (3.11) is taken into account, the expansion (3.9) takes on the form

$$\varphi(x) = \frac{1}{(2\pi)^{3/2}} \int dk \delta(k^2 - m^2) e^{ikx} \varphi(k). \quad (3.12)$$

Because of the presence under the integral sign of the  $\delta$ -function, the integration is in fact carried out not over the whole four-dimensional space but only over the two three-dimensional hyperboloids

$$k^0 = \pm \sqrt{\mathbf{k}^2 + m^2}, \quad (3.13)$$

one of which lies entirely within the upper light cone and the other entirely within the lower one. Noting further that the two hyperboloids are separately Lorentz-invariant, we arrive at the following Lorentz-invariant decomposition of the integral (3.12) into two terms:

$$\varphi(x) = \varphi^+(x) + \varphi^-(x), \quad (3.14)$$

$$\varphi^+(x) = \frac{1}{(2\pi)^{3/2}} \int dk e^{ikx} \delta(k^2 - m^2) \varphi^+(k), \quad (3.15)$$

$$\varphi^-(x) = \frac{1}{(2\pi)^{3/2}} \int dk e^{-ikx} \delta(k^2 - m^2) \varphi^-(k). \quad (3.16)$$

Here

$$\varphi^+(k) = \theta(k^0) \varphi(k) \quad \text{and} \quad \varphi^-(k) = \theta(k^0) \varphi(-k), \quad (3.17)$$

and  $\theta$  is the well-known step function

$$\theta(x) = \begin{cases} 1, & x > 0, \\ 0, & x < 0. \end{cases} \quad (3.18)$$

The functions  $\varphi^{(+)}(x)$  and  $\varphi^{(-)}(x)$  will in the future be referred to, in accordance with the indices introduced above, as the *positive-frequency* and *negative-frequency* parts of the function  $\varphi(x)$ . As may easily be seen the sign of the frequency is determined by the sign of the product  $kx$  (or, more precisely, by the sign of the "frequency" term  $k^0 x^0 = +x^0(k^2 + m^2)^{1/2}$ ) in the exponent of the integrand. In this connection we note that in the contemporary literature the converse notation is often used, i.e., the sign of the frequency is taken to be the same as that of the form

$$k_\mu x_\mu = kx + k_4 x_4 = -kx.$$

We have abandoned such a notation because (as will be seen in Chapter II) expressions such as (3.15) correspond in quantum theory to the creation of the field particles, while expressions such as (3.16) correspond to their annihilation. Therefore, in the system of notation adopted here, the signs (+) and (−) not only correspond to the sign of the frequency, but also symbolize the physical meaning of the corresponding quantum operators with (+) referring to creation and (−) to annihilation.

It will be shown later that the above decomposition also turns out to be very convenient for writing down the dynamic variables in the momentum representation, since they may be expressed as quadratic forms in  $\varphi^{(+)}(k)$  and  $\varphi^{(-)}(k)$ .

We also note that, since  $\varphi(x)$  is real, the rules of complex conjugation for  $\varphi^{(+)}(k)$  and  $\varphi^{(-)}(k)$  take the form

$$(\varphi^+(k))^* = \varphi^-(k), \quad (\varphi^-(k))^* = \varphi^+(k). \quad (3.19)$$

Carrying out the integration over  $k^0$  in (3.12) and (3.13), we obtain

$$\left. \begin{aligned} \varphi^+(x) &= \frac{1}{(2\pi)^{3/2}} \int \frac{dk}{2k^0} e^{ikx} \varphi^+(k), \\ \varphi^-(x) &= \frac{1}{(2\pi)^{3/2}} \int \frac{dk}{2k^0} e^{-ikx} \varphi^-(k) \end{aligned} \right\} (k^0 = +\sqrt{k^2 + m^2}). \quad (3.20)$$

If we invert (3.20), we can express  $\varphi^\pm(k)$  in terms of the field function in the coordinate

representation  $\varphi(x)$  and its time derivative  $\dot{\varphi}(x) = \varphi;_0(x)$ . The result is

$$\varphi^\pm(k) = \int (2\pi)^{-3/2} e^{\mp i k x} [k^0 \varphi(x) \mp i \dot{\varphi}(x)] dx; \quad k^0 = \sqrt{k^2 + m^2}. \quad (3.21)$$

Here it is convenient to renormalize the three-dimensional Fourier amplitudes. For this we introduce the notation

$$\left. \begin{aligned} \varphi^+(k) &= \frac{\varphi^+(k)}{\sqrt{2k^0}} \\ \varphi^-(k) &= \frac{\varphi^-(k)}{\sqrt{2k^0}} \end{aligned} \right\} (k^0 = +\sqrt{k^2 + m^2}), \quad (3.22)$$

and then expressions (3.20) will take on the form

$$\varphi^+(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{dk}{\sqrt{2k^0}} e^{i k x} \varphi^+(k), \quad (3.23)$$

$$\varphi^-(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{dk}{\sqrt{2k^0}} e^{-i k x} \varphi^-(k). \quad (3.24)$$

The choice of the normalizing constant in (3.22) will become clear from the expressions for the dynamic variables (see (3.26) below).

Substituting (3.14), (3.23), and (3.24) into the expression (3.5) for the energy density, and integrating over the configuration space, we obtain

$$\begin{aligned} P^0 &= \int T^{00} dx = \frac{1}{2} \int \left[ \sum_k \left( \frac{\partial \varphi}{\partial x^k} \right)^2 + m^2 \varphi^2 \right] dx = \\ &= \frac{1}{2} \int dx \left\{ \sum_k \left( \frac{\partial \varphi^+}{\partial x^k} \frac{\partial \varphi^+}{\partial x^k} + 2 \frac{\partial \varphi^+}{\partial x^k} \frac{\partial \varphi^-}{\partial x^k} + \frac{\partial \varphi^-}{\partial x^k} \frac{\partial \varphi^-}{\partial x^k} \right) + \right. \\ &\quad \left. + m^2 (\varphi^+(x) \varphi^+(x) + 2 \varphi^+(x) \varphi^-(x) + \varphi^-(x) \varphi^-(x)) \right\}. \end{aligned}$$

It is readily shown that the terms containing the products of the functions  $\varphi^\pm$  with the same parity do not contribute to the dynamic invariant  $P^0$ .

Since, for example,

$$\begin{aligned} \int dx \left\{ \sum_n \frac{\partial \varphi^+}{\partial x^n} \frac{\partial \varphi^+}{\partial x^n} + m^2 \varphi^+(x) \varphi^+(x) \right\} &= \\ = \int \frac{dk dk'}{2 \sqrt{k^0 k'^0}} \varphi^+(k) \varphi^+(k') e^{i(k^0 + k'^0)x^0} \cdot \left( m^2 - \sum_n k^n k'^n \right) \frac{1}{(2\pi)^3} \int dx e^{-i(k+k')x} &= \\ = \int \frac{dk}{2k^0} \varphi^+(k) \varphi^+(-k) e^{2ik^0 x^0} (m^2 - k^0 k^0 + k k) & \end{aligned}$$

and, since

$$m^2 - k^0 k^0 + \mathbf{k} \cdot \mathbf{k} = m^2 - k^2 = 0,$$

we obtain:

$$\int d\mathbf{x} \left\{ \sum_n \frac{\partial \varphi^+}{\partial x^n} \frac{\partial \varphi^+}{\partial x^n} + m^2 \varphi^+(x) \varphi^+(x) \right\} = 0.$$

A similar relation holds for the quadratic form in  $\varphi^{(-)}$ . Therefore

$$P^0 = \int d\mathbf{x} \left\{ \sum_k \frac{\partial \varphi^+}{\partial x^k} \frac{\partial \varphi^-}{\partial x^k} + m^2 \varphi^+(x) \varphi^-(x) \right\}. \quad (3.25)$$

A very similar calculation will now show that

$$P^0 = \int d\mathbf{k} \cdot k^0 \varphi^+(\mathbf{k}) \varphi^-(\mathbf{k}) \quad (k^0 = +\sqrt{\mathbf{k}^2 + m^2}).$$

The corresponding expression for the momentum vector has the form

$$P^\alpha = \int T^{0\alpha} d\mathbf{x} = \int d\mathbf{k} \cdot k^\alpha \varphi^+(\mathbf{k}) \varphi^-(\mathbf{k}) \quad (\alpha = 1, 2, 3).$$

Combining these expressions, we write them in the form which would also hold if, throughout the whole calculation, the functions  $\varphi^{(+)}$  and  $\varphi^{(-)}$  were treated as noncommuting:

$$P^n = \frac{1}{2} \int d\mathbf{k} \cdot k^n (\varphi^+(\mathbf{k}) \varphi^-(\mathbf{k}) + \varphi^-(\mathbf{k}) \varphi^+(\mathbf{k})). \quad (3.26)$$

This representation will turn out to be useful for the quantization of the scalar field.

We now see the point of choosing the particular normalization in (3.22). The amplitudes  $\varphi^+(\mathbf{k})$  and  $\varphi^{(-)}(\mathbf{k})$  were so chosen that the products  $\varphi^{(+)}(\mathbf{k})\varphi^{(-)}(\mathbf{k})$  could be interpreted as the average density of particles of momentum  $\mathbf{k}$ , energy  $k^0$ , mass  $m = \sqrt{(k^0)^2 - \mathbf{k}^2}$ , and no charge or spin. An example of such uncharged spinless particles is the pseudoscalar neutral pion.

The four-dimensional amplitudes  $\varphi^{(+)}(k)$  and  $\varphi^{(-)}(k)$  enable us to write the energy-momentum four-vector in the manifestly covariant form

$$P^n = \frac{1}{2} \int d\mathbf{k} \cdot k^n \theta(k^0) \delta(k^2 - m^2) (\varphi^+(k) \varphi^-(k) + \varphi^-(k) \varphi^+(k)). \quad (3.27)$$

*3.3. The Discrete Momentum Representation.* The discrete momentum representation is sometimes convenient. It is then assumed that the field is contained within a spatial cube of volume  $V = L^3$ , where  $L$  is the length of the cube edge. From the condition that  $\varphi$  should be periodic in the space coordinates with a period  $L$  we find that  $\varphi$  may be represented in the form of the sum

$$\begin{aligned} \varphi(x) = & \frac{1}{L^{3/2}} \sum_{n_1, n_2, n_3} \frac{\varphi_{n_1, n_2, n_3}^+}{\sqrt{2k^0}} \exp \left[ \frac{2\pi i}{L} (n_0 x^0 - n_1 x^1 - n_2 x^2 - n_3 x^3) \right] + \\ & + \frac{1}{L^{3/2}} \sum_{n_1, n_2, n_3} \frac{\varphi_{n_1, n_2, n_3}^-}{\sqrt{2k^0}} \exp \left[ -\frac{2\pi i}{L} (n_0 x^0 - n_1 x^1 - n_2 x^2 - n_3 x^3) \right], \end{aligned} \quad (3.28)$$

where  $n_1$ ,  $n_2$ , and  $n_3$  in this expression assume all integral values from  $-\infty$  to  $+\infty$ , and

$$k^0 = \frac{2\pi}{L} n_0, \quad n_0 = \sqrt{n_1^2 + n_2^2 + n_3^2 + \frac{L^2 m^2}{4\pi^2}}.$$

The summation in (3.28) is carried out over momenta whose components are integral multiples of  $2\pi/L$ . By letting the size  $L$  of the cube tend to infinity, it is evidently possible in the limit to go over to the continuous representation. For this limiting process

$$\left( \frac{2\pi}{L} \right)^3 \rightarrow dk, \quad (3.29)$$

and also

$$\left( \frac{L}{2\pi} \right)^{3/2} \varphi_n \rightarrow \varphi(k). \quad (3.30)$$

These rules can readily be used to show that the energy-momentum four-vector (3.26) assumes the following form in the discrete representation:

$$P^m = \sum_{n_1, n_2, n_3} k^m \varphi_{n_1, n_2, n_3}^+ \varphi_{n_1, n_2, n_3}^- \quad (3.31)$$

Expression (3.31) enables us to treat the field confined within the volume  $L^3$  as a system with an infinite number of degrees of freedom—a collection of an infinite number of particles, which from the point of view of the Hamiltonian formalism are usually treated by introducing the canonical coordinates and momenta as field oscillators with energy

$$k^0 = \frac{2\pi}{L} \sqrt{n_1^2 + n_2^2 + n_3^2 + \frac{L^2 m^2}{4\pi^2}}$$

and momentum

$$k = \frac{2\pi}{L} n \quad (n = n_1, n_2, n_3).$$

The quadratic combination of amplitudes  $\varphi_n^{(+)} \varphi_n^{(-)}$  then plays the role of the average number of particles with given energy and momentum.

**3.4. The Complex Scalar Field.** The theory of the complex scalar field is constructed by analogy with the theory of the real scalar field. The difference consists of the fact that the complex scalar field is described by the complex function



$$\varphi(x) = \varphi_1(x) + i\varphi_2(x),$$

i.e., essentially by two independent real functions  $\varphi_1$  and  $\varphi_2$ . However, the formalism becomes more convenient if instead of  $\varphi_1$  and  $\varphi_2$  their combinations  $\varphi$  and  $\varphi^* = \varphi_1 - i\varphi_2$  are used.

We write down the Lagrangian of the complex field in a form analogous to that of the Lagrangian of the real field

$$\mathcal{L} = \frac{\partial \varphi^*}{\partial x^\alpha} \frac{\partial \varphi}{\partial x_\alpha} - m^2 \varphi^*(x) \varphi(x). \quad (3.32)$$

Taking the variations of the functions  $\varphi$  and  $\varphi^*$  to be independent, we obtain from this expression with the aid of the formulas of §3: the field equations

$$(\square - m^2) \varphi(x) = 0, \quad (\square - m^2) \varphi^*(x) = 0;$$

the energy-momentum tensor

$$T_{kl} = \left( \frac{\partial \varphi^*}{\partial x^k} \frac{\partial \varphi}{\partial x^l} + \frac{\partial \varphi^*}{\partial x^l} \frac{\partial \varphi}{\partial x^k} \right) - g^{kl} \mathcal{L}; \quad (3.33)$$

the density of the energy-momentum four-vector

$$T^{00} = \sum_k \frac{\partial \varphi^*}{\partial x^k} \frac{\partial \varphi}{\partial x^k} + m^2 \varphi^* \varphi, \quad T^{0\alpha} = - \left( \frac{\partial \varphi^*}{\partial x^0} \frac{\partial \varphi}{\partial x^\alpha} + \frac{\partial \varphi^*}{\partial x^\alpha} \frac{\partial \varphi}{\partial x^0} \right) \quad (3.34)$$

and the current four-vector

$$J_k = i \left( \varphi^* \frac{\partial \varphi}{\partial x^k} - \frac{\partial \varphi^*}{\partial x^k} \varphi \right). \quad (3.35)$$

The spin angular momentum of the complex scalar field turns out to be zero by analogy with the same situation for the real scalar field.

Carrying out the decomposition into positive- and negative-frequency parts, and transforming to the momentum representation,

$$\begin{aligned} \varphi^\pm(x) &= \frac{1}{(2\pi)^{3/2}} \int dk \delta(k^2 - m^2) e^{\pm i k x} \varphi^\pm(k), \\ \varphi^{*\pm}(x) &= \frac{1}{(2\pi)^{3/2}} \int dk \delta(k^2 - m^2) e^{\pm i k x} \varphi^{*\pm}(k), \end{aligned}$$

we find after integration with respect to the variable  $k^0$ :

$$\varphi^{\pm}(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d\mathbf{k}}{\sqrt{2k^0}} e^{\pm i\mathbf{k}\cdot\mathbf{x}} \varphi^{\pm}(\mathbf{k}), \quad (3.36)$$

$$\dot{\varphi}^{\pm}(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d\mathbf{k}}{\sqrt{2k^0}} e^{\pm i\mathbf{k}\cdot\mathbf{x}} \dot{\varphi}^{\pm}(\mathbf{k}), \quad (3.37)$$

where by analogy with the previously considered case of the real field we have introduced the three-dimensional amplitudes

$$\left. \begin{aligned} \varphi^{\pm}(\mathbf{k}) &= \frac{\varphi^{\pm}(k)}{\sqrt{2k^0}}, \\ \dot{\varphi}^{\pm}(\mathbf{k}) &= \frac{\dot{\varphi}^{\pm}(k)}{\sqrt{2k^0}} \end{aligned} \right\} \quad (k^0 = \sqrt{\mathbf{k}^2 + m^2}).$$

The notation for the positive- and negative-frequency functions in the  $k$ -representation has been so introduced that, for example,  $\varphi^{*(+)}(k)$  denotes not the function which is the complex conjugate of  $\varphi^{(+)}(k)$ , but the positive-frequency part of  $\varphi^*(k)$ . The rules for complex conjugation in the momentum representation then have the form

$$(\varphi^{+}(k))^* = \dot{\varphi}^{-}(k), \quad (\varphi^{-}(k))^* = \dot{\varphi}^{+}(k). \quad (3.38)$$

Substituting the expansions (3.36) and (3.37) into the expressions for the energy-momentum density (3.34) and for the charge (3.35), we find after integrating over the configuration space:

$$P^l = \int d\mathbf{k} \cdot k^l \left\{ \dot{\varphi}^{+}(\mathbf{k}) \varphi^{-}(\mathbf{k}) + \dot{\varphi}^{-}(\mathbf{k}) \varphi^{+}(\mathbf{k}) \right\} \quad (l = 0, 1, 2, 3, k^0 = \sqrt{\mathbf{k}^2 + m^2}), \quad (3.39)$$

$$Q = \int d\mathbf{k} \left\{ \dot{\varphi}^{+}(\mathbf{k}) \varphi^{-}(\mathbf{k}) - \dot{\varphi}^{-}(\mathbf{k}) \varphi^{+}(\mathbf{k}) \right\}. \quad (3.40)$$

From these expressions for the energy-momentum four-vector and for the charge it follows that the product  $\varphi^{*(-)}(\mathbf{k})\varphi^{(+)}(\mathbf{k})$  may be interpreted as the density of the average number of particles of mass  $m$ , energy  $k^0$ , momentum  $\mathbf{k}$ , and charge  $-1$ , while the product  $\varphi^{*(+)}(\mathbf{k})\varphi^{(-)}(\mathbf{k})$  may be interpreted as the density of the average number of particles of energy  $k^0$ , momentum  $\mathbf{k}$ , and charge  $+1$ .

Accordingly, after quantization (§10),  $\varphi^{*(+)}(k)$  will describe the creation of a particle of mass  $m$ , momentum  $\mathbf{k}$ , and charge  $1$ , while  $\varphi^{(-)}$  will describe its annihilation. A similar correspondence also exists for  $\varphi^{(+)}$  and  $\varphi^{*(-)}$ .

Thus, the complex scalar field describes positively and negatively charged spinless particles.

**3.5. The Pion Field.** The three pions  $\pi^0$ ,  $\pi^+$ ,  $\pi^-$  form an *isospin triplet* (see Appendix 1B). This triplet is described by a three-component field function whose components form the isospin vector.

Two different representations of the pion field are usually employed. In the first of these, all three components are real:

$$\pi(x) = \{\pi_1(x), \pi_2(x), \pi_3(x)\}, \quad \pi_\alpha^*(x) = \pi_\alpha(x), \quad (3.41)$$

and in the second, the first and the third are mutually conjugate:

$$\varphi(x) = \{\varphi_1(x), \varphi_2(x), \varphi_3(x)\} \quad \varphi_2^* = \varphi_2, \quad \varphi_1^* = -\varphi_3. \quad (3.42)$$

The connection between the representations given by (3.41) and (3.42) can be taken in the form

$$\varphi_1 = -\frac{\pi_1 - i\pi_2}{\sqrt{2}}, \quad \varphi_2 = \pi_3, \quad \varphi_3 = \frac{\pi_1 + i\pi_2}{\sqrt{2}}. \quad (3.43)$$

The Lagrangian for the free pion field that is invariant under rotations in the three-dimensional isospin space in the representation given by (3.41), has the form

$$\mathcal{L} = \frac{1}{2} (\pi_{;n} \pi^{;n}) - \frac{m^2}{2} (\pi \pi). \quad (3.44)$$

Inverting (3.43), and using (3.42), we obtain

$$\mathcal{L} = \frac{1}{2} (\varphi_{;n}^* \varphi^{;n}) - \frac{m^2}{2} (\varphi^* \varphi) = \quad (3.45)$$

$$\left\{ \varphi_{1;n}^* \varphi_1^{;n} - m^2 \varphi_1^* \varphi_1 \right\} + \frac{1}{2} \left\{ \varphi_{2;n} \varphi_2^{;n} - m^2 \varphi_2 \varphi_2 \right\}. \quad (3.46)$$

The isospin invariance of the Lagrangian given by (3.44) leads to the new conservation law, namely, the conservation of the isotopic spin vector  $\mathbf{I}$ . Since the third component of the isospin vector is equal to the electric charge in the case of the pions [see (2.21) for  $Y = 0$ ], we shall consider rotation through the angle  $\alpha$  of the plane  $z_1 z_2$  in isospin space:

$$\pi'_1 = \cos \alpha \cdot \pi_1 - \sin \alpha \cdot \pi_2, \quad \pi'_2 = \sin \alpha \cdot \pi_1 + \cos \alpha \cdot \pi_2, \quad \pi'_3 = \pi_3. \quad (3.47)$$

It is shown in Appendix 1B that this transformation in the representation given by (3.42) can be written in the form

$$\varphi' = \exp \{-iT_3 \alpha\} \varphi, \quad (3.48)$$

where the matrix  $T_3$  is diagonal:

$$T_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Using the Noether theorem, we obtain the current four-vector

$$J_n(x) = i \bar{\varphi}^*(x) T_3 \varphi_n(x) \quad (3.49)$$

and the charge

$$Q = I_3 = i \int d\mathbf{x} (\bar{\varphi}_1 \varphi_{1;0} - \bar{\varphi}_3 \varphi_{3;0}). \quad (3.50)$$

The real component  $\varphi_2 = \pi_3$  is thus found to contribute nothing to the charge, and the complex conjugate components  $\varphi_1$  and  $\varphi_3$  provide contributions of opposite sign. The correspondence between the field components  $\varphi_a$  and the particles  $\pi^{\pm,0}$  is set in the following way:

$$\varphi_1 \sim \pi^+, \quad \varphi_2 \sim \pi^0, \quad \varphi_3 \sim \pi^-. \quad (3.51)$$

#### §4. The Vector Field

*4.1. Lagrangian, Subsidiary Condition, and Invariants.* The function describing the vector field consists of four components  $U_0$ ,  $U_1$ ,  $U_2$ , and  $U_3$  which together form a covariant four-vector, i.e., transforms under the Lorentz rotations

$$x'^k = x^k + \delta x^k, \quad \delta x^k = \omega^{kn} x_n, \quad \omega^{kn} + \omega^{nk} = 0$$

in accordance with the formulas

$$U'_k(x') = U_k(x) + \delta U_k, \quad \delta U^k = \omega^{kn} U_n(x). \quad (4.1)$$

The simplest generalization of the formalism developed in the preceding section to a vector field consists of choosing the Lagrangian in the form (for the sake of simplicity, we shall at first consider only the real vector field)

$$\mathcal{L} = -\frac{1}{2} \frac{\partial U_n}{\partial x_m} \frac{\partial U^n}{\partial x^m} + \frac{m^2}{2} U_k U^k, \quad (4.2)$$

i.e., in the form of a covariant sum of four "Lagrangians" each separately corresponding to the components  $U_0$ ,  $U_1$ ,  $U_2$ , and  $U_3$ . Such a Lagrangian obviously leads to Klein-Gordon equations for each of the components  $U_k$  and to dynamic variables that are natural generalizations of the corresponding expressions for the scalar field. However, it is easy to see that such a formulation also leads to negative energy terms that correspond to the component  $U_0$ . The way out of this difficulty consists of imposing the following invariant subsidiary condition on the components  $U_k$ :

$$\frac{\partial U^k}{\partial x^k} = 0. \quad (4.3)$$

This condition is the only possible invariant condition that is linear in the field functions.

It reduces the number of linearly independent components from four to three and, as will be shown below, it ensures that the energy of the vector field is positive definite. The three remaining independent components correspond to the three possible values of the component of the spin along a given direction. These are, respectively, 1, 0,  $-1$ , i.e., they describe particles with spin one. The imposition of the additional condition (4.3) corresponds to the exclusion of a particle of spin zero which, in this formulation, would lead to a negative energy.

Therefore, in the theory of the vector field, the Lagrangian is usually chosen [Wentzel, (1942), Pauli (1941)] in such a way that, in addition to the field equations, the automatic fulfillment of (4.3) is also guaranteed. We refer the reader to the literature for detailed calculations and merely state here that such a program may be based on the following Lagrangian:

$$\mathcal{L} = -\frac{1}{4} \left( \frac{\partial U_n}{\partial x_m} - \frac{\partial U_m}{\partial x_n} \right) \left( \frac{\partial U_n}{\partial x^m} - \frac{\partial U_m}{\partial x^n} \right) + \frac{m^2}{2} U_n U^n, \quad (4.4)$$

which differs from (4.2) by the term

$$\frac{1}{2} \frac{\partial U_n}{\partial x^m} \frac{\partial U^m}{\partial x_n}. \quad (4.5)$$

The arbitrariness in the choice of the Lagrangian is evidently related to the possibility of constructing several different invariants, using first derivatives of the components of the vector.

A similar arbitrariness occurs in the choice of the Lagrangian for the electromagnetic field, which is also described by a vector potential. However, in the theory of the electromagnetic field, because the mass  $m$  is equal to zero, it is not possible to construct a formalism in such a way as to guarantee that the subsidiary condition (4.3) should hold automatically. Therefore, in the classical theory the fulfillment of condition (4.3) is usually guaranteed by means of a suitable gauge transformation (see §5). However, when the electromagnetic field is quantized (see Chapter 2, §12) it is not possible to satisfy condition (4.3) in the form of a functional relationship between the components of the quantized potential since it turns out to be incompatible with the commutation relations. One therefore has to impose on the allowed states certain conditions that are equivalent to (4.3) holding only for average values.

Thus, in the theory of the electromagnetic field, the imposition of the subsidiary condition is carried out independently of the Lagrangian formalism. It therefore appears natural in the case of the vector field also not to associate condition (4.3) with the Lagrangian formalism. For example, it is possible to use a Lagrangian such as (4.2) and to impose condition (4.3) independently of it.

Here the problem arises of the lack of uniqueness in the structure of the Lagrangian and of the influence of this lack of uniqueness on the dynamic variables. We note in this connection that the action which corresponds to the difference (4.5) between the two

Lagrangians given above reduces to zero after integration by parts with condition (4.3) being taken into account, and consequently the actions corresponding to Lagrangians (4.2) and (4.4) coincide. The Lagrangians (4.2) and (4.4) lead to different energy-momentum tensors, angular-momentum tensors, and so on. However, it will be shown in the example of the energy-momentum tensor that the differences between these quantities may be represented, after the subsidiary condition and the field equations have been taken into account, in the form of corresponding divergences, as a result of which the dynamic variables of the type of the energy-momentum vector turn out to be equal. This will be sufficient for us, since, according to a remark made in §2, the unique definition of quantities such as the energy-momentum tensor falls outside the range of the topics discussed here.

Proceeding to the construction of the Lagrangian formalism of the vector field, we note that in agreement with general properties the real vector field describes neutral particles and the complex field describes charged particles. For the sake of brevity, we shall limit out discussion to the complex vector field, bearing in mind the fact that the corresponding calculations for the real field will lead to essentially different results only in calculations of the current and the charge.

In accordance with the foregoing, we shall choose the vector-field Lagrangian in the form

$$\mathcal{L} = -\frac{\partial \dot{U}_k}{\partial x^n} \frac{\partial U^k}{\partial x_n} + m^2 \dot{U}_n U^n \quad (4.6)$$

and shall impose independently of it the subsidiary conditions on  $U$  and  $U^*$ :

$$\frac{\partial U^n}{\partial x^n} \equiv \left( \frac{\partial U}{\partial x} \right) = 0, \quad \frac{\partial \dot{U}^n}{\partial x^n} \equiv \left( \frac{\partial \dot{U}}{\partial x} \right) = 0.$$

From the Lagrangian (4.6) we find with the aid of the formulas of §2: the field equations

$$(\square - m^2) U_n(x) = 0, \quad (\square - m^2) \dot{U}_n(x) = 0,$$

the energy-momentum tensor

$$T_{kl} = - \left( \frac{\partial \dot{U}_n}{\partial x^k} \frac{\partial U^n}{\partial x^l} + \frac{\partial \dot{U}_n}{\partial x^l} \frac{\partial U^n}{\partial x^k} \right) - g^{kl} \mathcal{L}, \quad (4.7)$$

the current vector

$$J_k = -i \left( \dot{U}_m \frac{\partial U^m}{\partial x^k} - \frac{\partial \dot{U}_m}{\partial x^k} U^m \right) \quad (4.8)$$

and the spin angular-momentum tensor

$$S_{lm}^n = \left( \dot{U}_m \frac{\partial U_l}{\partial x_n} + \frac{\partial \dot{U}_l}{\partial x_n} U_m - \dot{U}_l \frac{\partial U_m}{\partial x_n} - \frac{\partial \dot{U}_m}{\partial x_n} U_l \right). \quad (4.9)$$

Setting  $k = 0$  in (4.7), we obtain the spatial density of the energy-momentum four-vector

$$T^{00} = - \sum_k \frac{\partial \dot{U}_n}{\partial x^k} \frac{\partial U^n}{\partial x^k} - m^2 \dot{U}_n U^n, \quad (4.10)$$

$$T^{0\alpha} = \left( \frac{\partial \dot{U}_n}{\partial x^0} \frac{\partial U^n}{\partial x^\alpha} + \frac{\partial \dot{U}_n}{\partial x^\alpha} \frac{\partial U^n}{\partial x^0} \right), \quad \alpha = 1, 2, 3. \quad (4.11)$$

Similarly, starting from (4.8) and (4.9), we obtain the charge density

$$J^0 = -i \left( \dot{U}_n \frac{\partial U^n}{\partial x^0} - \frac{\partial \dot{U}_n}{\partial x^0} U^n \right) \quad (4.12)$$

and the spatial density of the components of the spin vector  $S$

$$S_y = \varepsilon_{y\alpha\beta} \int S_{\alpha\beta}^0 d\mathbf{x},$$

$$S_{\alpha\beta}^0 = \dot{U}_\beta \frac{\partial U_\alpha}{\partial x^0} + \frac{\partial \dot{U}_\alpha}{\partial x^0} U_\beta - \dot{U}_\alpha \frac{\partial U_\beta}{\partial x^0} - \frac{\partial \dot{U}_\beta}{\partial x^0} U_\alpha. \quad (4.13)$$

*4.2. Transition to the Momentum Representation.* For further calculation of dynamic variables we shall go over, as is usually done, to the momentum representation

$$U_n(x) = \frac{1}{(2\pi)^{3/2}} \int dk e^{ikx} \delta(k^2 - m^2) U_n(k),$$

$$\dot{U}_n(x) = \frac{1}{(2\pi)^{3/2}} \int dk e^{ikx} \delta(k^2 - m^2) \dot{U}_n(k)$$

and shall decompose the potentials  $U$ ,  $U^*$  into their positive- and negative-frequency parts:

$$U_n(x) = U_n^+(x) + U_n^-(x), \quad \dot{U}_n(x) = \dot{U}_n^+(x) + \dot{U}_n^-(x).$$

Integrating over the variable  $k^0$ , we obtain the three-dimensional momentum representation of the potentials in the form

$$\left. \begin{aligned} U_n^\pm(x) &= \frac{1}{(2\pi)^{3/2}} \int \frac{d\mathbf{k}}{\sqrt{2k^0}} e^{\pm i\mathbf{k}x} U_n^\pm(\mathbf{k}), \\ \dot{U}_n^\pm(x) &= \frac{1}{(2\pi)^{3/2}} \int \frac{d\mathbf{k}}{\sqrt{2k^0}} e^{\pm i\mathbf{k}x} \dot{U}_n^\pm(\mathbf{k}). \end{aligned} \right\} \quad (4.14)$$

where, as usual,

$$\left. \begin{aligned} U_n^\pm(\mathbf{k}) &= \frac{U_n^\pm(\mathbf{k})}{\sqrt{2k^0}}, \quad \dot{U}_n^\pm(\mathbf{k}) = \frac{\dot{U}_n^\pm(\mathbf{k})}{\sqrt{2k^0}} \quad (k^0 = \sqrt{\mathbf{k}^2 + m^2}), \\ U_n^\pm(\mathbf{k}) &= \theta(k^0) U_n(\pm \mathbf{k}), \quad \dot{U}_n^\pm(\mathbf{k}) = \theta(k^0) \dot{U}_n(\pm \mathbf{k}). \end{aligned} \right\} \quad (4.15)$$

We recall that, in accordance with the definitions of momentum amplitudes, the conditions of complex conjugation have the form

$$(U_n(\mathbf{k}))^* = \dot{U}_n(-\mathbf{k}) \quad \text{and consequently} \quad (U_n^\pm(\mathbf{k}))^* = \dot{U}_n^\pm(\mathbf{k}). \quad (4.16)$$

Substituting the expansions (4.14) into expressions (4.11), (4.12), and (4.13), and integrating with respect to  $\mathbf{x}$ , we obtain the energy-momentum four-vector

$$P^n = \int T^{0n} d\mathbf{x} = - \int d\mathbf{k} \cdot k^n \{ \dot{U}_m^-(\mathbf{k}) U^{+m}(\mathbf{k}) + \dot{U}_m(\mathbf{k}) U^{-m}(\mathbf{k}) \}, \quad (4.17)$$

the charge

$$Q = \int J^0 d\mathbf{x} = - \int d\mathbf{k} \{ \dot{U}_n^+(\mathbf{k}) U^{-n}(\mathbf{k}) - \dot{U}_n^-(\mathbf{k}) U^{+n}(\mathbf{k}) \} \quad (4.18)$$

and the spin vector

$$\mathbf{S} = i \int d\mathbf{k} \{ [\dot{\mathbf{U}}^+(\mathbf{k}) \times \mathbf{U}^-(\mathbf{k})] - [\dot{\mathbf{U}}^-(\mathbf{k}) \times \mathbf{U}^+(\mathbf{k})] \}. \quad (4.19)$$

It may be seen that the spin vector (4.19) is independent of time, due to the *symmetry* of the energy-momentum tensor (4.7).

Taking into account the conditions of complex conjugation (4.16) we note that according to (4.17) the term with  $m = 0$  in the expression for the energy  $P^0$  turns out to be negative as a result of which the energy is not positive definite. As was noted above, this difficulty is removed by imposing the subsidiary conditions which in the three-dimensional momentum representation (4.14) take on the form

$$k^n U_n^\pm(\mathbf{k}) = 0, \quad k^n \dot{U}_n^\pm(\mathbf{k}) = 0. \quad (4.20)$$

By virtue of these conditions, the components  $U_n$  are no longer independent. Expressing the component  $U_0$  in terms of the others with the aid of (4.20)

$$U_0^\pm(\mathbf{k}) = - \frac{1}{k^0} k^\alpha U_\alpha^\pm(\mathbf{k}), \quad \dot{U}_0^\pm(\mathbf{k}) = - \frac{1}{k^0} k^\alpha \dot{U}_\alpha^\pm(\mathbf{k}) \quad (\alpha = 1, 2, 3),$$

we obtain the following expression for the quadratic form in the integrand of (4.17), which depends only on the "space" components of the potential:



$$-\dot{U}_n^\pm(\mathbf{k}) U_n^\mp(\mathbf{k}) = \dot{U}_\alpha^\pm(\mathbf{k}) U_\alpha^\mp(\mathbf{k}) - \frac{1}{(k^0)^2} (k^\alpha \dot{U}_\alpha^\pm(\mathbf{k})) (k^\beta U_\beta^\mp(\mathbf{k})). \quad (4.21)$$

This may be diagonalized by the linear substitution

$$U(\mathbf{k}) = \mathbf{e}_1 a_1(\mathbf{k}) + \mathbf{e}_2 a_2(\mathbf{k}) + \frac{\mathbf{k}}{|\mathbf{k}|} \frac{k^0}{m} a_3(\mathbf{k}), \quad (4.22)$$

which represents a decomposition of the potential  $U_\alpha$  into a longitudinal and a transverse component. Here  $\mathbf{e}_1$  and  $\mathbf{e}_2$  are complex unit vectors orthogonal to the wave vector  $\mathbf{k}$  and to each other:

$$(\mathbf{e}_1 \cdot \mathbf{e}_1) = (\mathbf{e}_2 \cdot \mathbf{e}_2) = 1, \quad (\mathbf{e}_r \cdot \mathbf{k}) = (\dot{\mathbf{e}}_r \cdot \mathbf{k}) = (\mathbf{e}_r \cdot \mathbf{e}_s) = (\mathbf{e}_r \cdot \dot{\mathbf{e}}_s) = 0 \quad (r, s = 1, 2),$$

and are the transverse polarization vectors. The substitution given by (4.22) is none other than the transformation to the local frame in momentum space.

Substituting (4.22) into (4.21), we find after a simple calculation:

$$-\dot{U}_n^\pm(\mathbf{k}) U_n^\mp(\mathbf{k}) = \dot{a}_\alpha^\pm(\mathbf{k}) a_\alpha^\mp(\mathbf{k}).$$

Substituting this expression into (4.17) and (4.18), we obtain diagonal expressions for the energy momentum and for the charge, with the energy turning out to be manifestly positive definite in terms of the new variables:

$$P^l = \int d\mathbf{k} \cdot k^l \{ \dot{a}_\alpha^+(\mathbf{k}) a_\alpha^-(\mathbf{k}) + \dot{a}_\beta^-(\mathbf{k}) a_\beta^+(\mathbf{k}) \}, \quad (4.23)$$

$$Q = \int d\mathbf{k} \{ \dot{a}_\alpha^+(\mathbf{k}) a_\alpha^-(\mathbf{k}) - \dot{a}_\beta^-(\mathbf{k}) a_\beta^+(\mathbf{k}) \}. \quad (4.24)$$

*4.3. Spin of the Vector Field.* In order to make clear the connection between the amplitudes  $a$  and the spin variable, we shall consider the component of the spin vector along the direction of the propagation vector. Substituting (4.22) into (4.19) we shall obtain

$$S_3 \sim i \{ \dot{a}_1^+(\mathbf{k}) a_2^-(\mathbf{k}) - \dot{a}_2^+(\mathbf{k}) a_1^-(\mathbf{k}) + \dot{a}_2^-(\mathbf{k}) a_1^+(\mathbf{k}) - \dot{a}_1^-(\mathbf{k}) a_2^+(\mathbf{k}) \}. \quad (4.25)$$

This expression may be diagonalized by means of the linear transformation

$$\left. \begin{aligned} a_1^\pm &= \frac{b_1^\pm + b_2^\pm}{\sqrt{2}}, & a_2^\pm &= \frac{b_1^\pm - b_2^\pm}{i\sqrt{2}}, & a_3^\pm &= b_3^\pm, \\ \dot{a}_1^\pm &= \frac{\dot{b}_1^\pm + \dot{b}_2^\pm}{\sqrt{2}}, & \dot{a}_2^\pm &= i \frac{\dot{b}_1^\pm - \dot{b}_2^\pm}{\sqrt{2}}, & \dot{a}_3^\pm &= \dot{b}_3^\pm, \end{aligned} \right\} \quad (4.26)$$

which does not change the diagonal form of  $P$  and  $Q$ :

$$P^n = \int d\mathbf{k} \cdot k^n (\dot{b}_\alpha^-(\mathbf{k}) b_\alpha^+(\mathbf{k}) + \dot{b}_\alpha^+(\mathbf{k}) b_\alpha^-(\mathbf{k})), \quad (4.27)$$

$$Q = \int d\mathbf{k} (\dot{b}_\alpha^+(\mathbf{k}) b_\alpha^-(\mathbf{k}) - \dot{b}_\alpha^-(\mathbf{k}) b_\alpha^+(\mathbf{k})), \quad (4.28)$$

$$S_3 \sim [\dot{b}_1^+(\mathbf{k}) b_1^-(\mathbf{k}) - \dot{b}_1^-(\mathbf{k}) b_1^+(\mathbf{k}) + \dot{b}_2^-(\mathbf{k}) b_2^+(\mathbf{k}) - \dot{b}_2^+(\mathbf{k}) b_2^-(\mathbf{k})]. \quad (4.29)$$

It may be seen from these formulas that the quadratic combinations of the amplitudes  $b^{*(\pm)}$  and  $b^{(\pm)}$  may be considered as the densities of the average number of particles which have definite values of energy, momentum, charge, and component of spin along the direction of motion. Thus, for example, the quantity  $b_2^{*(-)}(\mathbf{k}) b_2^{(+)}(\mathbf{k})$  represents the density of particles with momentum  $\mathbf{k}$ , energy  $k^0$ , charge  $-1$ , and component of spin along the direction of motion equal to  $+1$ ; similarly  $b_3^{*(-)}(\mathbf{k}) b_3^{(+)}(\mathbf{k})$  is the density of particles with momentum  $\mathbf{k}$ , energy  $k^0$ , charge  $-1$ , zero component of spin, etc. As we shall see later (cf. Chapter II) after quantization, the amplitude  $b_2^{(+)}(\mathbf{k})$  describes the creation of a particle of energy  $k^0$ , momentum  $\mathbf{k}$ , charge  $-1$ , spin component  $+1$ , while the amplitude  $b_2^{*(-)}(\mathbf{k})$  describes its annihilation, and so on.

Therefore, in accordance with (4.22), (4.26), and (4.29), the amplitudes  $a_{1,2}$  correspond to linearly polarized and  $b_{1,2}$  to circularly polarized oscillations.

Thus the complex vector field describes positively and negatively charged particles of mass

$$m = \sqrt{k_0^2 - \mathbf{k}^2}$$

and with three possible values of the component of the spin vector along the direction of motion equal respectively to  $1, 0, -1$ . A rigorous foundation for this correspondence can, of course, be provided only by quantum theory. ↑

4.4. *Klein-Gordon Equations Written in the Form of a System of First-Order Equations.* We note that in the general case the system of equations

$$(\square - m^2) u_i(x) = 0 \quad (i = 1, 2, \dots, s) \quad (4.30)$$

may be replaced by a system of equations of the first order of the form

$$\left( i\Gamma^n \frac{\partial}{\partial x^n} - M \right) u(x) = 0, \quad (4.31)$$

where  $\Gamma^n$  and  $M$  are certain square matrices and the number  $r$  ( $r > s$ ) of components of the function  $u$  is equal to the rank of the matrices  $\Gamma$  and  $M$  whose properties are determined by the transformation laws of the function  $u$  and by the conditions of covariance of equations (4.31).

In particular, if the matrices  $\Gamma$  are determined by the conditions

$$\Gamma^n \Gamma^m + \Gamma^m \Gamma^n = 2g^{mn},$$

while  $M$  is diagonal, then, as will be shown in §6, the rank of the only irreducible representation of  $\Gamma$  is equal to four, and the function  $u$  is a four-component spinor.

The case when  $\Gamma$  are defined by the relations

$$\Gamma^k \Gamma^n \Gamma^m + \Gamma^m \Gamma^n \Gamma^k = g^{kn} \Gamma^m + g^{mn} \Gamma^k, \quad (4.32)$$

while the matrix  $M$  is diagonal has been investigated by Duffin (1938) and by Kemmer (1939) (see also Pauli (1941)). In this case, the rank of the matrices turns out to be equal to sixteen. The corresponding representation may be decomposed into three irreducible ones. In the first of these  $u$  has a single component equal to zero, in the second it has five components, and in the third it has ten components. The five-component function gives a scalar representation, corresponds to a scalar and its four-gradient, and describes particles of zero spin. The ten-component function gives a vector representation, corresponds to a four-vector  $U$  and to the six components of the tensor

$$H_{mn} = \frac{\partial U_n}{\partial x^m} - \frac{\partial U_m}{\partial x^n}$$

and describes particles of spin one.

The conditions of covariance of equations (4.31) under Lorentz transformation impose on the matrices  $\Gamma$  certain relations which establish the connection between the matrices  $\Gamma$  and the transformation laws of the wave function  $u$ . We shall examine the infinitesimal rotation transformation (under a transformation of translation, equations (4.31) are trivially covariant):

$$x^n \rightarrow x'^n = x^n + x_k \omega^{nk}, \quad \omega^{nk} + \omega^{kn} = 0,$$

under which the function  $u$  transforms in accordance with

$$u'(x') = \Lambda u(x). \quad (4.33)$$

As a result of the demand of covariance, equation (4.31) must retain its old form in terms of the new variables:

$$\left( i \Gamma^n \frac{\partial}{\partial x'^n} - M \right) u'(x') = 0.$$

If we express the derivatives with respect to the transformed coordinates  $x'$  occurring in the above expression in terms of the derivatives with respect to the original coordinates  $x$ , and then multiply on the left by  $\Lambda^{-1}$  and equate terms of the first order in the parameters  $\omega$ , we shall obtain the relations between the transformation matrix  $\Lambda$  and the coefficients  $\Gamma$  and  $M$  of the preceding equation

$$\left. \begin{aligned} \Lambda^{-1} M \Lambda &= M, \\ \Lambda^{-1} \Gamma^k \Lambda &= \Gamma^k + \Gamma_n \omega^{kn}, \end{aligned} \right\} \quad (4.34)$$

which guarantee the covariance of the equation under discussion.

Thus the tensor transformation laws of the field functions in the Duffin-Kemmer formalism take on the form given by (4.33), while the transformation matrix  $\Lambda$  turns out to be related to the matrix coefficients of the field equation. The tensor transformation law seems to take on "the external appearance" of a transformation of the spinor type (cf. §6). This fact is, of course, connected with the transition to a new system of "independent components" of the field function, which are in fact linear combinations of the components of the tensor and of their first derivatives.

Equation (4.31) may also be obtained by the variational method from a Lagrangian of the form

$$\mathcal{L} = \frac{i}{2} \left( \bar{u} \Gamma^n \frac{\partial u}{\partial x^n} - \frac{\partial \bar{u}}{\partial x^n} \Gamma^n u \right) - \bar{u} M u. \quad (4.35)$$

The function  $\bar{u}$  occurring in this expression is said to be "adjoint" to  $u$ , satisfies the equation

$$i \frac{\partial \bar{u}}{\partial x^n} \Gamma^n + \bar{u} M = 0$$

and is linearly related to the function  $u^*$ , which is complex-conjugate to  $u$ , by means of the matrix equation

$$\bar{u} = u^* \Gamma. \quad (4.36)$$

The matrix  $\Gamma$  of this last transformation is determined in terms of  $\Gamma^k$  and  $\Gamma^m$  by means of the condition that the Lagrangian (4.35) should be real. In this way we obtain:

$$(\Gamma)^{-1} \Gamma^{\dagger k} \Gamma = \Gamma^k, \quad (\Gamma)^{-1} M^{\dagger} \Gamma = M.$$

It is also evident that expressions for the dynamic variables may be obtained in the usual way from the Lagrangian (4.35).

As an illustration, consider the case of a real scalar field. The Klein-Gordon equation

$$(\square - m^2) \varphi(x) = 0$$

can be replaced by the first-order matrix equation

$$i \Gamma_n \frac{\partial u}{\partial x_n} - M u = 0,$$

where  $M$  is the diagonal matrix  $M_{ik} = m \delta_{ik}$ , and  $\Gamma_n$  have the form (points represent zeros)

$$\Gamma_0 = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot \end{pmatrix}, \quad \Gamma_1 = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & -1 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot \end{pmatrix}.$$

$$\Gamma_2 = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & -1 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot \end{pmatrix}, \quad \Gamma_3 = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & -1 \\ \cdot & \cdot & \cdot & 1 & \cdot \end{pmatrix}. \quad (4.37)$$

It is readily shown that the components of  $u$  can be expressed in the terms of  $\varphi_n$  as follows:

$$u_n = \frac{i}{\sqrt{m}} \frac{\partial \varphi}{\partial x^n}, \quad n = 0, 1, 2, 3, \quad u_4 = \sqrt{m} \varphi. \quad (4.38)$$

The transformation matrix  $\Gamma$  (4.36) then takes the form

$$\Gamma = \begin{pmatrix} 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & -1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & -1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & -1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 \end{pmatrix} \quad (4.39)$$

where

$$\bar{u} = {}^*u\Gamma = \{u_0^*, -u_1^*, -u_2^*, -u_3^*, u_4^*\},$$

and the Lagrangian (4.35) can be written in the form

$$\mathcal{L} = {}^*\bar{\varphi}(\square - m^2)\varphi, \quad (4.40)$$

which differs from (3.32) only by the term having the form of the four-divergence.

Thus, the Duffin-Kemmer formulation in the case of the free field is completely equivalent to the usual formulation. The difference appears when the interaction between the fields is introduced. For example, interaction with the electromagnetic field is usually introduced in the so-called minimum fashion so that the gradients (i.e., momenta in the sense of the usual quantum-mechanical correspondence) in the free Lagrangians are "extended" by the rule (through the introduction of the generalized momenta

$$p \rightarrow p - \frac{e}{c}A):$$

$$i \frac{\partial}{\partial x^n} \rightarrow i \frac{\partial}{\partial x^n} - eA_n. \quad (4.41)$$

This ensures gauge invariance of the total Lagrangian  $\mathcal{L}_{\text{tot}}$  (see also §8.2). This extension of the Lagrangian (4.35), which is a linear function of  $\partial/\partial x^n$ , leads to an interaction Lagrangian that is linear to the potential  $A_n$  of the electromagnetic field, whereas the extension of the free Lagrangian (3.1) leads to an interaction Lagrangian that is quadratic in  $A_n$ .

## §5. The Electromagnetic Field

5.1. *Potential of the Electromagnetic Field.* The electromagnetic field is described by the well-known Maxwell equations

$$\begin{aligned}\operatorname{curl} \mathbf{E} &= -\frac{\partial \mathbf{H}}{\partial x^0}, & \operatorname{curl} \mathbf{H} &= \frac{\partial \mathbf{E}}{\partial x^0}, \\ \operatorname{div} \mathbf{H} &= 0, & \operatorname{div} \mathbf{E} &= 0.\end{aligned}$$

For a more symmetric description of the electromagnetic field, a real covariant four-vector electromagnetic potential  $A_n$  ( $n = 0, 1, 2, 3$ ) is introduced in such a way that

$$\mathbf{E} = \operatorname{grad} A_0 - \frac{\partial \mathbf{A}}{\partial x^0}, \quad \mathbf{H} = \operatorname{curl} \mathbf{A}.$$

The components of the “four-dimensional curl” of the potential  $A$  form the well-known antisymmetric tensor of the electromagnetic field

$$H_{kl} = \frac{\partial A_l}{\partial x^k} - \frac{\partial A_k}{\partial x^l}, \quad (5.1)$$

whose components are related to the components of the vectors representing the electric and magnetic field strengths by means of the expressions

$$E_\alpha = H_{\alpha 0} \quad (\alpha = 1, 2, 3); \quad H_1 = H_{23}, \quad H_2 = H_{31}, \quad H_3 = H_{12}.$$

It is convenient to exhibit this relationship in the following explicit form:

$$H_{kl} = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & H_3 & -H_2 \\ E_2 & -H_3 & 0 & H_1 \\ E_3 & H_2 & -H_1 & 0 \end{pmatrix}.$$

Maxwell's equations may be rewritten with the aid of the tensor  $H_{kl}$  in the form

$$\frac{\partial H^{kl}}{\partial x_l} = 0, \quad (5.2)$$

$$\frac{\partial H_{kl}}{\partial x^m} + \frac{\partial H_{mk}}{\partial x^l} + \frac{\partial H_{lm}}{\partial x^k} = 0. \quad (5.3)$$

By making the further transition in these equations from  $H_{kl}$  to the potential  $A_l$  we

see that the four equations (5.3) are a consequence of the definition (5.1), and do not lead to any equation for  $A_I$ , while the four equations (5.2) yield:

$$\square A_n + \frac{\partial^2 A^h}{\partial x^h \partial x^n} = 0. \quad (5.4)$$

From this it follows that the formulation of the equations of the electromagnetic field in terms of the potential  $A_n$  is really simple, symmetric, and manifestly covariant.

5.2. *Gauge Transformation of the Second Kind and the Lorentz Condition.* We note further that the method of introducing the potential  $A$  is *nonunique to a high degree*. The point is that, in Maxwell's theory, the observable quantities, such as the vectors  $\mathbf{E}$ ,  $\mathbf{H}$ , the tensor  $H_{kl}$ , and also the equations of the electromagnetic field are all invariant under the so-called *gauge transformation of the second kind*:

$$A_n(x) \rightarrow A'_n(x) = A_n(x) + \frac{\partial f(x)}{\partial x^n}. \quad (5.5)$$

Therefore the potential  $A$  is not an observable quantity and is not unique. The function  $f(x)$  in (5.5) is an arbitrary function with partial derivatives of the first and second order.

This lack of uniqueness of the potential  $A$  may be used so as to satisfy some additional condition. The so-called *Lorentz condition*

$$\frac{\partial A^h}{\partial x^h} \equiv \left( \frac{\partial A}{\partial x} \right) = 0, \quad (5.6)$$

which is the only possible invariant condition linear in  $A$  is usually chosen as such a subsidiary condition. It is not difficult to show that  $f(x)$  may always be chosen in such a way as to satisfy condition (5.6).

If the Lorentz condition is taken into account, the field equations (5.4) assume the form

$$\square A_I = 0. \quad (5.7)$$

However, the additional Lorentz condition still does not completely determine the potential  $A$ . Equations (5.6) and (5.7) are invariant with respect to the "special gauge transformation of the second kind"

$$A_n(x) \rightarrow A'_n(x) = A_n(x) + \frac{\partial f_0(x)}{\partial x^n}, \quad (5.8)$$

which is characterized by the requirement that the arbitrary function  $f_0(x)$  must satisfy D'Alembert's equation\*

\*Ogiyevetskiy and Polubarinov (1961) have shown that an analogous invariance occurs for the neutral vector field with nonzero mass, described by the Lagrangian (4.2).

$$\square \hat{f}_0 = 0.$$

In any particular Lorentz frame of reference the function  $f_0$  may be chosen so that some one of the components of  $A$ , for example, the “scalar” potential  $A_0$ , vanishes. In this case the Lorentz condition assumes the form

$$\frac{\partial A_\alpha}{\partial x^\alpha} \equiv \operatorname{div} A = 0. \quad (5.9)$$

In order to establish the physical meaning of equation (5.9), it is convenient to go over to the momentum representation

$$A_n(x) = \frac{1}{(2\pi)^{3/2}} \int dk \delta(k^2) e^{ikx} A_n(k). \quad (5.10)$$

Substituting (5.10) into (5.9) we find:

$$k_\alpha A_\alpha(k) |_{k^2=0} = (k \cdot A(k)) |_{k^2=0} = 0. \quad (5.11)$$

Equation (5.11) is the condition that the *electromagnetic field is transverse*. Thus, in spite of the fact that the electromagnetic field is described by a four-component potential, only *the two linearly independent components* orthogonal to the propagation vector have physical meaning. It is important to stress that although the condition of transversality is not covariant, it is possible to arrange that it should be satisfied in any arbitrary particular Lorentz frame of reference by means of a suitable special gauge transformation (5.8).

We note further that the actual reduction of the four-component field to a two-component one, which takes place in consequence of the condition of invariance under a gauge transformation, is closely connected with the fact that the *rest mass* of the particles corresponding to this field, i.e., photons, *is zero*. It is precisely because of this important property that the potential  $A$  disappears from the field equations and the invariance of the electromagnetic field under gauge transformations is brought about.

**5.3. Lagrangian Formalism.** In going over to the Lagrangian formalism we note that, as we shall see subsequently (cf. Chapter 2), when the electromagnetic field is quantized it will no longer be possible to satisfy the subsidiary Lorentz condition (5.6) as a relation between the components of the quantized potential  $A_n$ . In the quantum theory of the electromagnetic field the Lorentz condition will have to be replaced by a certain condition on the wave function subjected to second quantization which guarantees only that the average value of the operator  $(\partial A / \partial x)$  taken over the allowed states will be equal to zero.

Therefore, as in the case of the vector field, and in contrast to the usual presentation, we shall not introduce the Lorentz condition into the Lagrangian formalism. We shall choose the Lagrangian function following the example of the corresponding function for the vector field in which we set  $m = 0$ :

$$\mathcal{L} = -\frac{1}{2} \frac{\partial A_m}{\partial x^n} \frac{\partial A^m}{\partial x_n}. \quad (5.12)$$



This Lagrangian actually corresponds to the Lagrangian of Dirac, Fock, and Podolsky (1932) (cf. also Wentzel (1942)):

$$\mathcal{L} = -\frac{1}{4} H_{kl} H^{kl} - \frac{1}{2} \left( \frac{\partial A}{\partial x} \right)^2,$$

from which it differs by the divergence

$$\frac{1}{2} \frac{\partial}{\partial x^l} \left[ A_k \frac{\partial A_l}{\partial x^k} - A_l \frac{\partial A_k}{\partial x^k} \right].$$

The Lagrangian (5.12) differs from the generally used gauge-invariant expression

$$\mathcal{L} = -\frac{1}{4} H_{kl} H^{kl} \quad (5.13)$$

by a quantity which, after being integrated over all space-time and after the Lorentz condition has been taken into account, reduces to zero, giving no contribution to the action of the system.

In a similar manner, the Lagrangian (5.12) leads to quantities of the type of the energy-momentum tensor which do not agree with the usual gauge-invariant expressions. However, as in the case of the vector field, the differences between the corresponding expressions may be expressed in the form of divergences, and after the field equations and the Lorentz condition have been taken into account, do not give any contributions to the dynamic characteristics of the system of the type of the energy-momentum tensor (and of other quantities analogous to it) we note that, in accordance with the remarks made in §2, this question falls outside the framework of this presentation.

From the Lagrangian (5.12) we obtain by the usual methods, and with the aid of (1.4), (2.9), and (2.15), the following expressions: the field equations

$$\square A_k = 0;$$

the energy-momentum tensor

$$T_{kl} = -\frac{\partial A_n}{\partial x^k} \frac{\partial A_n}{\partial x^l} - g^{kl} \mathcal{L};$$

the spatial energy-momentum density

$$T^{00} = -\frac{1}{2} \frac{\partial A_n}{\partial x^k} \frac{\partial A_n}{\partial x^k}, \quad T^{0\alpha} = \frac{\partial A_n}{\partial x^0} \frac{\partial A_n}{\partial x^\alpha} \quad (\alpha = 1, 2, 3); \quad (5.14)$$

the spin angular-momentum tensor

$$S_{lm}^n = \left( A_m \frac{\partial A_l}{\partial x_n} - A_l \frac{\partial A_m}{\partial x_n} \right)$$

and the spatial density of the spin vector

$$S_{\alpha\beta}^0 = A_\beta \frac{\partial A_\alpha}{\partial x^0} - A_\alpha \frac{\partial A_\beta}{\partial x^0}. \quad (5.15)$$

To complement the Lagrangian formalism, we shall impose the Lorentz condition

$$\left( \frac{\partial A}{\partial x} \right) = 0.$$

We note that the above Lorentz condition refers only to the unquantized electromagnetic field and, as we shall see later (in Chapter II), must be replaced upon quantization by certain conditions on the wave functions of the allowed states, which are equivalent to the Lorentz condition only in an average sense.

The expressions for the energy-momentum and angular-momentum tensors obtained in this way differ, as may be expected, from the corresponding gauge-invariant expressions commonly used in the theory of the electromagnetic field, which are obtained from the gauge-invariant Lagrangian (5.13). It is readily shown, however, that the differences between the corresponding tensors can be represented by the sum of the corresponding anti-symmetric divergences and terms that vanish when the field equations and the additional Lorentz condition are taken into account, so that they lead to identical expressions for the dynamic characteristics of the system.

In order to calculate the dynamic characteristics, we shall decompose the components of the potential into their positive- and negative-frequency parts

$$A_n(x) = A_n^+(x) + A_n^-(x)$$

and go over to the momentum representation. We write down directly the formula for the three-dimensional momentum representation, keeping in mind the fact that the three-dimensional momentum amplitudes  $A(\mathbf{k})$  are related to the four-dimensional amplitudes  $A(k)$  by usual expressions of type (4.15) and satisfy the following complex conjugate conditions:

$$(A_n^\pm(k))^* = A_n^\mp(k).$$

*5.4. Transverse, Longitudinal, and Time Components.* We now introduce a frame of reference associated with the momentum vector  $\mathbf{k}$ , i.e., we shall go over to a special frame of reference in momentum space by representing  $A(\mathbf{k})$  in the form of a sum of a time, a longitudinal and a transverse component:

$$A_n^\pm(k) = e_n^1 a_1^\pm(k) + e_n^2 a_2^\pm(k) + e_n^3 a_3^\pm(k) + e_n^0 a_0^\pm(k). \quad (5.16)$$

Here  $\mathbf{e}^1$  and  $\mathbf{e}^2$  are spatial polarization unit vectors orthogonal to each other and to the unit vector  $\mathbf{e}^3$  in the direction of the momentum vector:

$$(e^\alpha e^\beta) = \delta_{\alpha\beta} \quad (\alpha, \beta = 1, 2, 3), \quad e_0^\alpha = 0, \quad e^3 = \frac{k}{|k|},$$

$$[e^1 e^2] = e^3, \quad [e^2 e^3] = e^1, \quad [e^3 e^1] = e^2,$$

while  $e^0$  is a time unit vector:

$$e_n^0 = \delta_{n0}.$$

It is readily verified that under the transformation (5.16) the basic quadratic form which determines the energy-momentum four-vector [cf. (5.20)] does not change its form

$$A_n^+(\mathbf{k}) A^{-n}(\mathbf{k}) = g^{mn} a_m^+(\mathbf{k}) a_n^-(\mathbf{k}). \quad (5.17)$$

On further substituting the expansion (5.16) into the additional Lorentz condition, written separately for the positive- and the negative-frequency components,

$$k^n A_n^+(\mathbf{k}) = 0, \quad k^n A_n^-(\mathbf{k}) = 0,$$

we shall obtain two relations

$$|\mathbf{k}| a_2^\pm(\mathbf{k}) - k^0 a_0^\pm(\mathbf{k}) = 0.$$

Noting that since the mass  $m$  is equal to zero,  $|\mathbf{k}| = k^0$ , we obtain

$$a_3^+(\mathbf{k}) a_3^-(\mathbf{k}) - a_0^+(\mathbf{k}) a_0^-(\mathbf{k}) = 0. \quad (5.18)$$

This relation means that, because of the Lorentz condition, the average number  $a_3^{(+)} a_3^{(-)}$  of “longitudinal” photons is equal to the average number  $a_0^{(+)} a_0^{(-)}$  of “time-like” photons, and that their contributions to the energy-momentum four-vector are opposite in sign. One may therefore speak of the “longitudinal” and “time-like” photons “compensating” each other. Substituting (5.18) into (5.17), we obtain

$$-A_n^+(\mathbf{k}) A^{-n}(\mathbf{k}) = \sum_{\nu=1,2} a_\nu^+(\mathbf{k}) a_\nu^-(\mathbf{k}). \quad (5.19)$$

Further, on computing the energy-momentum four-vector

$$P^n = \int d\mathbf{k} \cdot k^n \{ -A_m^+(\mathbf{k}) A^{-m}(\mathbf{k}) \} = \sum_{\mu=1,2} \int d\mathbf{k} \cdot k^n a_\mu^+(\mathbf{k}) a_\mu^-(\mathbf{k}), \quad (5.20)$$

we find that in the case under consideration, as in the case of the vector field, the energy turns out to be positive definite only by virtue of the Lorentz subsidiary condition.

5.5. *Spin.* If we next determine the spin vector

$$\left. \begin{aligned} S_\alpha &= \varepsilon_{\alpha\beta\gamma} \int S_{\beta\gamma}^0 dx, \\ \mathbf{S} &= i \int d\mathbf{k} [\mathbf{A}^+(\mathbf{k}) \times \mathbf{A}^-(\mathbf{k})] = i \int d\mathbf{k} [\mathbf{a}^+(\mathbf{k}) \times \mathbf{a}^-(\mathbf{k})], \end{aligned} \right\} \quad (5.21)$$

which remains constant in time because of the symmetry of the tensor  $T^{kl}$ , we find its component along the direction of the propagation vector  $\mathbf{k}$  in the form

$$S_3 \sim i(a_1^+(\mathbf{k})a_2^-(\mathbf{k}) - a_2^+(\mathbf{k})a_1^-(\mathbf{k})). \quad (5.22)$$

Introducing the new amplitudes  $b_1^{(\pm)}$  and  $b_2^{(\pm)}$  as in §4, we obtain the “diagonal” expressions for  $P^n$  and  $S_3$ :

$$P^n = \int d\mathbf{k} \cdot \mathbf{k}^n (b_1^+(\mathbf{k})b_1^-(\mathbf{k}) + b_2^+(\mathbf{k})b_2^-(\mathbf{k})), \quad (5.23)$$

$$S_3 \sim (b_1^+(\mathbf{k})b_1^-(\mathbf{k}) - b_2^+(\mathbf{k})b_2^-(\mathbf{k})), \quad (5.24)$$

from which it follows directly that the products

$$b_i^+(\mathbf{k})b_i^-(\mathbf{k}) \quad (i = 1, 2)$$

may be regarded as average numbers of particles of zero mass, momentum  $\mathbf{k}$ , energy  $k^0 = |\mathbf{k}|$ , having a component of spin angular momentum along the direction of the propagation vector equal to  $+1 (i = 1)$  and to  $-1 (i = 2)$ . We are in fact dealing with *photons*.

Thus the transition to the momentum representation allows one to see directly that the electromagnetic field describes transverse photons with two possible values of the component of spin along the direction of motion. A rigorous foundation of this correspondence of course, given by the theory only as a result of second quantization.

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## §6. The Spinor Field.

### Dirac Matrices and Transformation Laws for Spinor Functions

*6.1. Factorization of the Klein-Gordon Operator.* We now proceed to examine the simplest spinor field which, as will be shown below, is characterized by the fact that it describes charged particles of spin  $\frac{1}{2}$ , for example, electrons and positrons. Following Dirac we shall obtain the corresponding equations by means of the “factorization” of the Klein-Gordon operator:

$$\square - m^2 = P_n P^n - m^2, \quad (6.1)$$

where for convenience the usual quantum mechanical notation

$$P_n = i \frac{\partial}{\partial x^n}.$$

has been adopted. The operator (6.1) is quadratic in the derivatives  $\partial/\partial x^n$ , and, as may be easily seen, cannot be represented in the form of two factors linear in  $P^n$  with numerical coefficients. Indeed, if by analogy with the formula for factoring the difference of two squares, one attempts to write (6.1) in the form

$$\square - m^2 = (P - m)(P + m),$$

where  $P$  is some linear combination of the operators  $P^k$  with coefficients  $\gamma^k$

$$P = P_k \gamma^k,$$

then one must demand that the relation

$$P_n P^n = (P_k \gamma^k)^2,$$

should hold, on expanding the right-hand side of which one finds the condition determining the coefficients  $\gamma$ :

$$\gamma^n \gamma^m + \gamma^m \gamma^n = 2g^{mn}. \quad (6.2)$$

Since, in accordance with this condition, the quantities  $\gamma^k$  with different indices anticommute, they are not ordinary numbers and may be chosen in the form of matrices.

With the aid of these quantities, the Klein-Gordon operator may be represented in the form of a product of two commuting matrix operators:

$$\square - m^2 = \left( i\gamma^n \frac{\partial}{\partial x^n} + m \right) \left( i\gamma^k \frac{\partial}{\partial x^k} - m \right), \quad (6.3)$$

and in order that the field function should satisfy the Klein-Gordon equation

$$(\square - m^2)\psi = 0, \quad (6.4)$$

we may demand that it should also satisfy one of the two first-order equations:

$$\left( i\gamma^n \frac{\partial}{\partial x^n} + m \right) \psi(x) = 0 \quad \text{or} \quad \left( i\gamma^n \frac{\partial}{\partial x^n} - m \right) \psi(x) = 0. \quad (6.5)$$

Equations (6.5) are, of course, less general than (6.4) and, although every solution of one of equations (6.5) also satisfies equation (6.4), the converse does not hold.

It is important to note that equations (6.5) are actually *quite new* equations, since the functions which satisfy them give a *spinor* representation of the Lorentz group.

One may therefore expect that equations (6.5) contain more detailed information than equation (6.4). It is well known that this is actually the case, since it was with the aid of just these equations (6.5) that Dirac succeeded in describing the spin of the electron.

Equations (6.5) are called *Dirac equations*, and the matrices  $\gamma$  defined by equations (6.2) are called the *Dirac matrices*.

A common property of all the solutions of the Klein-Gordon equation which we have considered until now, and which correspond to single-valued representations of the Lorentz group, is the fact that the particles described by them have integral spin (0 and 1 in the cases examined so far). Spinor representations have to be used to describe particles with half-integer spin, and the simplest of them corresponds to (6.5) and (6.2).

In view of the matrix character of the operators in (6.5) the wave function  $\psi$  which satisfies them is a many-component one, with the number of its components determined by the rank of the matrices  $\gamma$ .

**6.2. Dirac Matrices.** We now turn to a study of the properties of the four hypercomplex numbers  $\gamma^n$  defined by the relations (6.2). The rank of the irreducible representation of these quantities may be established on the basis of the following considerations.

If to the above four quantities we add all their possible products (including multiple ones) and all the linear combinations with all possible complex coefficients of all the hypercomplex numbers obtained in this way, we evidently shall obtain a set of elements for which the operations of addition of one element to another and of multiplication of an element by an element or by a complex number are defined. As it is usually stated, we obtain a certain *algebra*  $A$  over the field of complex numbers. In addition, we shall have to assume that a finite-dimensional matrix representation exists in this case.

By using a number of algebraic theorems\* it may be shown that the rank  $n$  of an irreducible matrix representation of a given system of hypercomplex numbers is related to the number  $h$  of linearly independent elements of algebra  $A$  by

$$h = n^2. \quad (6.6)$$

Going over to the matrix representation of the quantities  $\gamma$ , it may readily be seen that this relation expresses the simple fact that the number  $h$  of linearly independent square matrices of rank  $n$  is equal to the number of elements of these matrices.

Let us find the number  $h$ . From the four matrices  $\gamma^k$  one may construct, by multiplying them together, sixteen linearly independent matrices:

$$\left. \begin{array}{ll} \text{the unit matrix} & \mathbf{I} = g^{kk} \gamma^k \gamma^k \quad (\text{no summation}) \\ \text{the four matrices} & \gamma^k \quad (k = 0, 1, 2, 3), \\ \text{the six matrices} & \sigma^{kl} = i \frac{\gamma^k \gamma^l - \gamma^l \gamma^k}{2} = i \gamma^k \gamma^l \quad (k < l; k, l = 0, 1, 2, 3), \\ \text{the single matrix**} & \gamma^5 = -i \gamma^0 \gamma^1 \gamma^2 \gamma^3, \\ \text{the four matrices} & D^k = \gamma^k \gamma^5 \quad (k = 0, 1, 2, 3). \end{array} \right\} \quad (6.7)$$

It may readily be seen that all other products of the matrices (6.2), and consequently

\*cf., for example, Van der Waerden (1932), Ch. 2.

\*\*The symbol  $\gamma^4$  is usually reserved for the matrix which differs from  $\gamma^0$  by the factor  $i$ , i.e.,  $\gamma^4 = i\gamma^0$ .

all the linear combinations of such products, i.e., the whole algebra  $A$ , can be expressed by linear relations in terms of the matrices (6.7). Thus, for example, after suitable commutations have been performed we have

$$\gamma^2 \gamma^1 = i\sigma^{12}, \quad \gamma^1 \gamma^2 \gamma^3 \gamma^2 \gamma^0 = -D^2.$$

As may easily be shown, the matrices (6.7) are linearly independent. To prove this property, we must examine the traces of the matrices (6.7). Remembering that, by definition, the trace of a matrix is the sum of its diagonal elements, we arrive at the possibility of cyclic permutation of matrix factors under the trace sign. Thus, for example:

$$\text{Tr } ABC \equiv \sum_{\alpha, \beta, \gamma} A_{\alpha\beta} B_{\beta\gamma} C_{\gamma\alpha} = \sum_{\alpha, \beta, \gamma} C_{\gamma\alpha} A_{\alpha\beta} B_{\beta\gamma} = \text{Tr } CAB. \quad (6.8)$$

Using this property and the definition (6.2) we shall show that the traces of all the matrices (6.7) with the exception of the unit matrix are equal to zero.

Let us consider, for example, the trace of one of the matrices  $\gamma^k$  ( $k = 0, 1, 2, 3$ ). We multiply  $\gamma^k$  on the left by  $\gamma_n \gamma^n = 1$  ( $n \neq k$ ) and use the cyclic property. We then have:

$$\text{Tr } \gamma^k = \text{Tr } \gamma_n \gamma^n \gamma^k = \text{Tr } \gamma^n \gamma^k \gamma_n$$

from which we obtain with the aid of (6.2):

$$\text{Tr } \gamma^k = \frac{1}{2} \text{Tr } \gamma_n (\gamma^k \gamma^n + \gamma^n \gamma^k) |_{n \neq k} = 0. \quad (6.9)$$

In a similar way one can show that all the other traces are also equal to zero. To illustrate the method, we also calculate the trace of the matrix  $\gamma^5$ . By a cyclic permutation of the matrix  $\gamma^0$  which appears as a factor in  $\gamma^5$  we have:

$$\text{Tr } \gamma^5 = \text{Tr } \gamma^0 D^0 = \text{Tr } D^0 \gamma^0. \quad (6.10)$$

However, the result of a direct commutation of  $\gamma^0$  and  $D^{123}$  yields

$$\gamma^0 D^0 = -D^0 \gamma^0,$$

from which it follows directly that the trace of the matrix  $\gamma^5$  is equal to zero.

We shall now show that the linear independence of the matrices (6.7) follows from the fact that all their traces, except that of the unit matrix, are zero. To obtain the proof we assume the converse. Let

$$F = aI + \sum_n b^n \gamma^n + \sum_{m,k} c^{mk} \sigma^{mk} + \sum_l d^l D^l + e \gamma^5 = 0,$$

where  $a, b, c, d$ , and  $e$  are certain complex coefficients. Taking the trace of  $F$  we find  $a = 0$ . Then taking the trace of  $\gamma^k F$  we find that  $b^k = 0$ . In a similar manner by successively taking the traces of the products of  $F$  with each of the matrices (6.7), we convince ourselves that all the coefficients in  $F$  are zero. Thus the number of linearly independent matrices of the algebra under consideration is sixteen, and the rank of the irreducible representation of the matrices  $\gamma^k$  is equal to four in accordance with (6.6).

Thus the hypercomplex numbers  $\gamma^k$  may be represented by *square matrices of four rows and columns*. From the definition (6.2) it follows that the four matrices  $\gamma^k$  may be chosen to be unitary, if the condition of being Hermitian conjugate is imposed in the form

$$\gamma^{k\dagger} = \gamma_k. \quad (6.11)$$

(Here the matrix  $a^\dagger$  which is Hermitian conjugate to  $a$  is taken to be as usual the matrix which is obtained from  $a$  by the operation of taking the complex conjugate of each of its elements followed by an interchange of rows and columns, i.e.,  $a^\dagger_{\alpha\beta} = a^*_{\beta\alpha}$ .)

It may also be easily seen that the matrix  $\gamma^5$  introduced above anticommutes with  $\gamma^0, \gamma^1, \gamma^2, \gamma^3$ , while its square is equal to  $+1$ , i.e.,

$$\gamma^k \gamma^l + \gamma^l \gamma^k = 2g^{kl} \quad (k, l = 0, 1, 2, 3, 5), \quad (6.12)$$

where by definition

$$g^{55} = +1. \quad (6.13)$$

Also

$$\gamma^{k\dagger} = g^{kn} \gamma_n = \gamma_k \quad (k = 0, 1, 2, 3, 5). \quad (6.14)$$

From the fact that the matrices  $\gamma$  have four rows it follows that the trace of the unit matrix is equal to four, i.e.,

$$\text{Tr } \mathbf{I} = \text{Tr } \gamma^k \gamma_k = 4 \quad (\text{no summation}). \quad (6.15)$$

Taking into account the fact demonstrated above that the traces of the matrices  $\gamma^k$ , and also of their products  $\Gamma^{kl}, D^{klm}$ , are equal to zero, and using the basic formula (6.2), we arrive after some additional computation at the following set of formulas:

$$\left. \begin{aligned} \text{Tr } \gamma^k &= 0, \\ \text{Tr } \gamma^k \gamma^l &= 4g^{kl}, \\ \text{Tr } \gamma^k \gamma^l \gamma^m \gamma^n &= 4g^{kl} g^{mn} + 4g^{lm} g^{kn} - 4g^{km} g^{ln} \end{aligned} \right\} (k, l, m, n = 0, 1, 2, 3, 5) \quad (6.16)$$

and so on.



As a general rule, we find that the traces of the products of an odd number of matrices  $\gamma$  are always equal to zero, while traces of an even number of such matrices are up to a factor 4 equal to the antisymmetrized sums of products of a corresponding number of factors  $g^{kl}$ , with the signs of the individual terms of such sums being determined by whether the corresponding permutation of indices is odd or even.

In concluding our discussion of the Dirac matrices, we note that the basic relations (6.2), and all the properties of the matrices found above, are invariant under a unitary transformation

$$\gamma^k \rightarrow O \gamma^k O^{-1}, \quad (6.17)$$

where  $O$  is an arbitrary nonsingular matrix (i.e., one having an inverse) which may be regarded as unitary.

From this it follows that in general the matrices are determined only up to a unitary transformation, and the specific representation of the matrices may be chosen in a number of different ways. It is customary to use that representation of the Dirac matrices in which  $\gamma^0$  is diagonal:

$$\left. \begin{aligned} \gamma^0 &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \quad \gamma^2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}, \\ \gamma^3 &= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad \gamma^5 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}. \end{aligned} \right\} \quad (6.18)$$

This representation of the matrices  $\gamma$  is connected with the frequently used matrices  $\alpha$ ,  $\beta$ , and  $\sigma$ ,  $\rho$  by the relations

$$\left. \begin{aligned} \alpha_n &= \gamma_0 \gamma^n \quad (n = 1, 2, 3), \quad \beta = \gamma^0, \\ \sigma_v &= \gamma^5 \gamma^v \gamma^0 \quad (v = 1, 2, 3), \quad \sigma_1 = \sigma^{23}, \quad \sigma_2 = \sigma^{31}, \quad \sigma_3 = \sigma^{12}, \\ \varrho_1 &= -\gamma^5, \quad \varrho_2 = i\gamma^0 \gamma^5, \quad \varrho_3 = \gamma^0. \end{aligned} \right\} \quad (6.19)$$

All the other representations may be obtained from (6.18) by means of the transformation (6.17).

**6.3. The Dirac Equation.** Let us now return to an examination of Dirac's equations (6.5). Noting that the set of Dirac matrices is invariant with respect to a change of sign of  $\gamma$  (transformation (6.17) with  $O = \gamma^5$ ), we conclude that the sign in front of the mass term in the Dirac operator is not significant. Usually, the fundamental Dirac equation is written in the form

$$\left(i\gamma^n \frac{\partial}{\partial x^n} - m\right)\psi(x) = 0. \quad (6.20)$$

The second conjugate Dirac equation may be obtained from (6.20) in the following way. By taking the Hermitian conjugate of (6.20), and using (6.14), we obtain

$$i \frac{\partial \psi^*}{\partial x^n} \gamma^n + m \psi^* = i \sum_n g^{nn} \frac{\partial \psi^*}{\partial x^n} \gamma^n + m \psi^* = 0.$$

This equation for  $\psi^*$ , however, does not have the correct form, since it differs from the first equation in (6.5) by the factor  $g^{nn}$  under the summation sign. In order to remove this deficiency we may, for example, multiply this equation on the right by the matrix  $\gamma^0$ . After carrying out the commutation of Dirac matrices under the summation sign we obtain the equation

$$i \frac{\partial \bar{\psi}}{\partial x^n} \gamma^n + m \bar{\psi} = 0, \quad (6.21)$$

which has the correct form. The function  $\bar{\psi}$  which appears in this equation is defined by the relation

$$\bar{\psi}(x) = \psi^*(x) \gamma^0 \quad (6.22)$$

and is referred to as the *adjoint* (or, more precisely, the *Dirac-conjugate*) function with respect to  $\psi$ . In a similar fashion, equation (6.21) is said to be “adjoint” with respect to (6.20). It will be shown later (see §7) that, by analogy with (4.36), the function  $\bar{\psi}$  enables us to introduce a Lagrangian and to construct dynamic variables.

Since the rank of the Dirac matrices is equal to four, the wave functions  $\psi$  and  $\bar{\psi}$  are *four-component* ones, and are sometimes represented by a four-component column and row vector respectively. In spite of its simple appearance, the representation of  $\psi$  and  $\bar{\psi}$  by rows and columns may, however, lead to a certain amount of confusion (for example, in the introduction of the operation of charge conjugation in §13 of Chapter II). We shall not use it. It will be sufficient for us to remember that by writing out in detail the matrix form of Dirac equation (6.20), we obtain the four equations

$$\left(i\gamma_{\alpha\beta}^n \frac{\partial}{\partial x^n} - mI_{\alpha\beta}\right)\psi_\beta(x) = 0$$

for the four components  $\psi_\alpha$  ( $\alpha = 1, 2, 3, 4$ ). Obviously, the adjoint equation (6.21) may also be written out in component form.

**6.4. Transformation Properties of the Spinor Field.** We now turn to an examination of the transformations of the spinor wave functions under Lorentz transformations of the coordinates. We note that, in contrast to the tensor representations considered earlier,

for which the field equations were automatically covariant due to the covariance of the Klein-Gordon operator, the conditions for the covariance of the Dirac equations (6.20), (6.21) must be investigated separately.

It is well known that an equation is *covariant* if after a transformation, i.e., in terms of the transformed coordinates and functions, it has the same form as before the transformation.

Thus, for example, under the space-time translations

$$x \rightarrow x' = x^k + a^k$$

the Dirac operator retains its form, i.e.,

$$i\gamma^n \frac{\partial}{\partial x^n} - m = i\gamma^n \frac{\partial}{\partial x'^n} - m,$$

and therefore, in accordance with the remark made in §2, we must set

$$\psi'(x') = \psi(x),$$

since this ensures that the equation has the old form in terms of the new variables:

$$\left(i\gamma^n \frac{\partial}{\partial x'^n} - m\right) \psi'(x') = 0, \quad (6.23)$$

i.e., it remains covariant.

Under the infinitesimal rotations

$$x \rightarrow x' = x + \delta x, \quad \delta x^k = x^n \omega_{\phantom{n}n}^k = g^{kl} x^n \omega_{ln}, \quad \omega_{kn} + \omega_{nk} = 0 \quad (6.24)$$

the Dirac operator is no longer covariant:

$$\left(i\gamma^k \frac{\partial}{\partial x'^k} - m\right) = i\gamma^n \left(\frac{\partial}{\partial x^n} - \omega_{\phantom{n}n}^k \frac{\partial}{\partial x^k}\right) - m,$$

so that the transformation law for the spinor wave function has the more complicated matrix form (1.6):

$$\psi'(x') = \Lambda \psi(x). \quad (6.25)$$

To establish the connection between the transformation matrix  $\Lambda$  and the Dirac matrices and the transformation parameters, we start from the condition of covariance of the Dirac equation, and assume that it has the form (6.23) in terms of the transformed variables. Multiplying (6.23) on the left by  $\Lambda^{-1}$ , we obtain the original equation (6.20), provided the following relation holds:

$$\Lambda^{-1} \gamma^n \Lambda = \gamma^n + \gamma^k \omega^n_k \quad (n, k = 0, 1, 2, 3). \quad (6.26)$$

Here all terms of the first order in  $\omega$  cancel out, so that neglecting second-order terms, we obtain:

$$\Lambda^{-1} \left( i \gamma^k \frac{\partial}{\partial x'^k} - m \right) \psi'(x') = \left( i \gamma^k \frac{\partial}{\partial x^k} - m \right) \psi(x) = 0.$$

By representing the matrix  $\Lambda$  in the form

$$\Lambda = 1 + \lambda^{ik} \omega_{ik}, \quad (6.27)$$

where the expansion coefficients  $\lambda_{ik}$  are antisymmetric, i.e.,

$$\lambda^{ik} = -\lambda^{ki},$$

we find that, in view of (6.26),

$$\gamma^n \lambda^{kl} - \lambda^{kl} \gamma^n = g^{nk} \gamma^l,$$

which together with the conditions of antisymmetry yield

$$2\lambda^{kl} = \gamma^k \gamma^l - \gamma^l \gamma^k = -i\sigma^{kl} = i\sigma^{kl}. \quad (6.28)$$

Formulas (6.27) and (6.28) give us the explicit form of the transformation matrix  $\Lambda$  for the infinitesimal rotation transformations. However, it turns out that these formulas can also be used to obtain the explicit form of the matrix  $\Lambda$  for finite transformations. To achieve this, we make use of the *group* property of the operator  $\Lambda(\varphi)$  where  $\varphi$  is the angle of rotation in one of the planes  $x^k x^l$ . Since rotations are additive, we have

$$\Lambda(\varphi_1 + \varphi_2) \psi = \Lambda(\varphi_1) \Lambda(\varphi_2) \psi,$$

i.e.,

$$\Lambda(\varphi_1 + \varphi_2) = \Lambda(\varphi_1) \Lambda(\varphi_2).$$

Substituting  $\varphi_1 = \varphi$ ,  $\varphi_2 = d\varphi$  in the above expressions and rearranging:

$$\Lambda(\varphi + d\varphi) = \Lambda(\varphi) \Lambda(d\varphi), \quad \frac{\Lambda(\varphi + d\varphi) - \Lambda(\varphi)}{d\varphi} = \Lambda(\varphi) \frac{\Lambda(d\varphi) - 1}{d\varphi},$$

we obtain the differential equation for the group of rotations

$$\frac{d\Lambda(\varphi)}{d\varphi} = \Lambda(\varphi) \hat{p}_{ik}.$$

On integrating this equation, taking into account (6.28) and the initial condition  $\Lambda(0) = 1$  which arises from (6.27), we find that

$$\Lambda^{ik}(\varphi) = \exp \frac{1}{2} [\gamma^i \gamma^k \varphi] = e^{i\sigma^{ik} \frac{\varphi}{2}}. \quad (6.29)$$

The operator  $\Lambda(\varphi)$  thus produces a rotation through the "half-angle"  $\frac{1}{2}\varphi$ .

The calculation carried out above with the aid of which we have obtained (6.29) is a special case of an argument, well known in group theory, by means of which the group as a whole is reconstructed from infinitesimal rotation operators.

Let us examine special cases of (6.29). For a space rotation in the plane  $x^\alpha x^\beta$  ( $\alpha \neq \beta$ ,  $\alpha, \beta = 1, 2, 3$ )

$$x'^\alpha = x^\alpha \cos \varphi - x^\beta \sin \varphi, \quad x'^\beta = x^\beta \cos \varphi + x^\alpha \sin \varphi; \quad (6.30)$$

and from this we obtain

$$\left. \begin{aligned} \psi'(x') &= \Lambda_{\alpha\beta} \psi(x), \\ \Lambda_{\alpha\beta} &= e^{-i\sigma^{\alpha\beta} \frac{\varphi}{2}} = \cos \frac{\varphi}{2} - i\sigma^{\alpha\beta} \sin \frac{\varphi}{2}, \\ \Lambda_{\alpha\beta}^{-1} &= e^{i\sigma^{\alpha\beta} \frac{\varphi}{2}} = \cos \frac{\varphi}{2} + i\sigma^{\alpha\beta} \sin \frac{\varphi}{2}. \end{aligned} \right\} \quad (6.31)$$

For a Lorentz rotation in the plane  $x^0 x^\alpha$  ( $\alpha = 1, 2, 3$ )

$$\left. \begin{aligned} x'^0 &= x^0 \cosh \varphi + x^\alpha \sinh \varphi, \\ x'^\alpha &= x^\alpha \cosh \varphi + x^0 \sinh \varphi \end{aligned} \right\} \quad (\tanh \varphi = v) \quad (6.32)$$

we have, correspondingly,

$$\left. \begin{aligned} \Lambda_{0\alpha} &= e^{-i\sigma^{0\alpha} \frac{\varphi}{2}} = \cosh \frac{\varphi}{2} - i\sigma^{0\alpha} \sinh \frac{\varphi}{2}, \\ \Lambda_{0\alpha}^{-1} &= e^{i\sigma^{0\alpha} \frac{\varphi}{2}} = \cosh \frac{\varphi}{2} + i\sigma^{0\alpha} \sinh \frac{\varphi}{2}. \end{aligned} \right\} \quad (6.33)$$

where we have used the fact that  $(\sigma^{\alpha\beta})^2 = 1$ ,  $(\sigma^{0\alpha})^2 = -1$  (no summation over  $\alpha, \beta$ ).

As may be seen from (6.26), the expression  $\Lambda^{-1} \gamma^\mu \Lambda$  in the case of the infinitesimal transformation (6.24) is given in terms of the matrices  $\gamma^\mu$  by the same linear form that gives the transformed coordinates  $x'$  in terms of  $x$ . Forming the expressions  $\Lambda^{-1} \gamma^\mu \Lambda$  with

the aid of (6.31) and (6.33), and comparing them with (6.30) and (6.32), we find that the same holds for the finite rotation transformations.

The nonuniqueness of spinor functions follows directly from the formulas of space rotation (6.31). Setting  $\varphi = 2\pi$  in these formulas we find that the transformation matrix  $\Lambda(2\pi) = -1$  corresponds to a complete space rotation of the coordinate system, i.e., the field function changes sign under such a transformation. However, since the transformation of rotation through  $2\pi$  brings the coordinate system into its original position, i.e., coincides with the identity transformation, it follows that the spinor wave functions are always determined to within their sign.

We also exhibit the form of the matrix  $\Lambda$  for the transformations of reflection of different number of coordinate axes. Noting that the transformation formulas for reflections of an even number of different space axes, which reduce to rotations, follow from (6.31), we restrict ourselves to the transformation of reflection of all three space axes ( $P$  transformation)

$$\left. \begin{aligned} x'^\alpha &= -x^\alpha \quad (\alpha = 1, 2, 3), \quad x'^0 = x^0, \\ \psi'(x') &= \eta(P) \Lambda_{123} \psi(x), \quad \Lambda_{123} = \gamma^0, \end{aligned} \right\} \quad (6.34)$$

since the spinor representation is two valued, we have  $\eta^2(P) = \pm 1$ .

The transformation properties of the adjoint spinor  $\bar{\psi}$  follow from its definition (6.22). Taking the Hermitian conjugate of (6.25), and multiplying it on the right by  $\gamma^0$ , we obtain

$$\bar{\psi}'(x') \gamma^0 = \bar{\psi}(x) \gamma^0 \Lambda^\dagger \gamma^0, \quad (6.35)$$

i.e., the adjoint spinor is transformed by the matrix  $\gamma^0 \Lambda^\dagger \gamma^0$ . Further it is readily seen that the following relation always holds for any transformation in the full Lorentz group:

$$\gamma^0 \Lambda^\dagger \gamma^0 = \Lambda^{-1}, \quad (6.36)$$

so that the corresponding transformation law for the adjoint spinor is

$$\bar{\psi}'(x') = \bar{\psi}(x) \Lambda^{-1}. \quad (6.37)$$

To prove (6.36), we note that it obviously holds under reflection of the space axes (6.34), while its validity under rotations may be established with the aid of the relation

$$\gamma^0 \gamma^k \gamma^0 = \gamma^k \quad (6.38)$$

which follows from (6.2) and (6.14). Since, in accordance with (6.31) and (6.33), the transformation matrices of the rotations are linear functions of quadratic combinations of Dirac's matrices, we find with the aid of (6.38) that

$$\gamma^0 \Lambda^\dagger (\gamma^k \gamma^l) \gamma^0 = \gamma^0 \Lambda^\dagger (\gamma^l \gamma^k) \gamma^0 = \Lambda (\gamma^0 \gamma^l \gamma^0 \cdot \gamma^0 \gamma^k \gamma^0) = \Lambda (\gamma^l \gamma^k) = \Lambda (-\gamma^k \gamma^l),$$

which is equivalent\* to (6.36) in accordance with the second formulas of (6.31) and (6.33).

The foregoing discussion shows that under an arbitrary homogeneous Lorentz transformation of the coordinate system

$$x' = Lx, \quad x'^n = L^{nm}x_m \quad (6.40)$$

the spinor wave function  $\psi(x)$ , and the function  $\bar{\psi}(x)$  adjoint to it, transform with the aid of mutually inverse matrix operators  $\Lambda$  and  $\Lambda^{-1}$  which have the property

$$\Lambda^{-1}\gamma^n\Lambda = L^{nm}\gamma_m. \quad (6.41)$$

We shall now show that the transformation laws (6.25), (6.37), taken together with (6.41), ensure that *quadratic forms of spinors  $\psi$  and  $\bar{\psi}$  transform in accordance with tensor representations of the Lorentz group.*

To do this, consider the quadratic form

$$\bar{\psi}(x) O \psi(x), \quad (6.42)$$

where  $O$  is a certain, as yet arbitrary, matrix constructed from products of the  $\gamma$  matrices. Under the Lorentz transformation (6.40) we find, using (6.25) and (6.37), that

$$\bar{\psi}'(x') O \psi'(x') = \bar{\psi}(x) \Lambda^{-1} O \Lambda \psi(x).$$

Let us consider the simplest special cases: (a)  $O = 1$ ;  $\bar{\psi}'(x') \psi'(x') = \bar{\psi}(x)\psi(x)$ , from which it is clear that the form  $\bar{\psi}\psi$  is a scalar; (b)  $O = \gamma^n$  ( $n = 0, 1, 2, 3$ ); using (6.41) we find that

$$\bar{\psi}'(x') \gamma^n \psi'(x') = L^{mn} \bar{\psi}(x) \gamma_m \psi(x),$$

from which it follows that the four quantities  $\bar{\psi}\gamma^n\psi$  form a covariant four-vector.

Similarly, it may be shown that the quantities

$$\bar{\psi}(x) \frac{\gamma^n \gamma^m - \gamma^m \gamma^n}{2} \psi(x) = i \bar{\psi}(x) \sigma^{mn} \psi(x)$$

represent the components of an antisymmetric covariant tensor of rank two, and so on.

Of particular interest are those cases where the matrix  $\gamma^5$  occurs as a factor in  $O$ . The point is that, as may easily be shown,  $\gamma^5$  commutes with  $\Lambda$  under proper Lorentz transformations:

\*By carrying out a similar calculation for a transformation which includes a reflection of the time axis we would have obtained in place of (37) the relation

$$\bar{\psi}'(x)' = -\bar{\psi}(x) \Lambda^{-1}. \quad (6.39)$$

$$\Lambda^{-1}\gamma^5\Lambda = \gamma^5,$$

while under improper Lorentz transformations they anticommute:

$$\Lambda^{-1}\gamma^5\Lambda = -\gamma^5.$$

Therefore, the form  $\bar{\psi}(x)\gamma^5\psi(x)$  behaves like a scalar in the former case and changes sign in the latter. It is clear that it is a pseudoscalar. In a similar fashion, the four quantities

$$\bar{\psi}(x)\gamma^5\gamma^n\psi(x)$$

transform under rotations like the components of a four-vector, while under reflections they in addition change sign. They form a pseudovector.

Analogous arguments will also readily show the tensor nature of more complicated forms such as (6.42). However, we now conclude our examination of the properties of the  $\gamma$  matrices and of the transformation laws of spinor functions, since the material presented above will be sufficient for the purposes of our further discussion.

## §7. The Spinor Field.

### Properties of the Solutions and Dynamic Variables

*7.1. Momentum Representation and Matrix Structure.* We now proceed to discuss the properties of the solutions of the Dirac matrix equation (6.20)

$$\left(i\gamma^n \frac{\partial}{\partial x^n} - m\right)\psi(x) = 0.$$

The field function  $\psi$  has four components, and each of these components satisfies the Klein-Gordon equation. Indeed, applying the operator  $(i\hat{\partial} + m)$  from the left to (6.20) we find, on taking (6.3) into account, that

$$\left(i\gamma^n \frac{\partial}{\partial x^n} + m\right)\left(i\gamma^n \frac{\partial}{\partial x^n} - m\right)\psi(x) \equiv (i\hat{\partial} + m)(i\hat{\partial} - m)\psi(x) = (\square - m^2)\psi(x) = 0.$$

Therefore,  $\psi(x)$  may be written in the form

$$\psi(x) = \frac{1}{(2\pi)^{3/2}} \int dk e^{ikx} \delta(k^2 - m^2) \psi(k), \quad (7.1)$$

where, by definition, the momentum amplitude satisfies

$$(\hat{k} + m)\psi(k)|_{k^2=m^2} = 0. \quad (7.2)$$



Here we have introduced the notation

$$\hat{a} = a^n \gamma_n = (a\gamma) = a^0 \gamma - \alpha \gamma,$$

which we shall frequently employ in the subsequent discussion.

Decomposing, as usual, the function  $\psi(x)$  into its positive- and negative-frequency parts

$$\psi(x) = \psi^+(x) + \psi^-(x), \quad (7.3)$$

$$\psi^\pm(x) = \frac{1}{(2\pi)^{3/2}} \int dk e^{ikx} \delta(k^2 - m^2) \theta(\pm k^0) \psi(k), \quad (7.4)$$

and integrating over  $k^0$ , we obtain the formulas for the three-dimensional momentum representation:

$$\psi^\pm(x) = \frac{1}{(2\pi)^{3/2}} \int d\mathbf{k} e^{\pm i\mathbf{k}x} \psi^\pm(\mathbf{k}). \quad (7.5)$$

Here we have adopted the notation

$$\left. \begin{aligned} \psi^+(\mathbf{k}) &= \frac{\theta(k^0) \psi(k)}{2k^0}, \\ \psi^-(\mathbf{k}) &= \frac{\theta(k^0) \psi(-k)}{2k^0} \end{aligned} \right\} \quad (k^0 = \sqrt{\mathbf{k}^2 + m^2}). \quad (7.6)$$

The three-dimensional amplitudes  $\psi^{(+)}(\mathbf{k})$  and  $\psi^{(-)}(\mathbf{k})$  satisfy the matrix equations

$$(\mathbf{m} + \hat{\mathbf{k}}) \psi^+(\mathbf{k}) = 0, \quad (\mathbf{m} - \hat{\mathbf{k}}) \psi^-(\mathbf{k}) = 0. \quad (7.7)$$

The different signs of  $\hat{\mathbf{k}}$  in these equations arise from the difference between the signs of the exponents in the integrands of (7.5).

The matrix structure of  $\psi^{(+)}$  and  $\psi^{(-)}$  depends on the representation of the Dirac matrices  $\gamma$  and may be determined in the following manner. In view of the covariance of equation (7.2) established above, it may be discussed in some fixed system of coordinates, having in mind that we can always go over to any other system with the aid of the transformations discussed in the preceding section. Thus, if we take  $\mathbf{k} = 0$ , we find from (7.2) and (7.7) that

$$\begin{aligned} (\gamma^0 k^0 + m) \psi(k^0)|_{k^0=m} &= 0, \\ (\gamma^0 + \mathbf{I}) \psi^+(0) &= 0, \\ (\gamma^0 - \mathbf{I}) \psi^-(0) &= 0. \end{aligned}$$

From this we obtain in the representation (6.18)

$$\left. \begin{aligned} \psi_{\alpha}^{+}(0) &= c_3 \delta_{\alpha}^3 + c_4 \delta_{\alpha}^4, \\ \psi_{\beta}^{-}(0) &= c_1 \delta_{\beta}^1 + c_2 \delta_{\beta}^2. \end{aligned} \right\} \quad (7.8)$$

Here  $\alpha, \beta$  are matrix indices, and  $\delta_k^i$  are the Kronecker symbols.

The solution for an arbitrary  $k$  different from zero may be obtained from (7.8) by means of an appropriate Lorentz transformation.

The equations satisfied by  $\psi^{(+)}$  and  $\psi^{(-)}$  may also be written in the form

$$\left. \begin{aligned} (\gamma^0 k^0 - \boldsymbol{\gamma} \mathbf{k} + m) \psi^{+}(\mathbf{k}) &= 0, \\ (-\gamma^0 k^0 - \boldsymbol{\gamma} \mathbf{k} + m) \psi^{-}(-\mathbf{k}) &= 0 \end{aligned} \right\} \quad (k^0 = +\sqrt{\mathbf{k}^2 + m^2}),$$

in which they differ from each other only by the sign of  $k^0$ . As has been just shown, each of them has two linearly independent solutions. From this it follows that the Dirac equation has only *two linearly independent solutions* for each given value of the four-vector  $k$  (the sign of the  $k^0$  component being fixed!).

It is now not difficult to establish the transformation nature of the functions  $\psi^{(+)}$  and  $\psi^{(-)}$ . To do this we consider the set of transformations consisting of the three-dimensional purely spatial rotations and reflections of the space axes. This set of transformations forms a group  $G$  which is a subgroup of the Lorentz group. Since transformations in  $G$  do not act on the time coordinate  $x^0$ , they also leave invariant the matrix structure of the decomposition (7.3) of the field function into its positive- and negative-frequency parts. In other words, under three-dimensional rotations and spatial reflections, the two-component quantities  $\psi^{(+)}$  and  $\psi^{(-)}$  transform independently of each other. Therefore each of them provides a two-dimensional representation of the rotation group and of reflections in three-dimensional space. This representation is said to be a *spinor* representation, and quantities which transform in accordance with it are called *spinors in three-dimensional space*.

Thus the four-component field function  $\psi$ , which transforms in accordance with the spinor representation of the Lorentz group and which therefore represents a *spinor in four-dimensional pseudo-Euclidean space* over which the Lorentz group is defined, *decomposes with respect to the group of three-dimensional reflections and rotations into two irreducible parts which are spinors in three-dimensional space*.\*

The fact that the positive- and negative-frequency components of the field function  $\psi^{(+)}$  and  $\psi^{(-)}$  transform independently of each other under three-dimensional rotations and reflections may be directly checked in the representation (6.18) used by us in the following manner.

In accordance with (7.8), the field function  $\psi$  may be represented in the form

$$\psi = \begin{pmatrix} \psi^{-} \\ \psi^{+} \end{pmatrix}, \quad (7.9)$$

where  $\psi^{(-)}$  and  $\psi^{(+)}$  are two-component functions.

On the other hand, writing Dirac's matrices (6.18) with the aid of the *Pauli matrices*

\*We note that the terminology that we employ has been adopted from the book by Cartan (1938). Some authors use the term "spinor" to refer to the two-component spinor in three-dimensional space. In such cases, the four-component spinor of the Lorentz group is called a "bispinor."

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_1 \sigma_2 = -\sigma_2 \sigma_1 = i \sigma_3, \dots \quad (7.10)$$

in the “split-up form”\*

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma^\alpha = \begin{pmatrix} 0 & \sigma_\alpha \\ -\sigma_\alpha & 0 \end{pmatrix} \quad (\alpha = 1, 2, 3), \quad \gamma^5 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \quad (7.11)$$

we see that the transformation matrices of the three-dimensional rotations and reflections turn out, in accordance with (6.31) and (6.34), also to be diagonal in the “split-up” representation (7.9):

$$\gamma^1 \gamma^2 = -i \begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix}, \quad \gamma^2 \gamma^3 = -i \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_1 \end{pmatrix}, \quad \gamma^3 \gamma^1 = -i \begin{pmatrix} \sigma_2 & 0 \\ 0 & \sigma_2 \end{pmatrix}, \quad \gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (7.12)$$

from which the independence of the transformations  $\psi^{(+)}$  and  $\psi^{(-)}$  follows directly.

Accordingly, the adjoint spinor  $\bar{\psi}$  also has two linearly independent solutions for a definite sign of the frequency. Since the preceding solutions of Dirac's equation are complex, one may say that they describe positively and negatively charged particles. Since two linearly independent solutions exist, it follows now that these particles may exist in two different states which, as will be shown below, differ in the *sign of the component of spin* along the direction of motion.

**7.2. Decomposition into Spin States and Normalization and Orthogonality Relations.** Denoting the normalized linearly independent solutions for  $k^0 > 0$ , i.e., of the first of equations in (7.7), by  $\psi^{v,+}(\mathbf{k})$ , and those for  $k^0 < 0$ , i.e., of the second of equations in (7.7), by  $\psi^{v,-}(\mathbf{k})$  ( $v = 1, 2$ ), we write the expansions of the functions  $\psi^{(+)}(\mathbf{k})$  and  $\psi^{(-)}(\mathbf{k})$  into spin states in the form

$$\psi_\alpha^\pm(\mathbf{k}) = \sum_{v=1,2} a_v^\pm(\mathbf{k}) v_\alpha^{v,\pm}(\mathbf{k}). \quad (7.13)$$

Similarly, for the adjoint spinor

$$\bar{\psi}^\pm(\mathbf{k}) = \sum_{v=1,2} \bar{a}_v^\pm(\mathbf{k}) \bar{v}^{v,\pm}(\mathbf{k}). \quad (7.14)$$

Since  $\bar{\psi}^{(+)}$  and  $\bar{\psi}^{(-)}$  represent here, as elsewhere throughout this book, the positive- and negative-frequency parts of the function  $\bar{\psi}$ , the conditions of Hermitian conjugation for the normalized spinors  $v$  have the form

$$(v^{v,\pm}(\mathbf{k}))^* = \bar{v}^{v,\mp}(\mathbf{k}). \quad (7.15)$$

\*Here we have also adopted the abbreviated notation for the  $2 \times 2$  matrices:

$$\mathbf{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{0} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Therefore the conditions for the orthonormality of the spinors  $v$  may be written in the form

$$\dot{v}^{\nu,\pm}(k) v^{\mu,\mp}(k) \equiv \sum_{\alpha=1,2,3,4} \dot{v}_{\alpha}^{\nu,\pm}(k) v_{\alpha}^{\mu,\mp}(k) = \delta^{\mu\nu}. \quad (7.16)$$

By means of purely algebraic transformations, one may obtain from (7.15) and (7.16) and the Dirac equations a number of relations for quadratic forms in the spinors  $v$  and  $\bar{v} = v^* \gamma^0$ , the most important of which are the following:

the condition of orthonormality for the adjoint spinors

$$\bar{v}^{\nu,\pm}(k) v^{\mu,\mp}(k) = \dot{v}^{\nu,\pm}(k) \gamma^0 v^{\mu,\mp}(k) = \pm \frac{m}{k^0} \delta^{\mu\nu}; \quad (7.17)$$

the condition of mutual orthogonality for the spinors  $v$  with arguments differing only in their sign

$$v^{\nu,\pm}(k) v^{\mu,\pm}(-k) = 0; \quad (7.18)$$

the relations

$$\left. \begin{aligned} \dot{v}^{\pm}(k) [(k^{\alpha} \gamma^{\beta} - k^{\beta} \gamma^{\alpha}) - m \gamma^{\alpha} \gamma^{\beta}] v^{\pm}(-k) &= 0, \\ \sum_{\alpha=1,2,3} k^{\alpha} [\dot{v}^{\nu,\pm}(k) (\gamma^{\alpha} \gamma^{\beta} - \gamma^{\beta} \gamma^{\alpha}) v^{\mu,\mp}(k)] &= 0 \end{aligned} \right\} \quad (\alpha, \beta = 1, 2, 3) \quad (7.19a)$$

$$(7.19b)$$

and, finally, the formulas for summation over the spin index

$$\sum_{\nu=1,2} v_{\alpha}^{\nu,+}(k) \bar{v}_{\beta}^{\nu,-}(k) = \frac{(\hat{k} - m)_{\alpha\beta}}{2k^0}, \quad (7.20)$$

$$\sum_{\nu=1,2} v_{\alpha}^{\nu,-}(k) \bar{v}_{\beta}^{\nu,+}(k) = \frac{(\hat{k} + m)_{\alpha\beta}}{2k^0} \quad (k^0 = +\sqrt{k^2 + m^2}). \quad (7.21)$$

To prove relation (7.17), we begin with the Dirac equations

$$\left. \begin{aligned} (\gamma^0 k^0 - \gamma k + m) v^{\nu,+}(k) &= 0, \\ (\gamma^0 k^0 + \gamma k - m) v^{\mu,-}(-k) &= 0 \end{aligned} \right\} \quad (k^0 = \sqrt{k^2 + m^2}). \quad (7.22)$$

Multiplying the first of these on the left by  $v^{*\mu,-}(k)$ , we obtain

$$k^0 \bar{v}^{\mu,-}(k) v^{\nu,+}(k) - k \bar{v}^{\mu,-}(k) \gamma v^{\nu,+}(k) + m \bar{v}^{\mu,-}(k) v^{\nu,+}(k) = 0.$$

Taking the Hermitian conjugate, and making use of the Hermitian properties of the Dirac matrices (7.15), we obtain

$$k^0 \bar{v}^{\nu,-}(k) v^{\mu,+}(k) + k \bar{v}^{\nu,-}(k) \gamma v^{\mu,+}(k) + m \bar{v}^{\nu,-}(k) v^{\mu,+}(k) = 0,$$

a comparison of which with the original expression leads directly to

$$\bar{v}^{\nu,-}(\mathbf{k}) \gamma^0 v^{\mu,+}(\mathbf{k}) = -\frac{m}{k^0} \delta^{\mu\nu}.$$

The second of relations (7.17) is proved in a similar manner.

To prove the orthogonality properties (7.18), we multiply the first of equations (7.22) on the left by  $\bar{v}^{\mu,+}(-\mathbf{k})$ :

$$k^0 \bar{v}^{\mu,+}(-\mathbf{k}) v^{\nu,+}(\mathbf{k}) - \bar{v}^{\mu,+}(-\mathbf{k}) (\gamma^0 \boldsymbol{\gamma} \mathbf{k} - m \gamma^0) v^{\nu,+}(\mathbf{k}) = 0.$$

Multiplying the second of equations (7.22) on the left by  $\bar{v}^{\nu,-}(\mathbf{k})$ , and taking the Hermitian conjugate we arrive at the relation

$$k^0 \bar{v}^{\mu,+}(-\mathbf{k}) v^{\nu,+}(\mathbf{k}) + \bar{v}^{\mu,+}(-\mathbf{k}) (\gamma^0 \boldsymbol{\gamma} \mathbf{k} - m \gamma^0) v^{\nu,+}(\mathbf{k}) = 0,$$

and comparing this with the preceding relation, we have directly one of the relations in (7.18):

$$\bar{v}^{\mu,+}(-\mathbf{k}) v^{\nu,+}(\mathbf{k}) = 0.$$

The derivation of relations (7.19a) and (7.19b) is accomplished by the same method of multiplying the Dirac equations by products of the  $\gamma^n$  matrices and adjoint spinors and of taking the Hermitian conjugate of the expressions so obtained. In view of the relative awkwardness of such calculations, we omit the derivation of (7.19).

To obtain spin summation formulas (7.20) and (7.21), we consider the usual Green's function for the Dirac equation (7.2), which satisfies the inhomogeneous equation

$$G(\mathbf{k}) (\hat{\mathbf{k}} + m) = (\hat{\mathbf{k}} + m) G(\mathbf{k}) = 1 \quad (7.23)$$

and which consequently has the form

$$G(\mathbf{k}) = (\hat{\mathbf{k}} + m)^{-1} = \frac{\hat{\mathbf{k}} - m}{k^2 - m^2}. \quad (7.24)$$

In the last two equations  $k^0$  is not equal to the root  $\pm (k^2 + m^2)^{1/2}$  which we shall denote here by  $\lambda$ :

$$\lambda(\mathbf{k}) = \sqrt{k^2 + m^2}.$$

On the contrary, the number  $\pm \lambda$  has for  $k^0$  the nature of an eigenvalue of the corresponding homogeneous equation. Therefore  $G(\mathbf{k})$  is not even defined for  $k^0 = \pm \lambda$ . However, for  $k^0 \neq \pm \lambda$ ,  $G(\mathbf{k})$  may be expanded in terms of the complete system of solutions of the homogeneous equation

$$(\hat{\mathbf{k}} + m) v = 0,$$

which, in accordance with (7.22), consists of the four solutions  $v^{\nu,+}(\mathbf{k})$ ,  $v^{\nu,-}(\mathbf{k})$  ( $\nu = 1, 2$ ).

We then obtain the expansion

$$G_{\alpha\beta}(k) = \sum_{\nu=1,2} (v_{\alpha}^{\nu,+}(k) w_{\beta}^{\nu,+}(k) + v_{\alpha}^{\nu,-}(-k) w_{\beta}^{\nu,-}(k)).$$

If we substitute it into (7.23):

$$(\gamma^0 k^0 - \gamma \mathbf{k} + m) \left[ \sum_{\nu=1,2} (v_{\alpha}^{\nu,+}(k) w_{\beta}^{\nu,+}(k) + v_{\alpha}^{\nu,-}(-k) w_{\beta}^{\nu,-}(k)) \right] = 1$$

and use the field equations

$$(\gamma \mathbf{k} - m) v^{\nu,+}(k) = \gamma^0 \lambda v^{\nu,+}(k); \quad (\gamma \mathbf{k} - m) v^{\nu,-}(-k) = -\gamma^0 \lambda v^{\nu,-}(-k),$$

we obtain:

$$(k^0 - \lambda) \gamma^0 \sum_{\nu} v^{\nu,+}(k) w^{\nu,+}(k) + (k^0 + \lambda) \gamma^0 \sum_{\nu} v^{\nu,-}(-k) w^{\nu,-}(k) = 1. \quad (7.25)$$

Multiplying (7.25) on the left by  $\bar{v}^{\mu,+}(-k)$ , and taking into account the properties of normalization (7.16) and of orthogonality (7.18), we obtain an expression for the coefficient  $\omega^{\nu,+}$ :

$$w^{\nu,-}(k) = \frac{\bar{v}^{\nu,+}(-k)}{k^0 + \lambda(k)}.$$

Determining in a similar way the second coefficient  $\omega^{\nu,+}$ , we arrive at the following expression for  $G$ :

$$G(k) = \sum_{\nu} \frac{(k^0 + \lambda) v^{\nu,+}(k) \bar{v}^{\nu,-}(k) + (k^0 - \lambda) v^{\nu,-}(-k) \bar{v}^{\nu,+}(-k)}{(k^0)^2 - \lambda^2}. \quad (7.26)$$

Recalling then that  $\lambda^2 = \mathbf{k}^2 + m^2$ , and comparing (7.26) with (7.24) in the limits as  $k^0 \rightarrow +\lambda$  and  $k^0 \rightarrow -\lambda$ , we obtain formulas (7.20) and (7.21), respectively.

We shall need (7.17)–(7.19) for the calculation of the dynamic variables of the spinor field (this section), while formulas (7.20) and (7.21) will be useful for the quantization of the spinor field (Chapter 2) and for the calculation of the squares of matrix elements in the theory of interacting fields (Chapter 4).

**7.3. Lagrangian Formalism and Invariants.** We now turn to the Lagrangian formalism. Dirac equations (6.20) and (6.21) may be obtained by means of the variational principle from the following Lagrangian:

$$\mathcal{L} = \frac{i}{2} \left( \bar{\psi}(x) \gamma^n \frac{\partial \psi}{\partial x^n} - \frac{\partial \bar{\psi}}{\partial x^n} \gamma^n \psi(x) \right) - m \bar{\psi}(x) \psi(x). \quad (7.27)$$

In contrast to the Lagrangian of fields considered earlier, the spinor-field Lagrangian (7.27) reduces to zero if the functions  $\psi$  and  $\bar{\psi}$  occurring in it satisfy the field equations.

In the usual way, we obtain from (7.27) the energy momentum tensor

$$T^{kl} = \frac{i}{2} \left( \bar{\psi}(x) \gamma^k \frac{\partial \psi}{\partial x_l} - \frac{\partial \bar{\psi}}{\partial x_l} \gamma^k \psi(x) \right) \quad (7.28)$$

and the current four-vector

$$J^k(x) = \psi(x) \gamma^k \psi(x). \quad (7.29)$$

To calculate the spin tensor, we note that after carrying out the summation over the spin indices, the formula (2.15) for the spinor field may be written in the form

$$S^{k,lm} = - \frac{\partial \mathcal{L}}{\partial (\partial \psi / \partial x^k)} A^{\psi,lm} \psi(x) - \bar{\psi}(x) A^{\bar{\psi},lm} \frac{\partial \mathcal{L}}{\partial (\partial \bar{\psi} / \partial x^k)}.$$

The coefficients  $A^\psi$  and  $A^{\bar{\psi}}$  occurring above may be defined with the aid of formulas (6.27), (6.28), and of the rules for obtaining an adjoint spinor, in the following form:

$$A^{\bar{\psi},lm} = \frac{i}{2} \sigma^{lm}, \quad A^{\psi,lm} = -\frac{i}{2} \sigma^{lm},$$

where  $\sigma$  is the so-called "matrix spin tensor" (6.7):

$$\sigma^{lm} = i \frac{\gamma^l \gamma^m - \gamma^m \gamma^l}{2}. \quad (7.30)$$

With the aid of the formulas given above, we obtain from the Lagrangian (7.27) the following expression for the spin angular momentum tensor of the spinor field:

$$S^{ml,k} = \frac{1}{4} \bar{\psi}(x) \{ \gamma^k \sigma^{lm} + \sigma^{lm} \gamma^k \} \psi(x). \quad (7.31)$$

In order to carry out the integration over three-dimensional space and to obtain the dynamic variables, it is convenient to go over in the usual manner to the three-dimensional momentum representation

$$\psi^\pm(x) = \frac{1}{(2\pi)^{3/2}} \int dke^{\pm i k x} \sum_{\mathbf{v}=1,2} a_{\mathbf{v}}^\pm(\mathbf{k}) v^{\mathbf{v},\pm}(\mathbf{k}), \quad (7.32)$$

$$\bar{\psi}^\pm(x) = \frac{1}{(2\pi)^{3/2}} \int dke^{\pm i k x} \sum_{\mathbf{v}=1,2} \dot{a}_{\mathbf{v}}^\pm(\mathbf{k}) \bar{v}^{\mathbf{v},\pm}(\mathbf{k}). \quad (7.33)$$

Substituting (7.32), (7.33) into (7.28), setting  $k = 0$  in that expression, and integrating over three-dimensional space after taking into account the conditions of orthonormality (7.16), we obtain the energy-momentum four-vector

$$P^1 = \int d\mathbf{x} \cdot T^{01} = \int d\mathbf{k} \cdot k^1 \sum_{\mathbf{v}=1,2} (\dot{a}_{\mathbf{v}}^+(k) a_{\mathbf{v}}^-(k) - \dot{a}_{\mathbf{v}}^-(k) a_{\mathbf{v}}^+(k)), \quad (7.34)$$

where, as is always the case in this book,

$$k^0 = \sqrt{k^2 + m^2}.$$

Taking into account the fact that, in accordance with (7.32) and (7.33), the rules of Hermitian conjugation of the amplitudes  $a^\pm$  and  $a^{*\pm}$  have the form

$$(a^+)^* = \dot{a}^- \quad \text{and} \quad (a^-)^* = \dot{a}^+, \quad (7.35)$$

we see that, in the classical theory, the energy of the spinor field

$$P^0 = \int d\mathbf{k} \cdot k^0 \sum_{\mathbf{v}} (\dot{a}_{\mathbf{v}}^+(k) a_{\mathbf{v}}^-(k) - \dot{a}_{\mathbf{v}}^-(k) a_{\mathbf{v}}^+(k)) \quad (7.36)$$

is *not positive definite*. The positive definite nature of the energy of the spinor field is achieved in the quantum theory by Fermi-Dirac quantization.

Proceeding to the calculation of the spin vector, we note that, in accordance with (6.19), the components of the spin tensor  $\sigma^{lm}$  may be expressed in terms of the  $\alpha_i$  and  $\sigma_i$  matrices as follows:

$$\sigma^{lm} = \begin{pmatrix} 0 & i\alpha_1 & i\alpha_2 & i\alpha_3 \\ -i\alpha_1 & 0 & \sigma_3 & -\sigma_2 \\ -i\alpha_2 & -\sigma_3 & 0 & \sigma_1 \\ -i\alpha_3 & \sigma_2 & -\sigma_1 & 0 \end{pmatrix}. \quad (7.37)$$

Therefore, substituting  $k = 0$  and  $l, m = 1, 2, 3$  in (7.31), we see that the spatial density of the spin vector is given by the matrix "vector"  $\sigma$ :

$$S = \frac{1}{2} \int \dot{\psi}(x) \sigma \psi(x) d\mathbf{x}. \quad (7.38)$$

In contrast to the spins of the vector and the electromagnetic fields, the spin vector of the spinor field (7.38) is not conserved in time (which is related to the lack of symmetry in the energy-momentum tensor). However, when the field functions  $\psi, \bar{\psi}$  do not depend on some of the coordinates  $x^0, x', \dots$ , it is possible to ensure that the "equation of continuity" holds for some of the components of the tensor  $S$  and, consequently, the corresponding integrals remain conserved in time. Thus, substituting  $\partial/\partial x^1 = \partial/\partial x^2 = 0$ , we obtain

$$\frac{\partial S^{12,k}}{\partial x^k} = 0,$$



from which it follows that the component of the spin vector along the  $x^3$  axis

$$S_3 = \int d\mathbf{x} \cdot \mathbf{S}^{12,0}$$

is conserved in time. In the momentum representation, this statement is equivalent to "the conservation of the component of the spin vector along the direction of motion."

By passing in (7.38) to the three-dimensional momentum representation, and carrying out the integration over the three-dimensional configuration space, we obtain

$$\mathcal{S} \sim \frac{1}{2} \{ \dot{\psi}^+(k) \sigma \psi^-(k) + \dot{\psi}^-(k) \sigma \psi^+(k) + e^{2ih^0 x^0} \dot{\psi}^+(k) \sigma \psi^+(-k) + e^{2ih^0 x^0} \dot{\psi}^-(k) \sigma \psi^-(k) \}. \quad (7.39)$$

Restricting ourselves to the consideration of only the  $S_3$  component, we make use of relation (7.19) which we write in the form

$$\dot{v}^\pm(k) \sigma v^\pm(-k) = \frac{i}{m} \dot{v}^\pm(k) [\mathbf{k} \times \boldsymbol{\gamma}] v^\pm(-k).$$

This relation ensures that the time-dependent terms in  $S_3$  disappear in the coordinate system in which  $k_1 = k_2 = 0$ , and we obtain an expression in the following form:

$$\frac{1}{2} \sum_{\nu, \mu} \{ \dot{a}_\nu^+(k) a_\mu^-(k) \dot{v}^{\nu,+}(k) \sigma_3 v^{\mu,-}(k) + \dot{a}_\nu^-(k) a_\mu^+(k) \dot{v}^{\nu,-}(k) \sigma_3 v^{\mu,+}(k) \}, \quad (7.40)$$

for the further specification of which it is convenient to choose some special concrete representation of the Dirac matrices. In the representation given by (6.18), which we have used, the matrix  $\sigma_3$  has the form

$$\sigma_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Choosing the normalized spinors in the system  $k_1 = k_2 = 0$  in the form

$$\left. \begin{aligned} v^{1,-} &= N^{-1} \begin{pmatrix} 1, & 0, & \frac{k^3}{k^0+m}, & 0 \end{pmatrix}, \\ v^{2,-} &= N^{-1} \begin{pmatrix} 0, & 1, & 0, & -\frac{k^3}{k^0+m} \end{pmatrix}, \\ v^{1,+} &= N^{-1} \begin{pmatrix} \frac{k^3}{k^0+m}, & 0, & 1, & 0 \end{pmatrix}, \\ v^{2,+} &= N^{-1} \begin{pmatrix} 0, & -\frac{k^3}{k^0+m}, & 0, & 1 \end{pmatrix}, \end{aligned} \right\} \quad (7.41)$$

where the normalization constant is given by

$$N = \sqrt{1 + \left(\frac{k^3}{k^0 + m}\right)^2} = \sqrt{\frac{2k^0}{k^0 + m}},$$

we find that (7.40) assumes the form

$$\frac{1}{2} (\bar{a}_1^* a_1^- - \bar{a}_2^* a_2^- + \bar{a}_1^- a_1^+ - \bar{a}_2^- a_2^+). \quad (7.42)$$

Comparing expressions for energy-momentum (7.34), for the component of the spin vector (7.42), and for the charge

$$Q = \int \bar{\psi}^*(x) \psi(x) dx = \int dk \sum_v (\bar{a}_v^*(k) a_v^-(k) + \bar{a}_v^-(k) a_v^+(k)), \quad (7.43)$$

which follows directly from (7.29), we find\* that the spinor field describes charged particles with the values of the spin component along any given axis equal to  $\pm\frac{1}{2}$ . A more detailed classification of the possible values of energy-momentum, charge, and spin component will be given after quantization (§13), where it will receive a complete and unambiguous foundation.

**7.4. Spinor Field with Zero Mass.** The spinor field with zero mass, which corresponds to the neutrino, is of particular interest.

Substituting  $m = 0$  in (6.20), we obtain

$$i\gamma^n \frac{\partial \psi(x)}{\partial x^n} = 0. \quad (7.44)$$

This equation splits into two independent two-component equations. To show this, we use the fact that the operator in the Dirac equation anticommutes with the matrix  $\gamma^5$  in the absence of the mass term.

Introducing the projection operators

$$P_{\pm} = \frac{1 \pm \gamma^5}{2},$$

we obtain

$$P_{\pm} i\gamma^n \frac{\partial \psi}{\partial x^n} = i\gamma^n P_{\mp} \frac{\partial \psi}{\partial x^n} = 0,$$

i.e., two separate equations for  $\psi_{\pm} = P_{\pm} \psi$ :

\*Using the fact that quantization of the spinor field (§13) conserves the structure of (7.34), (7.42), and (7.43), changing only the signs of some terms.

$$i\gamma^n \frac{\partial \psi_{\pm}(x)}{\partial x^n} = 0. \quad (7.45)$$

In the representation given by (6.18),

$$\psi_{\pm} = \frac{1}{2} \begin{pmatrix} \psi_1 \mp \psi_3 \\ \psi_2 \mp \psi_4 \\ \mp \psi_1 + \psi_3 \\ \mp \psi_2 + \psi_4 \end{pmatrix}.$$

Therefore, each of the functions  $\psi_+$ ,  $\psi_-$  contains only two independent components and, in the ‘split’ form,

$$\psi_+ = \begin{pmatrix} \varphi_1 \\ -\varphi_1 \end{pmatrix}, \quad \psi_- = \begin{pmatrix} \varphi_2 \\ \varphi_2 \end{pmatrix} \quad (7.46)$$

can be expressed in terms of the two-component spinors

$$\varphi_1 = \frac{1}{2} \begin{pmatrix} \psi_1 - \psi_3 \\ \psi_2 - \psi_4 \end{pmatrix}, \quad \varphi_2 = \frac{1}{2} \begin{pmatrix} \psi_1 + \psi_3 \\ \psi_2 + \psi_4 \end{pmatrix}.$$

Using the “split” for of (7.11) for the Dirac matrices, we have instead of (7.45)

$$\left( \frac{\partial}{\partial x^0} + \sigma \frac{\partial}{\partial \mathbf{x}} \right) \varphi_1(x) = 0, \quad \left( \frac{\partial}{\partial x^0} - \sigma \frac{\partial}{\partial \mathbf{x}} \right) \varphi_2(x) = 0. \quad (7.47)$$

This type of equation was first put forward by Weyl in 1929. To elucidate the physical meaning of the two component functions  $\varphi_1$ ,  $\varphi_2$ , we transform to the momentum representation:

$$\varphi_{\alpha}(x) = \varphi_{\alpha}^{+}(x) + \varphi_{\alpha}^{-}(x), \quad \varphi_{\alpha}^{\pm}(x) = \frac{1}{(2\pi)^{3/2}} \int e^{\pm i p x} \phi_{\alpha}^{\pm}(p) dp. \quad (7.48)$$

The Weyl equations have the form

$$(p^0 - \sigma \mathbf{p}) \phi_1^{\pm}(p) = 0, \quad (7.49)$$

$$(p^0 + \sigma \mathbf{p}) \phi_2^{\pm}(p) = 0. \quad (7.50)$$

It was shown in §7.3 that the matrix “vector”  $\sigma$  describes the spin of a fermion. Since for the massless neutrino  $p^0 = |\mathbf{p}|$ , it is clear from (7.49) and (7.50) that the spin of the neutrino can either be parallel or antiparallel to its motion.

Helicity is defined as being equal to twice the projection of the fermion spin onto its direction of motion, i.e.,  $\sigma \mathbf{p}/|\mathbf{p}|$ . It has been established experimentally that the neutrino

spin is antiparallel to its momentum, i.e., the helicity of the neutrino is equal to  $-1$ . This is valid both for the electron and the muon neutrino. It will be shown later (§13.3) that the function  $\psi_-$  describes particles with negative helicity and  $\psi_+$  those with positive helicity.

The operator for the neutrino will therefore be defined as follows:

$$\mathbf{v}(x) = P_- \psi(x) = \psi_-(x).$$

It satisfies the Dirac equation

$$i\gamma_n \frac{\partial \mathbf{v}(x)}{\partial x_n} = 0 \quad (7.51)$$

and the additional condition

$$P_+ \mathbf{v}(x) = \frac{1 + \gamma^5}{2} \mathbf{v}(x) = 0. \quad (7.52)$$

The corresponding conjugate function

$$\bar{\mathbf{v}}(x) = \mathbf{v}^*(x) \gamma^0$$

satisfies the conjugate equation

$$i \frac{\partial \bar{\mathbf{v}}(x)}{\partial x^n} \gamma^n = 0 \quad (7.53)$$

and the additional condition

$$\bar{\mathbf{v}}(x) P_- = \bar{\mathbf{v}}(x) \frac{1 - \gamma^5}{2} = 0. \quad (7.54)$$

The Lagrangian for the neutrino field is written in the form

$$\mathcal{L}(x) = \frac{i}{2} (\bar{\mathbf{v}}(x) \gamma^n \mathbf{v}_{;n}(x) - \bar{\mathbf{v}}_{;n}(x) \gamma^n \mathbf{v}(x)). \quad (7.55)$$

The dynamic characteristics of the neutrino field will be discussed after quantization (§13.3).

## §8. Lagrangian of a System of Fields

In this section, without essentially raising the questions of the interaction of various fields, we shall consider briefly the general structure of the Lagrangians for systems of fields, and we shall obtain certain dynamic variables of such systems.

8.1. *The Interaction Lagrangian and Invariants of a System of Fields.* In considering problems connected with the interaction between different fields, one starts with the Lagrangian

$$\mathcal{L}_{\text{tot}} = \mathcal{L}_{\text{free}} + \mathcal{L}_{\text{int}}, \quad (8.1)$$

which is the sum of the appropriate Lagrangians of the free fields  $\mathcal{L}_{\text{free}}$  and the term  $\mathcal{L}_{\text{int}}$  which describes the interaction and which is called the *interaction Lagrangian*.

The interaction Lagrangian  $\mathcal{L}_{\text{int}}$  must satisfy the basic physical requirement—it must be relativistically invariant. It must therefore have the form of an arbitrary *invariant* algebraic or integral combination of the field functions of the interacting fields. Leaving aside the integral Lagrangians that lead to nonlocal theories which we are not going to consider, we conclude that the interaction Lagrangian  $\mathcal{L}_{\text{int}}$  may be formed from the field functions by contracting the product of two quantities of the same tensor dimensionality, e.g., two scalars, two pseudoscalars, two vectors, and so on.

Thus the interaction of a scalar field  $\varphi$  with a spinor field  $\psi$  may be chosen to be of the scalar-scalar type

$$\bar{\psi}(x) \psi(x) \varphi(x), \quad (8.2)$$

of the vector-vector type

$$\bar{\psi}(x) \gamma^\mu \psi(x) \varphi_{;\mu}(x), \quad (8.3)$$

etc. Similarly, the simplest forms of interaction of a pseudoscalar field  $\varphi(x)$  with a spinor field  $\psi(x)$  are the following:

$$\bar{\psi}(x) \gamma^5 \psi(x) \varphi(x), \quad (8.4)$$

$$\bar{\psi}(x) \gamma^5 \gamma^\mu \psi(x) \varphi'_{;\mu}(x). \quad (8.5)$$

Apart from relativistic invariance, which is an essential requirement in the theory, we can impose other restrictions on the Lagrangian as well.

For example, electric charge conservation forces the Lagrangian into a particular structure that admits simultaneous phase transformations of the complex-conjugate field functions describing charged particles:

$$\varphi \rightarrow e^{i\alpha} \varphi, \quad \bar{\varphi} \rightarrow e^{-i\alpha} \bar{\varphi}. \quad (8.6)$$

Conservation of baryon charge leads to further restrictions on the possible structure of the interaction Lagrangians. Such Lagrangians must be invariant under the special phase transformation

$$B \rightarrow e^{i\beta} B, \quad \bar{B} \rightarrow e^{-i\beta} \bar{B}, \quad (8.7)$$

where  $B$  represents all fields describing the baryons (particles with unit baryon charge) and  $\bar{B}$  all fields describing antibaryons. The laws of conservation of electric and baryon (and also lepton) charge are absolute laws, i.e., they are satisfied for all the interactions known at present.

The second group of conservation laws contains laws that are satisfied only in certain interactions. Thus, strong interactions are isotopically invariant and this empirical fact is reflected in the structure of the Lagrangian for the strong interactions, which must be invariant under rotations in the fictitious three-dimensional isospin space.

For example, the isospin invariant Lagrangian for the pion-nucleon interaction is

$$\mathcal{L}_{\text{int}}(x) = ig \bar{\Psi}(x) \gamma^5 \tau \Psi(x) \pi(x), \quad (8.8)$$

i.e., it has the structure of a pseudoscalar multiplied by a pseudoscalar under Lorentz transformations and a three-vector multiplied by a three-vector under isospin rotations. Isospin symmetry is not an absolute property and is not satisfied, for example, in electromagnetic interactions. Correspondingly, the interaction Lagrangians for nucleons and pions interacting with the electromagnetic field do not exhibit isospin symmetry.

Another example of symmetry breaking is the conservation of strangeness (hypercharge) which is valid for strong and electromagnetic interactions, but is not for weak interactions.

**8.2. Local Phase Transformations and Gauge Fields.** Transformations of field functions which depend not on constant parameters but on parameters that are functions of co-ordinates (parameter functions) play an important role in the theory of interacting fields.

The gauge transformation of the electromagnetic potential (5.5) which depends on the parameter function  $f(x)$  is an example of this type of transformation. Interaction between fields describing charged particles and the electromagnetic field can be introduced on the basis of a local generalization of the phase transformation (8.6).

The change in the gauge given by (8.6) signifies a change in the phase factor, i.e., a change that does not lead to any physical consequences. Invariance under the gauge transformation [Pauli (1941)] corresponds to the fact that only expressions that are bilinear in  $\varphi$  and  $\varphi^*$  are connected with physically measurable variables. We shall now demand that the theory should be invariant even when the transformation parameter in (8.6) depends on  $x$ :

$$\varphi(x) \rightarrow \varphi'(x) = e^{ie f(x)} \varphi(x), \quad \varphi^*(x) \rightarrow e^{-ie f(x)} \varphi^*(x), \quad (8.9)$$

i.e., the relative phase factor of the field function  $\varphi(x)$  at two different space-time points can be quite arbitrary. The *local phase transformation* (8.9) thus corresponds to the assumption that the arbitrariness in the choice of the phase factor can be given a purely local character.

It is readily verified that the Lagrangians for the above complex fields (3.32) and (7.27) are not invariant under the local phase transformations (8.9) because the corresponding gauges, apart from the phase factor, acquire an additive further term:

$$\partial_n \varphi(x) \rightarrow e^{ie f(x)} \left( \partial_n + ie \frac{\partial f(x)}{\partial x^n} \right) \varphi(x). \quad (8.10)$$

The invariance of the Lagrangian can be restored by introducing an additional vector field  $A_n$  which is transformed simultaneously with (8.9) in such a way that its transformation compensates the change in the Lagrangian under the influence of (8.9). This can be achieved by replacing all the derivatives  $\partial_n \varphi$  and  $\partial_m^* \varphi$  in the Lagrangian by the operators

$$D_n \varphi = (\partial_n - ie A_n(x)) \varphi(x); \quad \bar{D}_m^* \varphi = (\partial_m + ie A_m(x)) \bar{\varphi}(x), \quad (8.11)$$

subject to the condition that the transformation law for the field  $A_n$  is

$$A_n(x) \rightarrow A'_n(x) = A_n(x) + f_n(x). \quad (8.12)$$

The expressions  $D_n \varphi$ ,  $\bar{D}_m^* \varphi$  are called *covariant derivatives*. Under the simultaneous application of the phase transformation (8.9) of the complex field  $\varphi$  and the “compensating” gauge transformation (8.12) of the axillary vector field  $A_n$  these covariant derivatives transform similarly to (8.9), i.e., they change phase:

$$\left. \begin{aligned} D_n(A') \varphi'(x) &= e^{ief(x)} D_n(A) \varphi(x), \\ \bar{D}_n^*(A') \bar{\varphi}'(x) &= e^{-ief(x)} \bar{D}_n^*(A) \bar{\varphi}(x). \end{aligned} \right\} \quad (8.13)$$

The constant  $e$  in (8.11) and (8.12) can be identified with the electric charge, and the vector field  $A_n$  with the electromagnetic field considered in §5. The electromagnetic field introduced through the covariant derivatives  $D$ ,  $\bar{D}^*$ , is thus found to appear as a *compensating* field. Fields that compensate a change in the gauge of fields due to matter are also called *gauge fields*. Because of the Abelian (commutation) character of the group consisting of the transformations given by (8.12), the electromagnetic field is an Abelian gauge field.

Transition in the free Lagrangian from ordinary derivatives to covariant derivatives, i.e.,

$$\mathcal{L}_0(\varphi, \bar{\varphi}, \partial_n \varphi, \partial_m^* \varphi) \rightarrow \mathcal{L}_0(\varphi, \bar{\varphi}, D_n \varphi, \bar{D}_m^* \varphi) \quad (8.14)$$

leads to the appearance of terms representing the interaction between the original field  $\varphi$  and the electromagnetic field  $A$ .

The interaction introduced with the aid of the covariant derivatives (or, as is sometimes said, by “entending” the derivatives) is also called the *minimal electromagnetic interaction*.

In the case of spinor fields, when  $\mathcal{L}_0$  is linear in the derivatives, the minimal electromagnetic interaction is linear in the field  $A$ :

$$\mathcal{L}_{\text{int}}(x) = e \bar{\psi}(x) \gamma^\mu \psi(x) A_\mu(x). \quad (8.15)$$

To obtain the complete Lagrangian of the system (electromagnetic field plus the charged complex fields) we must add the Lagrangian for the free electromagnetic field to (8.14). This Lagrangian is invariant under the gauge transformation (8.12). For example, the total

Lagrangian of spinor electrodynamics (i.e., electron-positron and electromagnetic field systems) has the form

$$\mathcal{L} = \mathcal{L}_0(\psi, \bar{\psi})|_{(7.27)} + \mathcal{L}_0(A)|_{(5.13)} + \mathcal{L}_{\text{int}}(\psi, \bar{\psi}, A)|_{(8.15)}. \quad (8.16)$$

This expression is invariant under the transformations

$$\psi'(x) = \psi(x) e^{ieJ(x)}, \quad \bar{\psi}'(x) = \bar{\psi}(x) e^{-ieJ(x)}, \quad A'_n(x) = A_n(x) + \partial_n f(x). \quad (8.17)$$

For fields with integer spin, the minimal interaction in the usual formulation given by (3.32) and (4.6) contains quadratic terms. Thus, the Lagrangian for the complex scalar field (3.32) corresponds to

$$\mathcal{L}_{\text{int}}(x) = ie [\psi^*(x) \varphi_{;n}(x) - \varphi^*_{;n}(x) \varphi(x)] A^n(x) + e^2 \psi^*(x) \varphi(x) A^n(x) A_n(x). \quad (8.18)$$

However, when the Duffin-Kemmer formalism is used, the Lagrangian (4.35) turns out to be linear in the derivatives. Accordingly,

$$\mathcal{L}_{\text{int}}(x) = e\bar{u}(x) \Gamma^n u(x) A_n(x) = ie (\psi^* \varphi_{;n} - \varphi^*_{;n} \psi) A^n(x). \quad (8.19)$$

We have thus demonstrated that the electromagnetic field ensures the invariance of the total Lagrangian under local phase transformations that are generalizations of the “global” phase transformations (8.6) which lead to the conservation of electric charge. This point of view establishes a relationship between the properties of fields in space-time and the so-called “internal symmetries.” Isospin invariance is an important example of internal symmetry that directly generalizes charge symmetry.

The requirement of invariance under local isospin transformations leads to the necessity for a new vector field, namely, the Yang-Mills field (1954).

**8.3. The Yang-Mills Field.** Consider a field function  $\varphi$  that transforms under some representation of the gauge group

$$\varphi'_A = (\Lambda(\epsilon))_{AB} \varphi_B. \quad (8.20)$$

In the case of isospin transformations, this is the group  $SU(2)$  and the field  $\varphi_A$  corresponding to isospin  $I$  has  $2I + 1$  components, whereas  $\Lambda_{AB}$  is a  $(2I + 1) \times (2I + 1)$  matrix that depends on the three isospin rotation parameters  $\epsilon^\gamma$  ( $\gamma = 1, 2, 3$ ). Since the generators of rotations do not commute, the isospin group  $SU(2)$  is a non-Abelian group. Further information on compact Lie groups can be found in the book by Zhelobenko (1970) and in the review by Behrends et al. (1962).

The infinitesimal form of the transformation given by (8.20) is

$$\varphi'_A = \varphi_A + ig T_{AB}^\gamma \epsilon^\gamma \varphi_B. \quad (8.21)$$

where  $T_{AB}^\gamma$  are the generators of the representation of the gauge group transforming the field  $\varphi$ , and the common factor  $g$  has been taken out of the parameters  $\epsilon^\gamma$ . To ensure the



invariance of the Lagrangian for the field  $\varphi$  under the transformation (8.21) with the parameters  $\epsilon^\gamma(x)$  depending on the coordinates, we can proceed by analogy with electrodynamics and replace all the derivatives  $\partial_n \varphi$  in  $\mathcal{L}_0(\varphi)$  by the covariant derivatives

$$(D_n \varphi)_A = \partial_n \varphi_A - ig T_{AB}^\gamma B_n^\gamma \varphi, \quad (8.22)$$

which contain the new compensating vector field  $B$ . This field is the analog of the electromagnetic field  $A$ . However, whereas the field  $A$  is a scalar with respect to the gauge group, and does not therefore carry an electric charge, the field  $B$  has group indices and therefore transforms in a more complicated fashion:

$$B_n^{\alpha'} = B_n^\alpha + ig t^{\alpha\beta\gamma} \epsilon^\gamma(x) B_n^\beta + \partial_n \epsilon^\alpha(x). \quad (8.23)$$

Here the matrices  $t^{\alpha\beta\gamma}$  form the basis for the *associated* representation of the Lie algebra of the gauge group. For the  $SU(2)$  group, which we are considering,

$$t^{\alpha\beta\gamma} = \epsilon^{\alpha\beta\gamma},$$

where  $\epsilon$  is an absolute antisymmetric tensor.

If we use the fact that

$$[T^\alpha, T^\beta] = t^{\alpha\beta\gamma} T^\gamma, \quad (8.24)$$

we can readily verify that the covariant derivative (8.22) transforms under (8.21) and (8.23) similarly to (8.21):

$$[D_n(B') \varphi']_A = (D_n(B) \varphi)_A + ig T_{AB}^\gamma \epsilon^\gamma (D_n(B) \varphi)_B. \quad (8.25)$$

The analog of the electromagnetic field tensor (5.1) is now the tensor

$$F_{mn}^\alpha = \partial_m B_n^\alpha - \partial_n B_m^\alpha - ig t^{\alpha\beta\gamma} B_m^\beta B_n^\gamma. \quad (8.26)$$

In contrast to  $H_{mn}$  which is invariant under (8.12), the tensor  $F_{mn}^\alpha$  transforms in accordance with the associated representation of the gauge group under (8.23):

$$F_{mn}^{\alpha'} = F_{mn}^\alpha + ig t^{\alpha\beta\gamma} F_{mn}^\beta \epsilon^\gamma(x), \quad (8.27)$$

so that the invariant Lagrangian for the Yang-Mills field that does not interact with other fields is

$$\mathcal{L}_{YM} = -\frac{1}{4} F_{mn}^\alpha F^{mn, \alpha}. \quad (8.28)$$

This expression contains quadratic, cubic, and fourth degree terms in the fields  $B$ . The “free” Yang-Mills field *must therefore contain an interaction*.

Finite transformations corresponding to the infinitesimal transformations (8.23) can be written in a more compact form by introducing the matrix functions

$$(B_m(x))^{\beta\gamma} = B_m^\alpha(x) t^{\alpha\beta\gamma}, \quad (8.29)$$

the values of which belong to the associated representation of the gauge group for each  $x$ . The formula for the finite transformation is

$$B'_m(x) = \Lambda(\varepsilon) B_m(x) \Lambda^{-1}(\varepsilon) + \frac{i}{g} \Lambda(\varepsilon) \partial_m \Lambda^{-1}(\varepsilon). \quad (8.30)$$

**8.4. Dynamic Invariance of a System of Fields.** We now turn to a consideration of the invariants of a system of fields. Because of the relativistic invariance of the interaction Lagrangian, the total Lagrangian of the system is an invariant with respect to four translations and six rotations of the coordinate system. The invariants corresponding to these transformations—energy-momentum and angular-momentum—are determined by formulas which follow from Noether's theorem.

Of particular interest is the case in which  $\mathcal{L}_{\text{int}}$  does not depend on the derivatives of the field functions. In this case, in accordance with (2.9) and (2.16), the momentum and the spin of the system of fields are equal to the sum of the momenta and the sum of the spins of the individual fields, while the energy density of the system of fields is equal to the sum of the energy densities of the individual fields minus  $\mathcal{L}_{\text{int}}$ , i.e.,

$$\mathbf{P} = \sum_i \mathbf{P}_i, \quad P^0 = \sum_i P_i^0 - \int \mathcal{L}_{\text{int}}(x) dx, \quad \mathbf{S} = \sum_i \mathbf{S}_i. \quad (8.31)$$

The energy and momentum of a system of interacting quantum fields will be discussed in greater detail in Chapter II.

We now consider the phase transformation invariants. We take as our first example the system described by the Lagrangian (8.9). This system is invariant under the phase transformation of spinor functions of the form of (8.6):

$$\psi \rightarrow e^{i\alpha} \psi, \quad \bar{\psi} \rightarrow e^{-i\alpha} \bar{\psi}, \quad (8.6')$$

which is also a special case of the gauge transformation (8.9) for

$$ef(x) = \alpha = \text{const.}$$

The corresponding invariant (2.29) is identical with the charge of the free electron-positron field (7.43) by virtue of the fact that  $\mathcal{L}_{\text{int}}$  is independent of  $\partial\psi/\partial x$ .

Here is another example. Consider the system of four fields: proton, neutron, electron-positron, and neutrino fields. We shall describe protons, neutrons, electrons, and neutrinos by means of a spinor field of the type (7.27). Then

$$\begin{aligned} \mathcal{L} &= \mathcal{L}_0 + \mathcal{L}_{\text{int}} \\ \mathcal{L}_0 &= \mathcal{L}_P + \mathcal{L}_N + \mathcal{L}_e + \mathcal{L}_\nu, \end{aligned}$$

$$\begin{aligned}
\mathcal{L}_P &= \frac{i}{2} \left( \bar{\Psi}_P(x) \gamma^\alpha \frac{\partial \Psi_P}{\partial x^\alpha} - \frac{\partial \bar{\Psi}_P}{\partial x^\alpha} \gamma^\alpha \Psi_P(x) \right) - M \bar{\Psi}_P(x) \Psi_P(x), \\
\mathcal{L}_N &= \frac{i}{2} \left( \bar{\Psi}_N(x) \gamma^m \frac{\partial \Psi_N}{\partial x^m} - \frac{\partial \bar{\Psi}_N}{\partial x^m} \gamma^m \Psi_N(x) \right) - M \bar{\Psi}_N(x) \Psi_N(x), \\
\mathcal{L}_e &= \frac{i}{2} \left( \bar{\psi}(x) \gamma^k \frac{\partial \psi}{\partial x^k} - \frac{\partial \bar{\psi}}{\partial x^k} \gamma^k \psi(x) \right) - m \bar{\psi}(x) \psi(x), \\
\mathcal{L}_\nu &= \frac{i}{2} \left( \bar{\nu}(x) \gamma^l \frac{\partial \nu}{\partial x^l} - \frac{\partial \bar{\nu}}{\partial x^l} \gamma^l \nu(x) \right).
\end{aligned}$$

Here  $M$  is the nucleon mass,  $m$  is the electron mass,  $\Psi_P$  is the proton wave function,  $\Psi_N$  is the neutron wave function,  $\psi$  is the electron-positron function and  $\nu$  is the neutrino function.

The interaction Lagrangian of these four fields, which describes  $\beta$ -processes, is usually taken in the following form:

$$\mathcal{L}_{\text{int}} = G [(\bar{\Psi}_P O \Psi_N)(\bar{\psi} O \nu) + (\bar{\Psi}_N O \Psi_P)(\bar{\nu} O \psi)], \quad (8.32)$$

where  $O$  are the Dirac matrices which determine the form of the interaction.

The total Lagrangian  $\mathcal{L}_0$  is invariant with respect to several independent gauge transformations of the first kind; for example:

$$1) \quad \left. \begin{aligned} \psi(x) &\rightarrow e^{-i\chi} \psi(x), & \bar{\psi}(x) &\rightarrow e^{i\chi} \bar{\psi}(x), \\ \Psi_P(x) &\rightarrow e^{i\chi} \Psi_P(x), & \bar{\Psi}_P(x) &\rightarrow e^{-i\chi} \bar{\Psi}_P(x), \\ \Psi_N(x) &\rightarrow \Psi_N(x), & \bar{\Psi}_N(x) &\rightarrow \bar{\Psi}_N(x), \\ \nu(x) &\rightarrow \nu(x), & \bar{\nu}(x) &\rightarrow \bar{\nu}(x). \end{aligned} \right\} \quad (8.33)$$

The corresponding invariant is equal to the difference between the charges of the  $\Psi_P$  and  $\psi$  fields and represents electric charge.

$$2) \quad \left. \begin{aligned} \Psi_P &\rightarrow e^{i\beta} \Psi_P, & \bar{\Psi}_P &\rightarrow e^{-i\beta} \bar{\Psi}_P, \\ \Psi_N &\rightarrow e^{i\beta} \Psi_N, & \bar{\Psi}_N &\rightarrow e^{-i\beta} \bar{\Psi}_N, \\ \psi &\rightarrow \psi, & \bar{\psi} &\rightarrow \bar{\psi}, \\ \nu &\rightarrow \nu, & \bar{\nu} &\rightarrow \bar{\nu}. \end{aligned} \right\} \quad (8.34)$$

The corresponding invariant is equal to the sum of the charges of the  $\Psi_P$  and  $\Psi_N$  fields, and is the *baryon charge*.

There exists a simple correspondence between the form of the interaction Lagrangian and the type of elementary process described by this Lagrangian, which may be put on a rigorous basis by means of the theory of second quantization (Chapter 4).

It is shown in §9 that the field functions  $u$ ,  $u^\dagger$  decompose into creation and annihilation operators. The function  $u(x)$  decomposes into an annihilation operator for the basic particle

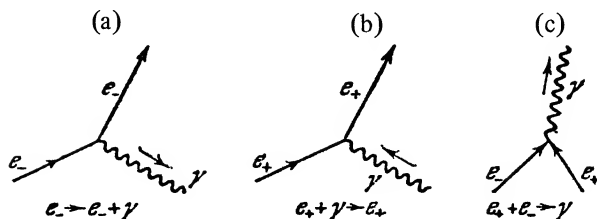


Fig. 1. (a) Emission of photon  $\gamma$  by electron  $e_-$ . (b) Absorption of photon  $\gamma$  by positron  $e_+$ . (c) Annihilation of an electron-positron pair with emission of a photon.

and a creation operator for the antiparticle;  $u^\dagger(x)$  decomposes into an annihilation operator for the antiparticle and the creation operator for the basic particle. Therefore a combination of functions of the type

$$u_1(x) u_2^\dagger(x) \dots u_k(x)$$

describes a process in which an antiparticle of the field  $u_1$  is created (or a particle of the field  $u_1$  is absorbed), a particle of the field  $u_2$  is created (or an antiparticle of the field  $u_2$  is absorbed), and so on.

For example, the interaction Lagrangian for the electron-positron and the electromagnetic fields, given by (8.15), describes elementary processes of the type shown in Fig. 1.

It may readily be seen with the aid of the above correspondence rules that the invariance of the above interaction Lagrangian under the transformation (8.6') guarantees the conservation of electric charge in the corresponding processes.



# Chapter 2

## QUANTUM THEORY OF FREE FIELDS

### §9. General Principles of Quantization of Wave Fields

*9.1. Operator Nature of Field Functions and State Amplitude.* So far, we have confined our attention to classical fields. We must now consider the quantization of wave fields. In the quantum theory of fields, the field functions describe sets of particles, the *mutual transformations* of which are contained explicitly in the theory. Accordingly, the wave functions for quantized fields are in fact *operators* and can be decomposed into particle *creation and annihilation operators* which obey certain commutation relations. The operator wave functions are determined by the field equations and the commutation relations to within a unitary transformation.

The field functions are therefore no longer functions in the classical sense, but are operators acting on the second-quantization wave function  $\Phi$  that is common to all the fields and is called the *state amplitude* (or the state vector).

As in ordinary quantum mechanics where the state of a system is specified by a wave function  $\psi$ , in the quantum field theory the physical state of a system is completely characterized by the state amplitude  $\Phi$ . The state amplitude  $\Phi$  is similar to  $\psi$  in that it may be looked upon as a vector in a certain linear space (i.e., the space in which the scalar product has been defined).

Since in the quantum field case the state of a system is characterized by quantities such as the four-momentum, the spin, . . . , and the number and type of particles, it is convenient from the mathematical point of view to consider generalized Hilbert spaces that include “improper” elements.

As in quantum mechanics, not all the vectors  $\Phi$  have a finite norm (for example, those corresponding to plane waves). However, the norms of physical states can always be chosen

so that they are real (these states are described by wave packets). Such state vectors are conveniently normalized to unity:

$$\Phi^* \Phi = 1. \quad (9.1)$$

In quantum field theory, expectation values and probabilities of states are given by quadratic forms of the state amplitude. We shall now consider the properties of the state amplitudes  $\Phi$  under the transformations of coordinates and field functions discussed in §2.

Since the specification of these properties is the basic quantization postulate for the wave fields, and this postulate can be formulated in a natural way in the spirit of correspondence with ordinary quantum mechanics, let us briefly review some of the basic ideas of quantum mechanics.

*9.2. Representations of the Schroedinger Equation.* The Schroedinger representation is the most widely used in quantum mechanics. In this representation, the behavior of a dynamic system in time is described with the aid of a time-dependent wave function  $\psi(t)$  satisfying the Schroedinger equation

$$i \frac{\partial \psi(t)}{\partial t} = H \psi(t), \quad t = x^0. \quad (9.2)$$

where  $H$  is the Hamiltonian operator corresponding to the total energy of the system, which is time independent for closed (conservative) systems. The dynamic variables of closed systems are characterized in the Schroedinger representation by operators that do not explicitly depend on time. Their expectation values

$$B_t = \psi^*(t) B \psi(t) \quad (9.3)$$

may, however, depend on time through the wave functions  $\psi(t)$ . Assuming that the operator  $H$  remains constant in time, we can formally integrate (9.2) and write

$$\psi(t) = U(t) \psi, \quad (9.4)$$

where

$$U(t) = \exp(-iHt) \quad (9.5)$$

and

$$\psi = \psi(0) = \text{const}$$

is independent of time.

Substituting this in (9.3), we obtain

$$B_t = \overset{*}{\psi} \overset{\dagger}{U}(t) B U(t) \psi = \overset{*}{\psi} e^{iHt} B e^{-iHt} \psi. \quad (9.6)$$

This formula may be interpreted as the expectation value of the time-dependent operator

$$B_H(t) = \overset{\dagger}{U}(t) B U(t) = e^{iHt} B_H(0) e^{-iHt}, \quad (9.7)$$

evaluated over the functions  $\psi$  that do not depend on time. We thus arrive at the Heisenberg representation in which the dynamic variables and not the wave functions depend explicitly on time. The two representations are completely equivalent from the point of view of calculating observable values of dynamic variables. The connection between the representations is obtained with the aid of the unitary operator (9.5), so that\*

$$\overset{\dagger}{U}(t) U(t) = 1. \quad (9.8)$$

Differentiating (9.7) with respect to time, we obtain the equations of motion in the Heisenberg representation:

$$i \frac{\partial B_H(t)}{\partial t} = [B_H(t), H], \quad (9.10)$$

where  $[a, b] = ab - ba$  are the quantum-mechanical Poisson brackets. We note that the formulas given by (9.4) and (9.7), which describe the evolution of the system between time 0 and time  $t$ , are in a way mutually *complementary*. To transform to time  $t$ , one must either transform the wave function  $\psi$  by means of (9.4), or transform the operator by means of (9.7).

**9.3. Transformation Properties of the State Amplitude and Field Operators.** Let us now consider the transformation properties of the state amplitudes  $\Phi$  under the transformations of coordinates and field functions considered in §2:

$$x \rightarrow x' = L(\omega) x, \quad u(x) \rightarrow u'(x') = \Lambda(\omega) u(x). \quad (9.11)$$

where  $\omega$  are the independent transformation parameters.

The transformation defined by (9.11) corresponds to a certain transformation of the state vector, which must be linear by virtue of the superposition principle:

$$\Phi \rightarrow \Phi' = U(\omega) \Phi. \quad (9.12)$$

To ensure that the norm of the state amplitude is invariant, the transformation operator  $U$ , which depends on the transformation parameters  $\omega$ , must satisfy the condition

$$\overset{\dagger}{U}(\omega) U(\omega) = 1. \quad (9.13)$$

\*In general, unitary operator  $V$  is defined by

$$(V\Phi_1)^* (V\Phi_2) = \Phi_1^* \Phi_2. \quad (9.9)$$



Consider the meaning of the unitary transformation (9.12). The formulas given by (9.12) and (9.13) correspond to (9.4) and (9.8). The transformation of the state vector defined by (9.12) is an *alternative* to the transformation of the operator wave function given by (9.11).

To evaluate the expectation value of a dynamic quantity in the new set of coordinates, we must either consider the expectation value of the transformed operator  $B'$  over the original state vectors  $\Phi$ , or the expectation value of the original operator  $B$  over the transformed state vectors  $\Phi'$ , i.e.,

$$B' = \hat{\Phi} B' \Phi = \hat{\Phi}' B \Phi'. \quad (9.14)$$

By  $B$  we understand both the operator field functions themselves and dynamic variables such as energy-momentum, charge, and so on, expressed in terms of bilinear combinations of them.

In the special case where  $B$  is the operator field function in the  $x$  representation, we have from (9.14)

$$\hat{\Phi} \varphi'(x) \Phi = \hat{\Phi}' \varphi(x) \Phi'$$

or, using (9.12),

$$u'(x) = \hat{U}(\omega) u(x) U(\omega) = U^{-1}(\omega) u(x) U(\omega). \quad (9.15)$$

The requirement of compatibility applied to (9.11) and (9.15) leads to important conditions which the operators must satisfy. They will be considered below.

Consider the special case of a transformation from the inhomogeneous Lorentz group

$$x \rightarrow x' = Lx = x + \delta x, \quad \delta x^k = a^k + \omega^{kn} x_n, \quad (9.16)$$

where  $a^k$  and  $\omega^{kn}$  are infinitesimal translation and rotation parameters.

The operator for the transformation of the state amplitude

$$\Phi \rightarrow \Phi' = U_L \Phi$$

will be written in the form

$$U_L = U(a, \omega) = 1 + \delta U_L. \quad (9.17)$$

The infinitesimal quantity  $\delta U_L$  must be anti-Hermitian by virtue of (9.13), and also linear in  $a$  and  $\omega$ . We therefore write

$$\delta U_L = i P_n a^n + \frac{i}{2} M_{nm} \omega^{nm}, \quad (9.18)$$

where  $P$  and  $M$  are Hermitian operators.

The transformation of the state amplitude

$$\Phi' = \left(1 + iP a + \frac{i}{2} M \omega\right) \Phi$$

has the same form as the infinitesimal transformation of a scalar field function

$$u'(x) = \left(1 + i p a + \frac{i}{2} m \omega\right) u(x),$$

which corresponds to (9.16), where the coefficients  $P$ ,  $M$  have the meaning of the usual quantum-mechanical operators of four-momentum and the angular momentum tensor

$$p_n = i \frac{\partial}{\partial x^n}, \quad m_{kn} = i \left( x_k \frac{\partial}{\partial x^n} - x_n \frac{\partial}{\partial x^k} \right). \quad (9.19)$$

This reflects the fact that the four-momentum and the angular momentum are generators of translations and rotations.

Using the correspondence principle, we shall therefore interpret  $P$  and  $M$  in (9.18) as operators corresponding to the energy-momentum four-vector and the angular momentum tensor, respectively. Of course, in quantum field theory, where the amplitude  $\Phi$  does not explicitly depend on the coordinates  $x$ , we cannot use formulas such as (9.19) and must take  $P$  and  $M$  to be certain operators acting on the state amplitude. By analogy, under the gauge transformations of field functions

$$u \rightarrow u' = e^{i\alpha} u, \quad u^* \rightarrow u'^* = e^{-i\alpha} u^*$$

the unitary transformation operator

$$\Phi' = U_\alpha \Phi$$

has the expansion

$$U_\alpha = 1 + i\alpha Q + \dots, \quad (9.20)$$

in which the Hermitian operator  $Q$  has to be interpreted as the charge operator.

**9.4. Quantization Postulate for Wave Fields.** We shall assume, as the basic postulate of quantization of wave fields, that the operators for the energy-momentum four-vector  $P$ , the angular momentum tensor  $M$ , the charge  $Q$ , and so on, which are the generators of infinitesimal transformations of state vectors (see (9.18) and (9.20)), can be expressed in terms of the operator functions of the fields by the same relations as in classical field theory, i.e., (2.10), (2.16), and (2.29), with the operators being, of course, arranged in

the appropriate order. This postulate is a further application of the correspondence principle and determines the transformation law for the second-quantization state amplitudes. We note that the operators  $U_L$  corresponding to the transformations  $L$  in the Lorentz group realize the representation of the group and, therefore, have the group property

$$U_{L_1 L_2} = U_{L_1} U_{L_2}. \quad (9.21)$$

If we use this property and the form of the infinitesimal transformation (9.18), we can obtain an expression for  $U_L$  in the case of finite transformations. To do this, we must write the group differential equation for  $U_L$  and then integrate it (the steps are similar to those in §6.4). In this way, we obtain, for example, the following result for finite space-time translations:

$$x \rightarrow x' = Lx, \quad Lx = x + a, \quad U_L = \exp(iPa). \quad (9.22)$$

Let us now consider the compatibility condition for the transformation properties of the operator wave function. From (9.11) and (9.15) we obtain

$$\Lambda_\omega u(L_\omega^{-1}x) = \dot{U}(\omega) u(x) U(\omega). \quad (9.23)$$

This is the most general operator condition of compatibility of the transformation properties of the operator field functions considered in §2, and of the transformation properties of the state amplitudes, which follow from the quantization postulate. They impose a number of important conditions on the field operators, which are conveniently considered in differential form (i.e., for infinitesimal transformations). We shall do this now.

We begin with the infinitesimal shift transformation

$$x' = x + \delta x, \quad \delta x^k = a^k.$$

If we use (9.18) and consider the first-order terms in  $a^k$  in (9.23), we obtain

$$-\frac{\partial u}{\partial x^k} a^k = -iP_k a^k u(x) + iu(x) P_k a^k,$$

i.e.,

$$i \frac{\partial u(x)}{\partial x^k} = [u(x), P_k^*]. \quad (9.24)$$

For infinitesimal rotations

$$x \rightarrow x' = x + \delta x, \quad \delta x^k = X_{(mn)}^k \delta \omega^{mn}, \quad \Lambda_\omega = \delta_{ij} + A_{i(kl)}^j \delta \omega^{kl}$$

we have, similarly,

$$i \left\{ \sum_j A'_{i(mn)} u_j - u_{i;k} (x_n \delta_m^k - x_m \delta_n^k) \right\} = [M_{mn}, u_i]. \quad (9.25)$$

Finally, for infinitesimal gauge transformations of the first kind, we obtain

$$u(x) = [u(x), Q], \quad -\dot{u}^*(x) = [\dot{u}^*(x), Q]. \quad (9.26)$$

It is also possible to obtain integral analogs of (9.24)–(9.26). Consider the simplest case of finite translations. Rewriting (9.23) in the form

$$u(L^{-1}x) = \hat{U}_L u(x) U_L,$$

and substituting (9.22) into it, we obtain

$$u(x-a) = e^{-iPa} u(x) e^{iPa}.$$

Hence, for  $a = x$ , we have

$$u(x) = e^{iPx} u(0) e^{-iPx}.$$

This is, clearly, a generalization of (9.7). It gives the dependence on the space-time coordinates in an explicit form. It follows from this formula that the expression

$$u_S(x) = e^{-iPx} u(0) e^{iPx} = u(x) |_{x^0=0} \quad (9.27)$$

can be looked upon as the field operator in the Schroedinger representation. Hence,  $u(x)$  which is related to (9.27) by

$$u(x) = e^{iHt} u_S(x) e^{-iHt}, \quad Ht = P_0 x^0, \quad (9.28)$$

is the field operator in the Heisenberg representation. The equations given by (9.24) are the equations of motion in the Heisenberg representation.

We have thus obtained equations (9.24)–(9.26), which are the operator conditions for the quantized functions of wave fields and do not depend on the particular form of the commutation relations. Of course, instead of the above quantization postulate we could have postulated the equations right at the outset. This procedure is sometimes employed in presentations of the theory on the basis of the canonical formalism (see §11.4). This is, however, less general than the above presentation. Equations (9.24)–(9.26) enable us to establish a number of properties for the operators  $u$  that is common to all fields.

*9.5. The Physical Meaning of the Positive- and Negative-Frequency Components and of Adjoint Functions.* From equations (9.24) we obtain directly the physical meaning of

the positive-frequency and the negative-frequency parts of the operator field function [Schwinger (1949a)]. Let us consider the field of particles of mass  $m$ , described by the function  $u(x)$ :

$$\begin{aligned} u(x) &= u^+(x) + u^-(x), \\ u^+(x) &= \int_{k^0 > 0} e^{ikx} \delta(k^2 - m^2) u(k) dk, \\ u^-(x) &= \int_{k^0 > 0} e^{-ikx} \delta(k^2 - m^2) u(-k) dk. \end{aligned}$$

Substituting separately  $u^{(+)}$  and  $u^{(-)}$  into condition (9.24) we find:

$$\left. \begin{aligned} [u^+(x), P^n] &= - \int_{k^0 > 0} e^{ikx} k^n \delta(k^2 - m^2) u(k) dk, \\ [u^-(x), P^n] &= \int_{k^0 > 0} e^{-ikx} k^n \delta(k^2 - m^2) u(-k) dk, \end{aligned} \right\} \quad (9.29)$$

from which it follows that:

$$[u^+(k), P^n] = -k^n u^+(k), \quad (9.30)$$

$$[u^-(k), P^n] = k^n u^-(k). \quad (9.31)$$

Here we have introduced the notation

$$u^+(k) = u(k)|_{k^0 > 0}, \quad u^-(k) = u(-k)|_{k^0 > 0}.$$

We now introduce a state with a definite value  $p^n$  of the energy-momentum four-vector described by the amplitude  $\Phi_p$

$$P^n \Phi_p = p^n \Phi_p. \quad (9.32)$$

Multiplying (9.30) on the right by  $\Phi_p$ , and taking into account (9.32), we obtain

$$P^n u^+(k) \Phi_p = (p^n + k^n) u^+(k) \Phi_p.$$

By the same method we find that from (9.31)

$$P^n u^-(k) \Phi_p = (p^n - k^n) u^-(k) \Phi_p,$$

where  $k^0 = (k^2 + m^2)^{1/2} > 0$  in both cases.

From the equations obtained above, it follows that the expression  $u^{(+)}(k) \Phi_p$  represents the amplitude of the state with the energy momentum  $k + p$ , while  $u^{(-)}(k) \Phi_p$  corresponds to the state with the energy momentum  $p - k$ . Since, at the same time, the relation  $k^2 = m^2$

$$\begin{aligned} k^2 &= m^2 \\ p^2 &= m^2 \\ (p-k)^2 &= m^2 \end{aligned}$$

holds, one may consider that *the operator  $u^{(+)}(k)$  describes the creation of a particle of mass  $m$  and four-momentum  $k$ , while the operator  $u^{(-)}(k)$  corresponds to the annihilation of such a particle.*

We emphasize that this property of the positive- and negative-frequency parts of the field operators is quite general and valid for fields of arbitrary tensor dimensionality, both real and complex, and does not depend on the specific form of the commutation relations.

Proceeding to the investigation of equation (9.25) we note that it explicitly contains the transformation matrix  $A$  of the components of the field functions which is determined by the tensor dimensionality of the field and gives a finite-dimensional representation of the Lorentz group. A detailed analysis of equation (9.25) leads to the decomposition of the operator field function into states with definite values of the total angular momentum, which corresponds to an expansion in spherical harmonics.

Let us also investigate equations (9.26). Introducing the amplitude  $\Phi_q$  for a state with given charge  $q$

$$Q\Phi_q = q\Phi_q,$$

and using (9.26), we obtain the two equations

$$\left. \begin{aligned} Qu\Phi_q &= (q-1)u\Phi_q, \\ Q^\dagger u\Phi_q &= (q+1)u\Phi_q, \end{aligned} \right\} \quad (9.33)$$

from which it follows that the operator  $u^\dagger$  increases the charge by 1 while  $u$  reduces it by the same amount. The positive- and negative-frequency parts of the operators  $u^\dagger$  and  $u$  have, individually, the same properties.

**9.6. Vacuum State and State Amplitude in the Fock Representation.** On the basis of the preceding investigation, we may in a natural way define the vacuum state and establish rules for constructing amplitudes which correspond to states with a definite number of various particles.

We shall consider a dynamic system consisting of several noninteracting quantized fields characterized by the operator functions

$$u_1(x), \dots, u_n(x).$$

For convenience of notation, we shall also include in this sequence the corresponding adjoint functions in those cases in which they differ from  $u_i$ .

Let us define the *vacuum state*  $\Phi^0$  for the given dynamic system. Since there are no particles in the vacuum, the energy and the momentum of the vacuum are equal to zero. Since the negative-frequency operators  $u^{(-)}$  diminish both the energy and the momentum, and the energy cannot be negative, the following relations must hold for all  $x$ :

$$u_1^-(x)\Phi_0 = \dots = u_n^-(x)\Phi_0 = 0.$$

Going over to the momentum representation, we obtain the corresponding relations

$$u_1^-(k_1) \Phi_0 = \dots = u_n^-(k_n) \Phi_0 = 0 \quad (9.34)$$

for all  $k_i$  such that  $k_i^2 = m_i^2$ , where  $m_i$  is the mass of the particle of the  $i$ th kind.

Relations (9.34) and their adjoint relations

$$\Phi_0^* u_1^\dagger(k_1) = \dots = \Phi_0^* u_n^\dagger(k_n) = 0 \quad (9.35)$$

together with the normalization condition

$$\Phi_0^* \Phi_0 = 1 \quad (9.36)$$

may be considered to be *the definition of the vacuum state of free fields*.

The amplitude of any arbitrary state of the dynamic system under consideration may now be represented with the aid of the vacuum amplitude introduced above and the creation operators for the appropriate particles. Thus the amplitude of the state which contains exactly  $s$  particles of the  $j_1, \dots, j_s$  kinds respectively will be given by an expression of the form

$$\Phi = \int F_s(k_1, \dots, k_s) \delta(k_1^2 - m_{j_1}^2) \dots \delta(k_s^2 - m_{j_s}^2) u_{j_1}^\dagger(k_1) \dots u_{j_s}^\dagger(k_s) dk_1 \dots dk_s \Phi_0.$$

Here  $F_s$  is a weighting function which characterizes the particle distribution with respect to the continuous state parameters—the energies and the momenta, while the indices  $j_1, \dots, j_s$  correspond to the discrete state characteristics (for example, the charge of the particles and the values of the spin components along a given direction).

The general amplitude for an arbitrary state will be given by a superposition of such expressions

$$\Phi = \sum_{\substack{(l_1, \dots, l_s) \\ s \geq 0}} \int F_s(k_1, \dots, k_s) \delta(k_1^2 - m_{j_1}^2) \dots \delta(k_s^2 - m_{j_s}^2) u_{j_1}^\dagger(k_1) \dots u_{j_s}^\dagger(k_s) dk_1 \dots dk_s \Phi_0. \quad (9.37)$$

Expression (9.37) may also be written in the configuration representation. In order to do this we shall first carry out the integration over all  $k_i^0$ :

$$\Phi = \sum_{\substack{(l_1, \dots, l_s) \\ s \geq 0}} \int F_s(\mathbf{k}_1, \dots, \mathbf{k}_s) u_{j_1}^\dagger(\mathbf{k}_1) \dots u_{j_s}^\dagger(\mathbf{k}_s) d\mathbf{k}_1 \dots d\mathbf{k}_s \Phi_0, \quad (9.38)$$

where, as usual,

$$u^\dagger(\mathbf{k}) = \frac{\theta(k^0) u(\mathbf{k})}{\sqrt{2k^0}} \quad (k^0 = \sqrt{\mathbf{k}^2 + m^2}),$$

and also

$$F_s(k_1, \dots, k_s) = \frac{F_s(k_1, \dots, k_s)}{\prod_{1 \leq i \leq s} \sqrt{2k_i^0}} \quad (k_i^0 = \sqrt{k_i^2 + m_i^2}).$$

Going over to the configuration representation with the aid of the relations

$$F_s(k_1, \dots, k_s) = \frac{1}{(2\pi)^{3s/2}} \int e^{i \sum_j (k_j x_j)} \varphi_s(x_1, \dots, x_s) dx_1 \dots dx_s,$$

$$u^+(k) = \frac{1}{(2\pi)^{3/2}} \int e^{ikx} u^+(0, x) dx, \quad u^+(0, x) = u^+(x)|_{x^0=0},$$

we obtain for the state amplitude the following expression in place of (9.38):

$$\Phi = \sum_{(l_1, \dots, l_s) \geq 0} \int \varphi_s(x_1, \dots, x_s) u_{l_1}^+(0, x_1) \dots u_{l_s}^+(0, x_s) dx_1 \dots dx_s \cdot \Phi_0. \quad (9.39)$$

The functions  $\varphi_s(x_1, \dots, x_s)$  occurring in the above expression are to be interpreted as ordinary wave functions for a system of  $s$  particles in configuration space. If we had a state in which exactly  $s$  particles were present, the state amplitude would have been completely characterized by one such function. In the general case, when the number of particles is not fixed, the state amplitude is characterized by a sequence of functions  $\psi_s$ . In fact, we then obtain the *Fock representation* for the state amplitude.

It should be emphasized that the dependence of  $\Phi$  on the time has dropped out in the above, which is quite natural, since in the representation chosen by us, the state amplitude in the absence of interaction turns out to be constant.

## §10. Setting Up the Commutation Relations

*10.1. Types of Commutation Relations.* We now proceed to set up the commutation relations for the operator wave functions.

In the classical theory of free fields with a quadratic Lagrangian, the canonical formalism is introduced by expressing the field functions linearly in terms of mutually conjugate generalized coordinates and momenta. The classical Poisson brackets for the field functions

$$\{u_\alpha(x), u_\beta(y)\}$$

turn out to be certain functions of  $x$  and  $y$  independent of  $u$ . Therefore, starting with the correspondence principle\*, it is customary to assume in the quantum theory of a free field

\*See §21 in the book by Dirac (1958).



that the commutation rule for the operator field functions has one of the two following forms:

$$\{u_\alpha(x), u_\beta(y)\}_- \equiv u_\alpha(x) u_\beta(y) - u_\beta(y) u_\alpha(x) = \Delta_{\alpha\beta}, \quad (10.1a)$$

or\*

$$\{u_\alpha(x), u_\beta(y)\}_+ \equiv u_\alpha(x) u_\beta(y) + u_\beta(y) u_\alpha(x) = \Delta_{\alpha\beta}, \quad (10.1b)$$

i.e., it is assumed that either the commutator or the anticommutator of two field operators is a  $c$ -number  $\Delta_{\alpha\beta}$ .

Relations of the first type are called Bose-Einstein commutation relations, while relations of the second type are called Fermi-Dirac commutation relations.

The quanta of fields satisfying (10.1a) obey the Bose-Einstein statistics. The corresponding particles are called bosons. The quanta of fields satisfying (10.1b) obey the Fermi-Dirac statistics and the corresponding particles are called fermions.

The precise form of the commutation functions  $\Delta$  for any particular field is determined by (9.24)-(9.26) and the structure of the energy operator for the given field. However, whatever the particular form of the commutation relations, it can be shown that the commutation function depends only on the difference  $x - y$ , i.e.,

$$\{u_\alpha(x), u_\beta(y)\} = \Delta_{\alpha\beta}(x - y). \quad (10.2)$$

To show this, consider the commutation relations given by (10.1) in the momentum representation. Since the formulas for the Fourier transformation are linear, the commutators or anticommutators of the frequency components in the momentum representation,  $u_\alpha^\pm(k)$ , must also be  $c$ -numbers. We shall first show that operators of the same frequency must strictly commute or anticommute:

$$\{u_\alpha^\pm(k), u_\beta^\pm(q)\} = 0. \quad (10.3)$$

As in (10.2), the symbol  $\{\dots\}$  represents either a commutator or an anticommutator.

Consider the state amplitude  $\Phi_p$  with a defined value of the four-momentum, satisfying (9.32). Applying the operators  $u_\alpha^+(k)$  and  $u_\beta^+(k)$  to this function, we obtain the new amplitudes

$$\Phi_1 = u_\beta^+(q) u_\alpha^+(k) \Phi_p, \quad \Phi_2 = u_\alpha^+(k) u_\beta^+(q) \Phi_p,$$

which, in accordance with the foregoing, satisfy the equations

\*The anticommutation property of field operators expressed by the commutation relation (10.1b) has the consequence that the commutators of dynamic variables which are bilinear forms of the field operators are expressed in terms of the commutation functions  $\Delta_{\alpha\beta}$ . The correspondence principle for the second case should be understood in this sense.

$$P^n \Phi_{1,2} = (p^n + k^n + q^n) \Phi_{1,2}.$$

We now add and then subtract these equations and obtain

$$P^n (\Phi_2 \pm \Phi_1) = P^n \{u_\alpha^\pm(k), u_\beta^\pm(q)\} \Phi_p = (p^n + k^n + q^n) \{u_\alpha^\pm(k), u_\beta^\pm(q)\} \Phi_p.$$

If we now suppose that  $\{u_\alpha^\pm, u_\beta^\pm\}$  is a nonzero  $c$ -number, then by cancelling it out we obtain

$$P^n \Phi_p = (p^n + k^n + q^n) \Phi_p,$$

i.e., we are in conflict with the original equation (9.32). We have thus proved (10.3). In precisely the same way we can show that

$$\{u_\alpha^\pm(k), u_\beta^\mp(q)\} = 0 \quad \text{for} \quad k \neq q. \quad (10.4)$$

The relations given by (10.3) and (10.4) have a simple physical interpretation. They correspond to the fact that events involving the *creation* of particles of any field do not interfere with one another, and this also applies to particle *annihilation* events. Events involving the creation and annihilation of particles with different momenta do not interfere either.

To prove (10.2), it is now sufficient to consider the commutators (anticommutators) of the different-frequency parts of the operator field functions. Using the explicit form of the Fourier transformation,

$$\{u_\alpha^\pm(x), u_\beta^\mp(y)\} = \frac{1}{(2\pi)^3} \int e^{i(px - qy)} \{u_\alpha^\pm(p), u_\beta^\mp(q)\} dp dq,$$

we verify that if we demand that the commutator (anticommutator) in the momentum representation be proportional to the  $\delta$ -function [see (10.4)]

$$\{u_\alpha^\pm(p), u_\beta^\mp(q)\} \sim \delta(p - q), \quad (10.5)$$

we find that in the coordinate representation, the corresponding quantity is a function of the difference  $x - y$ :

$$\{u_\alpha^\pm(x), u_\beta^\mp(y)\} = \Delta_{\alpha\beta}^\pm(x - y).$$

Consequently, the complete commutator (anticommutator) will be a function of this difference. We have thus proved (10.2). We note that (10.2) expresses the translational invariance of the commutation relations. Naturally, to prove this formula, we have used the properties of the energy-momentum four-vector operator which is the generator of shift transformations.

There is no difficulty in showing that in the case of complex fields, the commutators (anticommutators) of the operators referring to particles with different charge are also always equal to zero, i.e.,

$$\{u^+, u^-\} = \{u^{+\ast}, u^{-\ast}\} = 0. \quad (10.6)$$

and (10.4) also holds. To show this, we must consider the state amplitude  $\Phi_p$  corresponding to a given charge, and proceed by analogy with the derivation of (10.3). We leave this to the reader.

The physical meaning of (10.6) is that events involving the creation and annihilation of particles with different charges are mutually independent.

We have thus established that for any (complex) field, the only nonzero  $c$ -numbers are

$$\{u_{\alpha}^-(p), u_{\alpha}^{+\ast}(p)\} \text{ and } \{u_{\alpha}^-(p), u_{\alpha}^+(p)\},$$

i.e., the (anti)commutators of the creation and annihilation operators referring to particles with the same energy-momentum four-vectors and the same charge. The corresponding expressions in the coordinate representation are translationally invariant:

$$\{u^+(x), u^{+\ast}(y)\} = \Delta^+(x-y), \quad \{u^-(x), u^{-\ast}(y)\} = \Delta^-(x-y),$$

and their sum gives the total commutation function

$$\{u(x), u^{\ast}(y)\} = \Delta(x-y) = \Delta^+(x-y) + \Delta^-(x-y).$$

To determine the normalization in the commutation relations given by (10.5) and the explicit form of the commutation functions, we must turn to the explicit form of the energy-momentum four-vector operator and use (9.24)–(9.26).

*10.2. Fermi-Dirac and Bose-Einstein Commutation Relations.* To establish the specific form of the commutation relations, we now use (9.24) with the energy-momentum operator in the form

$$P^n = \sum_{\alpha} \int dq q^n [a_{\alpha}^{+\ast}(q) a_{\alpha}^-(q) \pm a_{\alpha}^{+\ast}(q) a_{\alpha}^+(q)]. \quad (10.7)$$

In accordance with the classical expression (3.26), (3.39), (4.23), (5.20), and (7.34), we have expressed this operator in terms of the independent amplitudes  $a_{\alpha}^{\pm}$ ,  $a_{\alpha}^{\pm\ast}$  and linear relations of the form

$$u_{\alpha}^{\pm}(k) = v_{\alpha\beta}^{\pm}(k) a_{\beta}^{\pm}(k), \quad (10.8)$$

where the coefficients  $v_{\alpha\beta}^{\pm}$  are  $c$ -numbers.

Since the operators  $a^{\pm}$ ,  $a^{\pm\ast}$  do not commute, their order in (10.7) corresponds to the order of the functions  $u^{\ast}$  and  $u$  in the Lagrangians. We recall that the operators  $a^{\pm}$  and

$\bar{a}^{\pm}$  are related by the conditions for complex (in this case Hermitian) conjugation:

$$(a^{\pm}(\mathbf{k}))^* = \bar{a}^{\mp}(\mathbf{k}). \quad (10.9)$$

The upper sign in front of the second term on the right-hand side of (10.7) refers to fields with integer spin (scalar, vector, and electromagnetic fields), and the lower sign to fields with half-integer spin (spinor fields).

We now note that (10.3), (10.6) and (10.8) lead to the following properties of the commutators of the operators  $a^{\pm}$ ,  $\bar{a}^{\pm}$  and their bilinear forms in (10.7):

$$\begin{aligned} [a^+(\mathbf{k}), \bar{a}^+(q) a^-(q)]_- &= [a^-(\mathbf{k}), \bar{a}^-(q) a^+(q)]_- = 0, \\ [a^+(\mathbf{k}), \bar{a}^-(q) a^+(q)]_- &= \{a^+(\mathbf{k}), \bar{a}^-(q)\} a^+(q), \\ [a^-(\mathbf{k}), \bar{a}^+(q) a^-(q)]_- &= \{a^-(\mathbf{k}), \bar{a}^+(q)\} a^-(q), \end{aligned} \quad (10.10)$$

and there are analogous relationships for the commutators of  $\bar{a}^{\pm}$ , which can be obtained from the above with the aid of the Hermitian conjugation properties. Substituting (10.7) in (9.30) and (9.31), and using (10.10), we obtain

$$\begin{aligned} k^n a_{\alpha}^-(\mathbf{k}) &= \int d\mathbf{q} q^n \{a_{\alpha}^-(\mathbf{k}), \bar{a}_{\beta}^+(q)\} a_{\beta}^-(q), \\ -k^n a_{\alpha}^+(\mathbf{k}) &= \mp \int d\mathbf{q} q^n \{a_{\alpha}^+(\mathbf{k}), \bar{a}_{\beta}^-(q)\} a_{\beta}^+(q), \end{aligned}$$

and hence

$$\{a_{\alpha}^-(\mathbf{k}), \bar{a}_{\beta}^+(q)\} = \delta_{\alpha\beta} \delta(\mathbf{k} - \mathbf{q}), \quad (10.11)$$

$$\{a_{\alpha}^+(\mathbf{k}), \bar{a}_{\beta}^-(q)\} = \mp \delta_{\alpha\beta} \delta(\mathbf{k} - \mathbf{q}). \quad (10.12)$$

We have thus obtained two versions of the commutation relations for each type of field. The requirement that these relations must be symmetric in the sign of the electric charge, or, more precisely, with respect to the replacement of particles by antiparticles, i.e.,

$$a_{\alpha}^+(\mathbf{k}) \leftrightarrow \bar{a}_{\alpha}^+(\mathbf{k}) \quad (10.13)$$

will uniquely define the quantization recipe in each case. The symmetry property (10.13) reflects the fact that the choice between the “main” field function  $u$  and its complex conjugate  $\bar{u}$  is a matter of convention, and the opposite choice, namely,

$$w(x) = \bar{u}(x), \quad \bar{w}(x) = u(x)$$

leads only to the replacement of the “main” particles by their antiparticles. This replacement will affect the charge operator which is not invariant under (10.13), but should have no effect on the equation of motion (9.24) or the expression for the energy-momentum

four-vector. The conditions of symmetry under (10.13) will also ensure the correct transition from the complex to the real field:

$$\overset{*}{u}(x) = u(x), \quad a^\pm(k) = \overset{*}{a}^\pm(k).$$

We note that the transformation (10.13) is called charge conjugation and the corresponding symmetry is called charge symmetry (see §§13 and 14).

If we use symmetry under (10.13), we find that *Bose-Einstein quantization is consistent for fields with integer spin [upper sign on the right-hand side of (10.12)], whereas Fermi-Dirac quantization is consistent for spinor fields (lower sign).*

We also note that if instead of symmetry under (10.13) we demand that the metric in Hilbert's space must be positive, i.e.,

$$\overset{*}{\Phi} \overset{\dagger}{A} A \Phi = \overset{*}{\Phi} |A|^2 \Phi > 0,$$

we can reach these conclusions only with the aid of (10.12). We leave it to the reader to verify this.

We thus find that fields with integer spin are quantized by the Bose-Einstein rules

$$[a_\alpha^-(k), \overset{*}{a}_\beta^+(q)]_- = \delta_{\alpha\beta} \delta(k - q), \quad [\overset{*}{a}_\alpha^-(k), a_\beta^+(q)]_- = \delta_{\alpha\beta} \delta(k - q), \quad (10.14)$$

whereas those with half-integer spin are quantized by the Fermi-Dirac rules

$$[a_\alpha^-(k), \overset{*}{a}_\beta^+(q)]_+ = \delta_{\alpha\beta} \delta(k - q), \quad [\overset{*}{a}_\alpha^-(k), a_\beta^+(q)]_+ = \delta_{\alpha\beta} \delta(k - q). \quad (10.15)$$

We note that the commutation relations (10.14) and (10.15) uniquely define the normalization of the operator field functions.

*10.3. Connection of Spin with Statistics. Pauli's Theorem.* The above results form a special case of the fundamental Pauli theorem which establishes the relationship between the transformation properties of the field and the method of its quantization (relation between spin and statistics):

*Fields describing particles with integer spin are quantized by the Bose-Einstein rules, and those describing particles with half integer-spins are quantized by the Fermi-Dirac rules.*

Pauli's theorem is valid for fields with arbitrary (as high as desired) spin. In proving Pauli's theorem we used the fact that the metric was positive and (or) the symmetry under (10.13). However, there are other possibilities. Violation of the connection between spin and statistics, which is established by Pauli's theorem, leads to a number of other fundamental contradictions.

Thus, for example, quantization of a field with integer spin according to the Fermi-Dirac rules leads to the violation of macroscopic causality. We shall show this for a complex scalar field (Pauli's analysis).

To do this, we evaluate the frequency parts of the commutation function in the coordinate representation in an explicit form.

Transforming to the coordinate representation, we obtain the following expressions on the right-hand sides of (10.14):

$$\frac{1}{(2\pi)^3} \int \frac{d\mathbf{k} d\mathbf{q}}{\sqrt{2k^0} \sqrt{2q^0}} e^{iqy - i\mathbf{k}x} \delta(\mathbf{k} - \mathbf{q}) = \frac{1}{(2\pi)^3} \int \frac{d\mathbf{k}}{2k^0} e^{-i\mathbf{k}(x-y)} = \frac{1}{i} D^-(x-y), \quad (10.16)$$

where we have introduced standard notation for the frequency parts of the Pauli-Jordan commutation function:

$$\frac{1}{(2\pi)^3} \int \frac{d\mathbf{k}}{2k^0} e^{i\mathbf{k}(x-y)} = iD^+(x-y) = \frac{1}{i} D^-(y-x). \quad (10.17)$$

$$D(x) = D^+(x) + D^-(x) = \frac{i}{(2\pi)^3} \int e^{-i\mathbf{k}x} \delta(k^2 - m^2) \varepsilon(k^0) d^4k, \quad (10.18)$$

and

$$\varepsilon(k^0) = \theta(k^0) - \theta(-k^0) = \begin{cases} +1 & \text{for } k^0 > 0, \\ -1 & \text{for } k^0 < 0. \end{cases}$$

After transformation to the coordinate representation, the Bose-Einstein quantization of this complex scalar field yields, according to (10.14),

$$[\varphi^-(x), \varphi^{*+}(y)]_- = \frac{1}{i} D^-(x-y), \quad (10.19)$$

$$[\varphi^+(x), \varphi^{*-}(y)]_- = \frac{1}{i} D^+(x-y). \quad (10.20)$$

The total commutator in the coordinate representation is, therefore, expressed in terms of the Pauli-Jordan function:

$$[\varphi(x), \varphi^{*}(y)]_- = \frac{1}{i} D(x-y). \quad (10.21)$$

On the other hand, in the Fermi-Dirac quantization, we find from (10.11) and from (10.12), using the upper sign, that

$$[\varphi^-(x), \varphi^{*+}(y)]_+ = \frac{1}{i} D^-(x-y), \quad [\varphi^+(x), \varphi^{*-}(y)]_+ = iD^+(x-y).$$

The total anticommutator in this case is

$$[\varphi(x), \varphi^{*}(y)]_+ = i(D^+(x-y) - D^-(x-y)). \quad (10.22)$$

The frequency parts  $D^\pm$  outside the light cone for  $\lambda \equiv (x-y)^2 < 0$  have the following explicit form [see § 16, formulas (16.10) and (16.11)]:

$$D^-(x-y) = -D^+(x-y) = \frac{mi}{4\pi^2 \sqrt{-\lambda}} K_1(m \sqrt{-\lambda}), \quad \lambda < 0.$$

It follows that the Pauli-Jordan function, and together with it the commutator (10.21), will vanish outside the light cone. This important property ensures that events separated by space-like intervals are independent, i.e., it ensures causality in space-time (also called the locality property). At the same time, the anticommutator (10.22) is not zero outside the light cone. Thus, the Fermi-Dirac quantization of the scalar field leads to a conflict with the property of causality.

Accounts of Fermi-Dirac quantization of the spinor field frequently appeal to the negative sign in front of the second term in the classical expression for the energy (7.34), which leads to an uncertainty in the sign in the case of Bose-Einstein quantization. We shall not repeat this well-known analysis, especially since we have essentially used this sign in our main argument.

Finally, we note that we have established the Pauli theorem for the idealized case of free fields. An analogous result has been proved in recent years for interacting fields within the framework of the so-called axiomatic quantum field theory [see, for example, Chapter IV of Streater and Wightman (1964) and also Chapter V of Bogolyubov, Logunov, and Todorov (1969)].

**10.4. Normal Product of Operators and the Form of Dynamic Variables.** We now introduce the concept of an operator written in normal form and the concept of the normal product of operators.

The *normal form of an operator* is said to be the form in which in each term all the creation operators  $u^{(+)}$  (or correspondingly  $a^{(+)}$ ) in the momentum representation are written to the left of all the annihilation operators  $u^{(-)}$  (or, correspondingly,  $a^{(-)}$ ).

It is easy to see that the normal form of operators is the most convenient one from the point of view of carrying out calculations. Indeed, in order to calculate the matrix element  $\Phi^* A \Phi$  of any arbitrary operator  $A$  in its normal form, it is necessary merely to commute all the operators  $u^{(-)}$  occurring in  $A$  with all the  $u^{(+)}$  occurring in the state amplitude  $\Phi$ , and all the  $u^{(+)}$  occurring in  $A$  with all the  $u^{(-)}$  occurring in  $\Phi^*$  until one of the  $u^{(-)}$  operates on  $\Phi_0$ , or one of the  $u^{(+)}$  operates on  $\Phi_0^*$ , which gives zero.

Consider an example. Let us write down in normal form the product of two Bose operators  $\varphi^*(x)$  and  $\varphi(y)$ . We then obtain in turn:

$$\begin{aligned} \Phi^*(x) \varphi(y) &= \Phi^{*+}(x) \varphi^+(y) + \Phi^{*-}(x) \varphi^-(y) + \Phi^{*+}(x) \varphi^+(y) + \Phi^{*-}(x) \varphi^-(y) = \\ &= \Phi^{*+}(x) \varphi^+(y) + \Phi^{*+}(x) \varphi^-(y) + \varphi^+(y) \Phi^{*+}(x) + \Phi^{*-}(x) \varphi^-(y) - i\Delta^-(x-y). \end{aligned}$$

It is evident that in the more general case by reducing to the normal form the product of a certain number of operator wave functions  $u$ , we shall obtain a sum of products of the components  $\varphi^+$ ,  $\varphi^-$  and of commutator  $\Delta^{(-)}$ -functions. The general prescription for such a product is considered by us later (§17) and forms the content of an important theorem due to Wick. The whole expression may be conventionally regarded as a "polynomial" in powers of  $\Delta^{(-)}$ -functions. The zero degree term of this polynomial, i.e., the sum of terms not involving any  $\Delta^{(-)}$ -functions, is called the *normal product* of the original operator

wave functions. The normal product can also be defined as the original product reduced to normal form with all the commutator functions assumed equal to zero in the process of reduction.

The normal product of operators  $u_1, u_2, \dots, u_n$  is represented by the symbol

$$: u_1 u_2 \dots u_n :.$$

As another example, consider the normal product of the two operators  $\psi^{*+}(x), \psi(y)$  obeying the Fermi-Dirac rules. Clearly,

$$: \psi^*(x) \psi(y) : = \psi^*(x) \psi^+(y) + \psi^*(x) \psi^-(y) - \psi^+(y) \psi^*(x) + \psi^-(y) \psi^*(x).$$

We now agree that all dynamic variables that depend quadratically on operators with the same arguments, such as the Lagrangian, the energy-momentum, current, and so on, will be written in the form of normal products. For example, the Lagrangian for the scalar complex field (3.32) will be written in the form

$$\mathcal{L} = : \bar{\varphi} \cdot \varphi : - m^2 : \bar{\varphi} \varphi :.$$

It is readily seen that by definition of the vacuum amplitude  $\Phi_0$ ,

$$\varphi^-(x) \Phi_0 = 0, \quad \bar{\varphi}^-(x) \Phi_0 = 0$$

and the conjugate relation  $\bar{\Phi}_0 \varphi^+(x) = 0, \bar{\Phi}_0 \bar{\varphi}^+(x) = 0$ , the expectation values of all the dynamic variables are zero for the vacuum state:

$$\bar{\Phi}_0 P^n \Phi_0 = 0, \quad \bar{\Phi}_0 Q \Phi_0 = 0, \text{ and so on.}$$

This procedure excludes pseudophysical quantities such as zero-point energy, zero-point charge, and so on, which usually appear in the course of quantization. It is also quite obvious that all the conservation laws established in the classical theory for the variables introduced above remain in force in the present context too, because the corresponding proofs involve the use of algebraic identities that are valid for normal products as well.

We note that the transition to the normal product may violate the property of positiveness.

In order to write down the commutation relations in a relativistically invariant form we shall go over to the four-dimensional operators by means of the relations

$$a^\pm(k) = \frac{\theta(k^0) a(\pm k)}{\sqrt{2k^0}} \quad (k^0 = \sqrt{k^2 + m^2}),$$

from which it also follows that:

$$a^\pm(k) = \int \sqrt{2k^0} \theta(k^0) \delta(k^2 - m^2) a(\pm k) dk^0.$$



Therefore, representing the right-hand terms of (10.14) and (10.15) in the form

$$\delta_{ij} \int dk^0 \int dk'^0 2k^0 \theta(k^0) \delta(k - k') \delta(k^2 - m^2),$$

and equating the integrands, we obtain

$$\begin{aligned} \theta(k^0) \theta(k'^0) \delta(k^2 - m^2) \delta(k'^2 - m^2) \{a_i(-k), a_j(k')\}^* = \\ = \theta(k^0) \theta(k'^0) \delta(k^2 - m^2) \delta(k'^2 - m^2) \{a_i(-k), a_j^*(k')\} = \\ = \delta_{ij} \delta(k - k') \delta(k^2 - m^2) \theta(k^0). \end{aligned}$$

From this it follows that for the Fermi-operators

$$\delta(k^2 - m^2) \delta(k'^2 - m^2) \{a_i(k), a_j^*(k')\}_+ = \delta_{ij} \delta(k + k') \delta(k^2 - m^2) \quad (10.23)$$

and for the Bose-operators

$$\begin{aligned} \delta(k^2 - m^2) \delta(k'^2 - m^2) \{a_i(k), a_j^*(k')\}_- = \\ = -(\theta(k^0) - \theta(k'^0)) \delta_{ij} \delta(k + k') \delta(k^2 - m^2). \end{aligned} \quad (10.24)$$

*10.5. Commutation Relations in the Discrete Momentum Representation.* We shall also give the form of the commutation relations in the discrete momentum representation mentioned in §3. Using the discrete expansion for the field functions of the type (3.28),

$$u(x) = \frac{1}{L^{3/2}} \sum_{(n_1, n_2, n_3)} \left( \frac{u^{(+)}(n)}{\sqrt{2k^0}} e^{\frac{2\pi i}{L}(nx)} + \frac{u^{(-)}(n)}{\sqrt{2k^0}} e^{-\frac{2\pi i}{L}(nx)} \right)$$

and the discrete representation of the energy-momentum four-vector of the type (3.31)

$$P^\mu = \sum_{(\alpha, n)} (a_{\alpha}^{(-)}(n) a_{\alpha}^{(+)}(n) + a_{\alpha}^{(+)}(n) a_{\alpha}^{(-)}(n)) k^\mu,$$

with

$$\begin{aligned} u(n) = u(n_1, n_2, n_3), \quad a(n) = a(n_1, n_2, n_3), \\ k^\alpha = \frac{2\pi}{L} n^\alpha \quad (\alpha = 1, 2, 3) \quad \text{and} \quad k^0 = \frac{2\pi}{L} \sqrt{n_1^2 + n_2^2 + n_3^2 + \frac{L^2}{4\pi^2} m^2}, \end{aligned}$$

we obtain after a calculation similar to the one given above the following discrete commutation relations:

in the case of the spinor field

$$\{a_{\alpha}^{(+)}(n), a_{\beta}^{(-)}(n')\}_+ = \{a_{\alpha}^{(-)}(n), a_{\beta}^{(+)}(n')\}_+ = \delta_{\alpha\beta} \delta_{nn'}; \quad (10.25)$$

in the case of all the other fields

$$\{a_{\alpha}^{(-)}(n), a_{\beta}^{(+)}(n')\}_- = \{a_{\alpha}^{(+)}(n), a_{\beta}^{(-)}(n')\}_- = \delta_{\alpha\beta} \delta_{nn'}. \quad (10.26)$$

Here  $\delta_{nn'}$  is the product of Kronecker symbols

$$\delta_{nn'} = \delta_{n_1 n'_1} \delta_{n_2 n'_2} \delta_{n_3 n'_3}$$

while the discrete operators  $a(n)$  are related to the continuous  $a(k)$  in the limit of large  $L$  by expressions of type (3.30):

$$\lim_{L \rightarrow \infty} \left( \frac{L}{2\pi} \right)^{1/2} a(n) = a(k).$$

In the second quantization theory, quadratic combinations of the discrete creation and annihilation operators of the type

$$a^{(-)*} a^{(+)}, \quad a^{(-)*} a^{(-)}$$

play the role of operators for the number of particles. Let us examine the case of quantization according to Bose-Einstein rules. We shall then obtain from (10.26)

$$a^{(-)*} a^{(+)} - a^{(+)*} a^{(-)} = 1. \quad (10.27)$$

Denoting the operator  $a^{*(-)} a^{(-)}$  by  $n$ :

$$n = a^{(+)*} a^{(-)}, \quad (10.28)$$

we shall show that the eigenvalues  $N$  of the operator  $n$

$$n\Phi = N\Phi \quad (10.29)$$

are integers. To see this we consider the expression

$$(\Phi (a^{(+)*})^k (a^{(-)})^k \Phi). \quad (10.30)$$

After carrying out successive commutations of the operators  $a^{*(-)}$  and  $a^{(-)}$  and taking into account the relation

$$a^{(+)*} n = (n-1) a^{(+)*},$$

which follows from (10.27) and (10.28), we obtain successively:

$$\begin{aligned} a^{(+)*} a^{(+)*} a^{(-)} a^{(-)} &= a^{(+)*} n a^{(-)} = (n-1) a^{(+)*} a^{(-)} = n(n-1), \\ a^{(+)*} a^{(+)*} a^{(+)*} a^{(-)} a^{(-)} a^{(-)} &= a^{(+)*} n(n-1) a^{(-)} = (n-1) a^{(+)*} (n-1) a^{(-)} = n(n-1)(n-2), \\ &\dots \dots \dots \end{aligned}$$

from which it follows that

$$(\Phi (a^{(+)*})^k (a^{(-)})^k \Phi) = (\Phi n(n-1) \dots (n-k+1) \Phi) = N(N-1) \dots (N-k+1) (\Phi \Phi). \quad (10.31)$$

Since the operator

$$(a^{(+)*})^k (a^{(-)})^k$$

represents the product of an operator and its conjugate, the matrix element (10.30) cannot be negative. On the other hand, for a nonintegral  $N$  a value of  $k$  may be found

$$k > N + 1,$$

for which (10.31) will become less than zero, which is impossible. Therefore  $N$  is an integer.

As may be easily seen, in the discrete representation the four-momentum and the charge are expressed by sums of the type

$$\sum_{\nu} \omega_{\nu} n_{\nu},$$

where  $\omega_{\nu}$  is the four-momentum or the charge of a single particle in the state  $\nu$ . It is therefore natural to consider  $n_{\nu}$  as an operator which represents the number of particles in the given state.

We now turn to the Fermi-Dirac quantization rule:

$$a^{(-)*} a^{(+)} + a^{(+)*} a^{(-)} = 1. \quad (10.32)$$

Denoting

$$a^{(+)*} a^{(-)} = n,$$

we shall investigate the eigenvalues  $N$  of the operator  $n$ . In order to do this we examine the expression

$$\Phi n^2 \Phi = N^2 \Phi \Phi.$$

Commuting the operators  $a^{(-)}$  and  $a^{(+)*}$  and taking into account the relations

$$a^{(-)} n = (1 - n) a^{(-)}, \quad a^{(-)} a^{(-)} + a^{(-)} a^{(-)} = 0,$$

we obtain:

$$n^2 = a^{(+)*} a^{(-)} a^{(+)*} a^{(-)} = a^{(+)*} a^{(-)} n = a^{(+)*} (1 - n) a^{(-)} = n - a^{(+)*} a^{(+)*} a^{(-)} a^{(-)} = n,$$

i.e.,

$$\Phi n^2 \Phi = \Phi n \Phi,$$

from which it follows that

$$N^2 = N,$$

and consequently  $N$  may equal either 1, or 0, i.e.,

$$N=0, 1. \quad (10.33)$$

Using the expressions for the energy-momentum and the charge operators, we further find without difficulty that the numbers  $N_{\nu}$  which correspond to the operators

$$n_{\nu} = a_{\nu}^{(+)*} a_{\nu}^{(-)}, \quad n_{\nu}^* = a_{\nu}^{(+)} a_{\nu}^{(-)*},$$

are indeed equal to the number of particles with the corresponding momenta and charge, and the operators  $n_{\nu}$  and  $n_{\nu}^*$  are therefore the operators for the number of particles.

We therefore see that quantization in accordance with Bose-Einstein rules leads to quite a different physical picture from that obtained from quantization in accordance with Fermi-Dirac rules. When quantization is carried out according to (10.27), the occupation numbers  $N$  may take on arbitrarily large integral values. In the case of Bose-Einstein statistics, an arbitrarily large number of particles may exist in the same state (characterized by a given four-momentum, charge and spin). On the other hand, when quantization is carried out in accordance with (10.32), the occupation numbers may only take on the two values 0 and 1 in accordance with (10.33). Relation (10.33) is an expression of the Pauli exclusion principle: in a system of particles obeying Fermi-Dirac statistics, not more than one particle may exist in any given state.

We have examined briefly the usual discrete representation of the field functions which was used primarily in the early literature of quantum field theory. Its virtues are the simplicity and the directness of the method of introducing the occupation numbers and the possibility of expressing the state amplitude as a function of these numbers.

However, since it is noncovariant, we shall not use it in our presentation, but we shall work with the continuous representation.

It is not difficult to see that in this case the operator expressions

$$a^{(+)*}(\mathbf{k}) a^{(-)}(\mathbf{k}), \quad a^{(+)}(\mathbf{k}) a^{(-)*}(\mathbf{k})$$

play the role of the density of the number of particles in three-dimensional momentum space.

## §11. Scalar and Vector Fields

In this section, we shall examine the simplest boson fields, real and complex (pseudo) scalar fields, the pion field, and the field of charged vector mesons.

*11.1. Real and Complex Scalar Fields.* In accordance with the formulas of §§10.2 and 10.3, the commutators of the real scalar field have the form

$$[\varphi^-(\mathbf{k}), \varphi^+(\mathbf{q})]_- = \delta(\mathbf{k} - \mathbf{q}), \quad (11.1)$$

$$[\varphi^-(x), \varphi^+(y)]_- = \frac{1}{i} D^-(x - y) = i D^+(y - x), \quad (11.2a)$$

$$[\varphi(x), \varphi(y)]_- = \frac{1}{i} D(x - y), \quad (11.2b)$$

where the Pauli-Jordan commutation function  $D$  and its frequency parts  $D^\pm$  are defined by (10.16), (10.17), and (10.18).

Differentiating (11.2b) with respect to  $x^0$ , and using (10.16) and (10.18), we have for  $y^0 = x^0, \mathbf{y} = \mathbf{x}'$

$$[\dot{\phi}(\mathbf{x}, x^0), \phi(\mathbf{x}', x^0)] = \frac{1}{i} \delta(\mathbf{x} - \mathbf{x}'). \quad (11.3)$$

We shall obtain the principal dynamic variables of the quantized real scalar field from the general prescription of §10 by rewriting the corresponding expressions for the unquantized field in terms of normal products:

$$\left. \begin{aligned} \mathcal{L} &= \frac{1}{2} : \frac{\partial \phi}{\partial x^n} \frac{\partial \phi}{\partial x_n} : - \frac{m^2}{2} : \phi^2 : , \\ T_{mn} &= : \frac{\partial \phi}{\partial x^m} \frac{\partial \phi}{\partial x^n} : - g^{mn} \mathcal{L} , \\ P^n &= \int d\mathbf{k} k^n \phi^+(\mathbf{k}) \phi^-(\mathbf{k}). \end{aligned} \right\} \quad (11.4)$$

We note that the arguments of §9 which established the meaning of the positive- and negative-frequency parts of the field function  $\phi^{(+)}$  and  $\phi^{(-)}$  did not determine their normalization. This normalization may now be determined by means of (11.4) for the energy-momentum four-vector. We consider the amplitude of the state containing one scalar particle with an unnormalized momentum distribution function  $c(\mathbf{k})$

$$\Phi_1 = \int c(\mathbf{k}) \phi^+(\mathbf{k}) d\mathbf{k} \Phi_0, \quad (11.5)$$

and we calculate the expectation value of the operator (11.4) for this state. After carrying out the commutations of the operators, we obtain

$$\langle P^n \rangle_1 \equiv \frac{\Phi_1^* P^n \Phi_1}{\Phi_1^* \Phi_1} = \frac{\int \bar{c}(\mathbf{k}) c(\mathbf{k}) k^n d\mathbf{k}}{\int \bar{c}(\mathbf{k}) c(\mathbf{k}) d\mathbf{k}}. \quad (11.6)$$

The transition to a state with a given value of the four-momentum  $k^n$  may be carried out in (11.5) and (11.6) by localizing the function  $c(\mathbf{k})$  in a small region around the value  $\mathbf{k} = \mathbf{K}$  (for example, by means of a limiting process  $c(\mathbf{k}) \rightarrow \sim \delta(\mathbf{k} - \mathbf{K})$ ). In the above  $k^n \rightarrow K^n$  and we obtain

$$\langle P^n \rangle_1 \rightarrow K^n.$$

Thus, the expectation value of the operator  $P^n$  for a state with the given value of the four-momentum  $K^n$  is exactly equal to  $K^n$  (which corresponds to the results of the preceding section) and, consequently, the normalization of the operators  $\phi^{(+)}$  and  $\phi^{(-)}$  in (11.4) is correct.

In contrast to the real field the complex scalar field is characterized by two mutually conjugate functions  $\phi$  and  $\phi^*$  and describes charged particles of both signs of charge.

Proceeding to the examination of the positive- and negative-frequency parts of the functions  $\varphi$  and  $\varphi^*$ , we find, in accordance with the properties established in §10 of the function  $\varphi$  decreasing and of the function  $\varphi^*$  increasing the charge of the field, that the operator  $\varphi^{(+)}$  describes the creation of a particle of negative charge, the operator  $\varphi^{(-)}$  describes the annihilation of a particle of positive charge, the operator  $\varphi^{*(+)}$  describes the creation of a particle of positive charge, and the operator  $\varphi^{*(-)}$  describes the annihilation of a particle of negative charge.

It was also shown in §10 that the scalar field is quantized according to Bose-Einstein rules, and that the operators which refer to particles of different sign always (anti) commute among themselves. From this it follows that the commutation rules for the operators of the complex scalar field have the following form:

$$\left. \begin{aligned} [\varphi^-(\mathbf{k}), \varphi^+(\mathbf{k}')]_- &= \delta(\mathbf{k} - \mathbf{k}'), \\ [\varphi^-(\mathbf{k}), \varphi^{*+}(\mathbf{k}')]_- &= \delta(\mathbf{k} - \mathbf{k}'), \end{aligned} \right\} \quad (11.7)$$

while all the other commutators are equal to zero.

The corresponding formulas in the  $x$ -representation were obtained in §10.3 [(10.19), (10.20), (10.21)].

The basic dynamic variables can be obtained by writing (3.32), (3.34), (3.39), and (3.40) with the aid of normal products:

$$\mathcal{L} = : \frac{\partial \varphi}{\partial x^n} \frac{\partial \varphi}{\partial x_n} : - m^2 : \varphi \varphi : , \quad (11.8)$$

$$T_{mn} = : \left( \frac{\partial \varphi}{\partial x^m} \frac{\partial \varphi}{\partial x^n} + \frac{\partial \varphi}{\partial x^n} \frac{\partial \varphi}{\partial x^m} \right) : - g^{mn} \mathcal{L} , \quad (11.9)$$

$$P^n = \int d\mathbf{k} \cdot k^n (\varphi^{*+}(\mathbf{k}) \varphi^-(\mathbf{k}) + \varphi^+(\mathbf{k}) \varphi^{*-}(\mathbf{k})), \quad (11.10)$$

$$Q = \int d\mathbf{k} (\varphi^{*+}(\mathbf{k}) \varphi^-(\mathbf{k}) - \varphi^+(\mathbf{k}) \varphi^{*-}(\mathbf{k})). \quad (11.11)$$

It follows from the structure of the operators  $P^n$  and  $Q$  that  $\varphi^{*+}(\mathbf{k})$  is the operator for the creation of a particle with energy-momentum  $k$  and charge  $+1$ ;  $\varphi^{*-}(\mathbf{k})$  is the annihilation operator for the same particle;  $\varphi^{*+}(\mathbf{k})$  is the creation operator for a particle with energy-momentum  $k$  and charge  $-1$ ;  $\varphi^{*-}(\mathbf{k})$  is the annihilation operator for the same particle.

As noted in § 2.4, the simplest complex (pseudo) scalar field corresponds to  $K$ -mesons. It is well known that the mesons form two isospin doublets, i.e., the doublet of "basic" particles ( $K^0$ ,  $K^-$ ) and the doublet of antiparticles ( $\bar{K}^0$ ,  $K^+$ ). The formalism presented above corresponds to the charge components of these doublets ( $K^+$ ,  $K^-$ ). To describe all four  $K$ -mesons, we must replace the isotopic scalar  $\varphi$  by the two-component isospinor

$$\Phi = \begin{pmatrix} K_0 \\ K_- \end{pmatrix}, \quad \Phi^* = (K_0^*, K_+^*) \quad (11.12)$$

and take into account the relationships given by (2.24) between the third component of isotopic spin and electric charge. Instead of (11.8), (11.10), and (11.11) we then obtain

$$\mathcal{L} = : \dot{\Phi}_{;n} \Phi^{;n} : - m_K^2 : \dot{\Phi} \Phi : , \quad (11.13)$$

$$\begin{aligned} P^n &= \int dq q^n \{ \dot{\Phi}^+(q) \Phi^-(q) + \Phi^+(q) \dot{\Phi}^-(q) \} = \\ &= \int dq q^n \{ K_-^+(q) K_-^-(q) + K_0^+(q) K_0^-(q) + K_0^+(q) K_0^-(q) + K_+^+(q) K_+^-(q) \}, \end{aligned} \quad (11.14)$$

$$\begin{aligned} Q &= \int dq \left\{ \dot{\Phi}^+(q) \frac{\tau_3 - I}{2} \Phi^-(q) - \Phi^+(q) \frac{\tau_3 - I}{2} \dot{\Phi}^-(q) \right\} = \\ &= \int dq [K_+^+(q) K_+^-(q) - K_-^+(q) K_-^-(q)]. \end{aligned} \quad (11.15)$$

Here  $K_0^\pm$ ,  $K_\pm^\pm$  are the creation and annihilation operators for the “basic” doublet ( $K_0$ ,  $K_-$ ) and  $K_0^\pm$ ,  $K_+^\pm$  are the corresponding operators for the antidoublet ( $\bar{K}_0$ ,  $K_+$ ).

**11.2. Pion Field.** As mentioned in §3.5, the pion field is described by the three real components  $\pi_\alpha$  ( $\alpha = 1, 2, 3$ ) of the isotopic vector  $\pi(x)$ , or the complex linear combinations

$$\varphi_1 = -\frac{\pi_1 - i\pi_2}{\sqrt{2}}, \quad \varphi_2 = \pi_3, \quad \varphi_3 = \frac{\pi_1 + i\pi_2}{\sqrt{2}}. \quad (3.43)$$

We recall that the transition  $\pi_i$  to  $\varphi_k$  conserves the diagonal property of the quadratic form [see (3.44) and (3.45)].

Proceeding now to the frequency parts of the functions  $\varphi$ , we find that, in accordance with §9.5, the operators  $\varphi$  describe:

- $\varphi_1^+$  — the creation of the positive pion
- $\varphi_1^-$  — the annihilation of the positive pion
- $\varphi_2^+$  — the creation of the neutral pion
- $\varphi_2^-$  — the annihilation of the neutral pion
- $\varphi_3^+$  — the creation of the negative pion
- $\varphi_3^-$  — the annihilation of the negative pion.

The commutation relations have the form

$$[\varphi_m^-(k), \varphi_n^+(q)]_- = [\pi_m^-(k), \pi_n^+(q)]_- = \delta_{mn} \delta(k - q), \quad (11.16)$$

$$[\varphi_m^-(x), \varphi_n^+(y)]_- = [\pi_m^-(x), \pi_n^+(y)]_- = \frac{\delta_{mn}}{i} D^-(x - y), \quad (11.17)$$

$$[\varphi_m(x), \varphi_n(y)]_- = [\pi_m(x), \pi_n(y)]_- = \frac{\delta_{mn}}{i} D(x - y). \quad (11.18)$$

We shall obtain the basic dynamic variables for the three-pion field by writing the corresponding expressions from §3.5 in terms of the normal products:

$$\mathcal{L} = \frac{1}{2} : (\pi; {}_n\pi; {}_n) : - \frac{m^2}{2} : \pi\pi : = \frac{1}{2} : \varphi_\alpha : {}_n\varphi_\alpha^* : - \frac{m^2}{2} : \varphi_\alpha\varphi_\alpha : , \quad (11.19)$$

$$P^n = \int dq q^n \pi^+(q) \pi^-(q) = \int dq q^n \varphi_\alpha^+(q) \varphi_\alpha^-(q), \quad (11.20)$$

$$Q = \int dq \{ \varphi_1^+(q) \varphi_1^-(q) - \varphi_3^+(q) \varphi_3^-(q) \} = \int dq \varphi^+(q) \tau_3 \varphi^-(q). \quad (11.21)$$

When we pass from (11.19) to (11.20) and (11.21) we must take into account the inter-relationship between the complex functions  $\varphi_\alpha$  given by (3.44).

**11.3. Complex Vector Field.** When we quantize the vector field we shall, as in §4, restrict our attention to the complex vector field. We note that the transition to the real vector field is performed in the formulas given below with allowance for the change in the form of the commutation relations [see the transition from (11.7) to (11.1)].

To establish the quantization rules for the four-potential of the vector field, we note that the mechanical generalization of the quantization of the scalar field to this case, i.e., the independent quantization of each component of the potential  $U_n$ , following the example of the scalar field, turns out to be impossible since this procedure would not guarantee that the expectation value of the energy will be positive. The procedure is also incompatible with the subsidiary condition (4.3).

The quantization procedure must therefore take cognizance of the subsidiary condition which, as we have seen in §4, automatically guarantees that  $P^0$  shall be positive. In particular, it was shown that as a result of going over from  $U_n$  to the longitudinal and transverse components in accordance with formula (4.22) the energy-momentum four-vector may be expressed in terms of three linearly independent amplitudes  $a_\alpha(k)$  in the following manner:

$$P^n = \int dk \cdot k^n \sum_{\alpha=1,2,3} (a_\alpha^+(k) a_\alpha^-(k) + a_\alpha^-(k) a_\alpha^+(k)). \quad (4.23)$$

This form of writing down the classical four-momentum takes into account the subsidiary condition (4.3) and guarantees that the unquantized  $P^0$  shall be positive. It is also clear that the quantization of the three independent amplitudes  $a_\alpha$  directly in accordance with Bose-Einstein rules will guarantee that the expectation value of the energy operator will be positive definite.

As a result of this, the operators  $a_\alpha$  must be made to satisfy the following commutation relations:

$$\left. \begin{aligned} [a_\alpha^-(k), a_\beta^+(k')]_- &= \delta_{\alpha\beta} \delta(k - k'), \\ [a_\alpha^+(k), a_\beta^-(k')]_- &= \delta_{\alpha\beta} \delta(k - k') \end{aligned} \right\} \quad (10.14)$$

(all the other commutators are equal to zero).

Rewriting in normal form the expressions for the energy-momentum four-vector (4.27), for the charge (4.28), and for the component of the spin vector along the direction of motion (4.29), all expressed in terms of the amplitudes  $b_\alpha$  (which are related to  $a_\alpha$  by expressions (4.26)), we obtain:



$$P^n = \int d\mathbf{k} \cdot \mathbf{k}^n \sum_{\alpha} (\bar{b}_{\alpha}^{\dagger}(\mathbf{k}) b_{\alpha}^{-}(\mathbf{k}) + b_{\alpha}^{\dagger}(\mathbf{k}) \bar{b}_{\alpha}^{-}(\mathbf{k})), \quad (11.22)$$

$$Q = \int d\mathbf{k} \sum_{\alpha} (\bar{b}_{\alpha}^{\dagger}(\mathbf{k}) b_{\alpha}^{-}(\mathbf{k}) - b_{\alpha}^{\dagger}(\mathbf{k}) \bar{b}_{\alpha}^{-}(\mathbf{k})), \quad (11.23)$$

$$S_3 = \int d\mathbf{k} \{ \bar{b}_1^{\dagger}(\mathbf{k}) b_1^{-}(\mathbf{k}) - b_1^{\dagger}(\mathbf{k}) \bar{b}_1^{-}(\mathbf{k}) + b_2^{\dagger}(\mathbf{k}) \bar{b}_2^{-}(\mathbf{k}) - \bar{b}_2^{\dagger}(\mathbf{k}) b_2^{-}(\mathbf{k}) \}. \quad (11.24)$$

By evaluating the corresponding expectation values we convince ourselves that  $b_1^{*(+)}(\mathbf{k})$  and  $b_1^{(-)}(\mathbf{k})$  are respectively the creation and annihilation operators for a particle of momentum  $\mathbf{k}$ , charge +1, and component of spin along the direction of motion +1,  $b_2^{*(+)}(\mathbf{k})$ ,  $b_2^{(-)}(\mathbf{k})$  are the creation and annihilation operators for a particle of momentum  $\mathbf{k}$ , charge +1 and spin component -1,  $b_3^{*(+)}(\mathbf{k})$  and  $b_3^{(-)}(\mathbf{k})$  are the creation and annihilation operators for a particle of momentum  $\mathbf{k}$ , charge +1 and zero spin component. The meaning of the operators  $b_{\alpha}^{(*)}$  and  $b_{\alpha}^{*(-)}$  may be obtained from the foregoing by means of the following rule: *the transition from  $b_{\alpha}^{(\pm)}$  to  $b_{\alpha}^{*(-\pm)}$  corresponds to a change in the sign of the charge and in the sign of the spin component.*

Thus, for example, if we calculate the expectation values of the operators (11.22), (11.23), (11.24) for a single-particle state described by an amplitude of the form of (11.5)

$$\Phi_1 = \int c(\mathbf{k}) b_1^{\dagger}(\mathbf{k}) d\mathbf{k} \Phi_0,$$

taking into account the rule for Hermitian conjugates  $(b_1^{(+)*})^* = b_1^{(-)}$ , we obtain as a result of going over to  $c(\mathbf{k})$  localized near  $\mathbf{k} = \mathbf{K}$

$$\langle P^n \rangle_1 = K^n, \quad \langle Q \rangle_1 = -1, \quad \langle S_3 \rangle_1 = -1, \text{ and so on.}$$

We see from the above that the vector field (4.3), (4.6), describes charged particles of mass  $m$  and with three possible values of the component of spin (1, 0, -1) along a given direction. More briefly, it is said that this field describes vector mesons of spin 1.

The amplitudes  $a_{\alpha}$  also have a simple interpretation. Thus, the amplitudes  $a_3$  correspond to particles with zero spin component along the direction of motion, while  $a_1$  and  $a_2$  describe combinations of states with spin components 1 and -1, which correspond to linear polarizations.

Let us also write down the commutation relations for the four dependent amplitudes  $U_n$ . With the aid of (4.22) we find without difficulty the commutation relations for the three amplitudes  $U_{\alpha}(\mathbf{k})$  ( $\alpha = 1, 2, 3$ ):

$$[\bar{U}_{\alpha}^{\dagger}(\mathbf{k}), U_{\beta}^{\dagger}(\mathbf{k}')]_{-} = [\bar{U}_{\alpha}^{\dagger}(\mathbf{k}), \bar{U}_{\beta}^{\dagger}(\mathbf{k}')]_{-} = \delta(\mathbf{k} - \mathbf{k}') \left( \delta_{\alpha\beta} + \frac{k_{\alpha} k_{\beta}}{m^2} \right).$$

Utilizing the relation between  $U_0$  and  $U_{\alpha}$

$$U_0(\mathbf{k}) = -\frac{1}{k_0} k^{\alpha} U_{\alpha}(\mathbf{k}),$$

we then determine the commutators involving  $U_0$ :

$$\begin{aligned} [\hat{U}_0^*(k), U_\alpha^+(k')]_- &= [U_0^-(k), \hat{U}_\alpha^+(k')]_- = -\frac{k^0 k^\alpha}{m^2} \delta(k - k'), \\ [\hat{U}_0^*(k), U_0^+(k')]_- &= [U_0^-(k), \hat{U}_0^+(k')]_- = \left(\frac{k^0 k^0}{m^2} - 1\right) \delta(k - k'). \end{aligned}$$

Combining all the expressions obtained above, we obtain the formula

$$[\hat{U}_n^*(k), U_l^+(k')]_- = [U_n^-(k), \hat{U}_l^+(k')]_- = (-g_{nl} + k_n k_l / m^2) \delta(k - k'), \quad (11.25)$$

which is relativistically symmetric and compatible with the subsidiary condition (4.3). Going over to the coordinate representation, we obtain from the above:

$$[\hat{U}_l^*(x), U_n^+(y)]_- = [U_l^-(x), \hat{U}_n^+(y)]_- = \left(g_{ln} + \frac{1}{m^2} \frac{\partial^2}{\partial x^l \partial x^n}\right) iD^-(x - y), \quad (11.26)$$

$$[\hat{U}_l^*(x), U_n(y)]_- = \left(g_{ln} + \frac{1}{m^2} \frac{\partial^2}{\partial x^l \partial x^n}\right) iD(x - y), \quad (11.27)$$

where, as usual,  $D(x - y)$  is the Pauli-Jordan commutation function and  $D^{(-)}$  is its negative-frequency part.

It is not difficult to verify that the commutation relations given by (11.27) are compatible both with the field equations and with the subsidiary conditions. Thus, operating on both sides of (11.27) with the Klein-Gordon operator  $\square_x - m^2$ , we obtain the identity  $0 = 0$ , since according to (11.18)

$$(\square_x - m^2) D(x - y) = 0.$$

Applying the operation  $\sum_l \partial / \partial x_l$  to (11.27) we obtain a similar result, since

$$\frac{\partial}{\partial x_l} \left( g_{ln} + \frac{1}{m^2} \frac{\partial^2}{\partial x^l \partial x^n} \right) D(x - y) = \left( \frac{\partial}{\partial x^n} - \frac{\square}{m^2} \frac{\partial}{\partial x^n} \right) D(x - y) = 0.$$

In practice, an important special case of the vector field is the field describing the isospin triplet ( $\rho^+$ ,  $\rho^0$ ,  $\rho^-$ ) of vector rho-mesons.

*11.4. Hamiltonian Formalism and Canonical Quantization.* For the purposes of comparison with the above quantization scheme, which is based on the Lagrangian formalism, we shall now consider the canonical quantization scheme for the real scalar field.

The canonical quantization procedure is founded on the canonical (i.e., Hamiltonian) formalism of classical mechanics. Let us briefly recall the essence of this in relation to systems with an infinite number of degrees of freedom (classical fields).

In the canonical formalism, the basic quantity is the Hamiltonian function  $H$ , looked upon as a function of the canonical variables, i.e., the generalized coordinates  $q_i$  and the

generalized momenta  $p_i$ . In the absence of external fields, the Hamiltonian function is equal to the energy of the system  $P_0$ . In our case, we can take the generalized coordinates to be the value of the field function  $\varphi(x)$  at the sites of the space lattice

$$q_i(t) = \varphi(\mathbf{x}_i, t). \quad (11.28)$$

This implies the subdivision of the entire three-dimensional volume into cells of size  $\Delta V_i$  around the lattice sites  $\mathbf{x}_i$ . We can now define the canonical momentum by

$$P_i(t) = \frac{\partial \Lambda(t)}{\partial \dot{q}_i(t)},$$

where

$$\dot{q}_i(t) = \frac{\partial \varphi(\mathbf{x}_i, t)}{\partial t},$$

In these expressions  $\Lambda$  is the Lagrange function. In accordance with (1.2), this function can be written in the form

$$\Lambda(t) = \sum_i \mathcal{L}(q_i(t), \dot{q}_i(t)) \Delta V_i.$$

We therefore finally have

$$p_i(t) = \frac{\partial \mathcal{L}}{\partial \dot{q}_i(t)} \Delta V_i. \quad (11.29)$$

Using (2.9), (2.10), (11.28), and (11.29), we can write down the Hamiltonian function (the Hamiltonian) in the following way:

$$H = \sum_i T^{00}(\mathbf{x}_i, t) \Delta V_i = \sum_i p_i(t) \dot{q}_i(t) - \Lambda(t). \quad (11.30)$$

It is implied in this expression that the  $\dot{q}_i$  are expressed as functions of the generalized momenta  $p_i$ .

Before we pass to the continuous case ( $\Delta V \rightarrow 0$ ), it is also convenient to introduce another canonical momentum (the momentum density, in fact)

$$\pi(\mathbf{x}, t) = \frac{\partial \mathcal{L}}{\partial \dot{\varphi}(\mathbf{x}, t)}. \quad (11.31)$$

Instead of (11.30) we now have

$$H = \int d\mathbf{x} \{ \pi(\mathbf{x}, t) \dot{\varphi}(\mathbf{x}, t) - \mathcal{L}(\mathbf{x}, t) \}, \quad (11.32)$$

where, again, the velocities are expressed in terms of the momenta.

The canonical variables  $q, p$  satisfy the equations of motion in the Hamilton form

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = - \frac{\partial H}{\partial q_i}, \quad (11.33)$$

which are also called the *canonical equations*. Now suppose that  $A(p, q)$  is a dynamic quantity written as a function of the canonical variables, which does not depend explicitly on time. Using (11.33), we can write its total derivative with respect to time in the form

$$\frac{dA(p, q)}{dt} = \sum_i \left[ \frac{\partial H}{\partial p_i} \frac{\partial A}{\partial q_i} - \frac{\partial H}{\partial q_i} \frac{\partial A}{\partial p_i} \right] \equiv \{H, A\}, \quad (11.34)$$

where we have introduced the abbreviated notation for the so-called *classical Poisson brackets*

$$\{A, B\} = \sum_i \left[ \frac{\partial A}{\partial p_i} \frac{\partial B}{\partial q_i} - \frac{\partial A}{\partial q_i} \frac{\partial B}{\partial p_i} \right]. \quad (11.35)$$

The Poisson brackets for the canonical variables have the form

$$\{q_i, q_j\} = \{p_i, p_j\} = 0, \quad \{p_i, q_j\} = \delta_{ij}. \quad (11.36)$$

In the canonical formalism, the quantization postulate consists of the replacement of the classical Poisson brackets by the *quantum Poisson brackets*

$$\{A, B\} \rightarrow i[A, B] \equiv i(AB - BA). \quad (11.37)$$

After quantization, the operators  $q_i$  and  $p_i$  satisfy the commutation relations

$$[q_i, q_j] = [p_i, p_j] = 0, \quad [q_i, p_j] = i\delta_{ij}.$$

Passing to the continuous limit, we obtain

$$[q(x), q(x')] = [\pi(x), \pi(x')] = 0, \quad (11.38)$$

$$[\pi(x, t), q(x', t)] = -i\delta^3(x - x'), \quad (11.39)$$

and

$$\frac{dA}{dt} = i[H, A]. \quad (11.40)$$

Since for the Lagrangian (3.1)

$$q(x) = \varphi(x), \quad \pi(x) = \dot{\varphi}(x),$$

we verify that (11.39) is identical with (11.3). The “relativization” of (11.39) and the transition to the usual (not equal-time) commutation relations (11.3) can be performed by transforming to the momentum representation.

The equation of motion given by (11.40) is the analog of the zero component in (9.24).

We note that difficulties may be encountered when the method of canonical quantization is applied to other fields. We have in mind here, above all, the cases of the so-called *singular Lagrangians*. A Lagrangian is called singular if the canonical momentum determined in accordance with (11.31) is identically zero.

The gauge-invariant Lagrangian of the electromagnetic field

$$\mathcal{L} = -\frac{1}{4} H_{mn} H^{mn} = -\frac{1}{4} (\partial_m A_n - \partial_n A_m) (\partial^m A^n - \partial^n A^m), \quad (5.13)$$

is an example of a singular Lagrangian.

In this case, it is natural to consider the four components of the potential  $A_m$  as independent canonical coordinates. However, the momentum  $\pi_0$  that is the canonical conjugate of the coordinate  $A_0$  turns out to be zero:

$$\pi_0(x) = \frac{\partial \mathcal{L}}{\partial \dot{A}_0(x)} = 0.$$

The second example of a singular Lagrangian is provided by the Lagrangian corresponding to the Yang-Mills field (8.28). The standard procedure of canonical quantization must be modified in this case. We refer readers interested in this point to the book by Dirac (1964), and note that, physically, the singular Lagrangians correspond to systems with non-holonomic constraints, and these must be broken before quantization can be performed.

## §12. The Electromagnetic Field

*12.1. Singularities of the Electromagnetic Field and the Quantization Procedure.* In the quantization of the electromagnetic field, it is necessary to satisfy simultaneously the requirement of positive energy density, the Lorentz subsidiary condition, and the condition of transversality. Moreover, the whole formulation must have the property of relativistic covariance.

We encountered a similar situation in the quantization of the vector meson field. The difference consists of the fact that vector mesons may exist in three spin states while photons, because of their transversality, may exist only in two spin states, and also of the fact that, in contrast to mesons, the mass of the photons is zero. The first of the above two circumstances ensures that the components of the potential of the electromagnetic field contain, to an even greater extent than in the case of the vector field, “superfluous” variables, since we have four components and only two states in which real photons may exist.

The second difference ( $m = 0$ ) is that the procedure used for the quantization of the vector field cannot be applied to the electromagnetic field. Indeed, the quantization of a hypothetical vector field with vanishingly small mass, i.e., of a field which differs from the electromagnetic field by not having a gauge transformation and therefore has three components, encounters fundamental difficulties. Attempts to carry out such a quantization result in meaningless expressions, first of all when we diagonalize the energy-momentum

by means of the substitution (4.22), and, second of all, when we set up commutation relations for the components of the potential  $A_n$  (11.25)–(11.27), because of the appearance of the vanishingly small mass in the denominators of the expressions enumerated above.

The quantization of the electromagnetic field is therefore carried out as follows. We regard the components of the vector potential as independent quantities, thereby giving up the Lorentz condition in operator form. Then, in accordance with the general prescription for quantization (§9), starting with the structure of the energy-momentum four-vector expressed in terms of its longitudinal, transverse, and time-like components  $a_n(k)$

$$P^n = \int d\mathbf{k} \cdot \mathbf{k}^n (-a_m^+ (\mathbf{k}) a^{-m} (\mathbf{k})), \quad (12.1)$$

we arrive at the commutation relations

$$[a_m^-(\mathbf{k}), a_n^+(\mathbf{k}') ]_- = -g^{mn} \delta (\mathbf{k} - \mathbf{k}'). \quad (12.2)$$

If we use the formulas for the transformation from the momentum to the coordinate representation

$$A_n^\pm (x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d\mathbf{k}}{\sqrt{2k_0}} e^{\pm i\mathbf{k}\mathbf{x}} A_n^\pm (\mathbf{k}) = \frac{1}{(2\pi)^{3/2}} \int \frac{d\mathbf{k}}{\sqrt{2k_0}} e^{\pm i\mathbf{k}\mathbf{x}} e_n^m (\mathbf{k}) a_m^\pm (\mathbf{k}), \quad (12.3)$$

we obtain

$$[A_m^-(x), A_n^+(y)] = ig_{mn} D_0^-(x-y),$$

where  $D^{(-)}$  is the usual negative-frequency commutation function for zero mass:

$$D_0^-(x) = \frac{i}{(2\pi)^3} \int d\mathbf{k} \delta(k^2) \theta(-k^0) e^{i\mathbf{k}\mathbf{x}}.$$

The complete commutator for the components  $A_n$  has the following form:

$$[A_n(x), A_m(y)]_- = ig^{mn} D_0(x-y), \quad (12.4)$$

$$D_0(x) = \frac{1}{(2\pi)^3 i} \int e^{i\mathbf{k}\mathbf{x}} \delta(k^2) \varepsilon(k^0) d\mathbf{k}.$$

The quantization given by (12.2) evidently does not guarantee that the expectation value of the energy will be positive. We shall leave this question open for the time being, and will return to it later when we formulate the Lorentz subsidiary condition.

The quantization given by (12.2) allows us to regard the operators  $a_n^{(\pm)}$  as creation and annihilation operators for four independent kinds of photons—two transverse kinds, “longitudinal,” and “time-like.” However, the following difficulty arises in this quantization. The component  $a_0$  satisfies the relation

$$[a_0^-(k), a_0^+(k')]_- = -\delta(k - k'),$$

a comparison of which with (10.14) shows that the creation and annihilation operators for the “time-like” photons behave as if they interchanged places, which occurs because the term  $a_0^{(+)}a_0^{(-)}$  in (12.1) has a negative sign.

However, this situation is incompatible with the assumption that the field is real. Thus, if we use this formula to determine the vacuum expectation value, we obtain, using (9.29),

$$\hat{\Phi}_0 [a_0^-(k), a_0^+(k')]_- \Phi_0 = \hat{\Phi}_0 a_0^-(k) a_0^+(k') \Phi_0 = -\delta(k - k').$$

Multiplying the left-hand side by  $\hat{f}(k) f(k')$ , integrating with respect to  $k$  and  $k'$ , and using the fact that  $(a_0^+)^* = a_0^-$ , we find that

$$\int dk \hat{\Phi}_0 a_0^-(k) \hat{f}(k) \int dk' f(k') a_0^+(k') \Phi_0 = \hat{\Phi}_0 \left| \int a_0^+(k) f(k) dk \right|^2 \Phi_0 > 0.$$

Performing the same operations on the right-hand side, we arrive at the negative expression

$$-\int \delta(k - k') \hat{f}(k') f(k') dk' dk = -\int |f(k)|^2 dk < 0$$

and, consequently, we have a contradiction.

**12.2. Indefinite Metric.** In order to eliminate the difficulty outlined above we shall use the formal approach of Bleuler (1950) and Gupta (1950), which is essentially based on the fact that the “time-like photons” that correspond to the zero component of the potential do not actually exist, but that their appearance in the intermediate steps of the argument is connected with the transition from observable quantities (the vectors  $E$  and  $H$ ) to the nonobservable four-potential  $A$ , which was made to ensure that the theory was relativistically symmetric and covariant.

We shall now assume that the  $a_0$  component, in contrast to the other components, is anti-Hermitian, i.e.,

$$a_n^* = -a^n$$

(this assumption is equivalent to introducing a Hermitian  $a_4$ ,  $a_0 = ia_4$ ). We then obtain the usual commutation relations

$$\begin{aligned} [a_m^-(k), a_n^+(k')]_- &= \delta(k - k') \delta_{nm}, \\ [\hat{A}_m(x), A_n(y)]_- &= \delta_{nm} \frac{1}{i} D_0(x - y). \end{aligned}$$

In order to retain the self-conjugate property for the operator  $a_n$  we shall have to introduce an *indefinite metric* in the space of the state amplitudes. To do this, we introduce the Hermitian operator  $\eta$  defined by

$$\eta a_n = -a^n \eta, \quad \eta^2 = 1, \quad \eta \Phi_0 = \Phi_0, \quad (12.5)$$

i.e., one which commutes with  $a_\alpha$  ( $\alpha = 1, 2, 3$ ) and anticommutes with  $a_0$ . By introducing a new definition for the adjoint state amplitude

$$\Phi^\dagger = \check{\Phi}^\dagger \eta, \quad (12.6)$$

i.e., by defining the expectation value as

$$\langle f \rangle = \check{\Phi}^\dagger \eta f \Phi = \check{\Phi}^\dagger f \Phi, \quad (12.7)$$

we obtain the self-conjugate property for the operator  $a_n$ :

$$(\check{\Phi}^\dagger a_n \Phi)^* = (\check{\Phi}^\dagger \eta a_n \Phi)^* = \check{\Phi}^\dagger a_n^* \eta \Phi = \check{\Phi}^\dagger \eta a_n \Phi = \check{\Phi}^\dagger a_n \Phi.$$

Let us now consider the question of the subsidiary condition. In the course of quantization we assume that the  $A_n$  are independent, and consequently we may not impose the Lorentz condition on the operators  $A_n$ . It is readily seen that it is likewise impossible to impose the Lorentz condition on the allowed state amplitudes, i.e., to demand that

$$\frac{\partial A_n(x)}{\partial x_n} \Phi = 0,$$

for such a condition would, for example, contradict the definition of the vacuum state. Indeed, by setting  $\Phi = \Phi_0$  we obtain

$$\left(\frac{\partial A}{\partial x}\right) \Phi_0 = \left(\frac{\partial A^+}{\partial x}\right) \Phi_0 = 0.$$

Multiplying on the left by  $A_k^{*(-)}(y)$  we obtain

$$\begin{aligned} \check{A}_k^-(y) \frac{\partial A_n^+(x)}{\partial x_n} \Phi_0 &= \frac{\partial}{\partial x_n} (\check{A}_k^-(y) A_n^+(x)) \Phi_0 = \\ &= \frac{\partial}{\partial x_n} (A_n^+(x) \check{A}_k^-(y)) \Phi_0 - \frac{\partial}{\partial x_n} D_0^+(x-y) g^{nk} \Phi_0 = -\frac{\partial}{\partial x_k} D_0^+(x-y) \Phi_0 \neq 0, \end{aligned}$$

i.e., a contradiction.

We shall therefore formulate the Lorentz condition as a condition on allowed states but in a weaker form:

$$\frac{\partial A_n^-(x)}{\partial x_n} \Phi \equiv \left(\frac{\partial A^-}{\partial x}\right) \Phi = 0, \quad (12.8)$$

and its adjoint relation in the form

$$\check{\Phi}^\dagger \left(\frac{\partial A^+}{\partial x}\right) = 0. \quad (12.9)$$



These conditions guarantee that the Lorentz condition is satisfied on the average

$$\left\langle \left( \frac{\partial A}{\partial x} \right) \right\rangle = \dot{\Phi} \left( \frac{\partial A}{\partial x} \right) \Phi = 0,$$

which is quite sufficient to establish correspondence with the classical field.

Let us now turn to the question of the positive observed energy density. For this we rewrite the subsidiary conditions in the momentum representation. We obviously have

$$(k^0 a_0^-(\mathbf{k}) - |\mathbf{k}| a_3^-(\mathbf{k})) \Phi = 0,$$

$$\dot{\Phi} (k^0 a_0^+(\mathbf{k}) - |\mathbf{k}| a_3^+(\mathbf{k})) = 0,$$

or, since  $k^0 = |\mathbf{k}|$ ,

$$(a_0^-(\mathbf{k}) - a_3^-(\mathbf{k})) \Phi = 0, \quad (12.10)$$

$$\dot{\Phi} (a_0^+(\mathbf{k}) - a_3^+(\mathbf{k})) = 0. \quad (12.11)$$

From this it follows that in the allowed states the total energy and momentum of the longitudinal and time-like pseudophotons are equal to zero, since

$$\langle a_3^+ a_3^- - a_0^+ a_0^- \rangle = \dot{\Phi} (a_3^+ a_3^- - a_0^+ a_0^-) \Phi = \dot{\Phi} (a_3^+ - a_0^+) a_3^- \Phi = 0.$$

We therefore obtain for the energy-momentum vector

$$\langle P^n \rangle = \int d\mathbf{k} \cdot k^n \langle -a^{+m}(\mathbf{k}) a_m^-(\mathbf{k}) \rangle = \int d\mathbf{k} \cdot k^n \langle a_1^+(\mathbf{k}) a_1^-(\mathbf{k}) + a_2^+(\mathbf{k}) a_2^-(\mathbf{k}) \rangle. \quad (12.12)$$

This ensures that the expectation value of energy is positive. We shall now show that the re-defined norm (12.7) gives the same results for the expectation values of observables as the old norm.

To do this, we write the local expansion (5.16) in the form

$$A_n^-(\mathbf{k}) = \sum_{\sigma=1,2} e_n^\sigma a_\sigma^-(\mathbf{k}) + \left( \frac{k_n}{|\mathbf{k}|} - \delta_n^0 \right) a_3^-(\mathbf{k}) + \delta_n^0 a_0^-(\mathbf{k}) = A_n^{\text{tr},-}(\mathbf{k}) + k_n \Lambda(\mathbf{k}) + \delta_{n0} F^-(\mathbf{k}), \quad (12.13)$$

where  $F^-(\mathbf{k})\Phi = 0$  and the term  $k_n \Lambda$  has the structure of a four-gradient and can be eliminated by the gauge transformation, whereas  $A^{\text{tr}}$  is the transverse component of  $A$ . We may therefore suppose that

$$A_n^-(\mathbf{k}) \Phi = A_n^{\text{tr},-}(\mathbf{k}) \Phi, \quad \dot{\Phi} A_n^+(\mathbf{k}) = \dot{\Phi} A_n^{\text{tr},+}(\mathbf{k}),$$

and, consequently,

$$\dagger\Phi A_n\Phi = \dagger\Phi A_n^{\text{tr}}\Phi.$$

It is not difficult to show that the more general assertion also holds:

$$\dagger\Phi K\Phi = \dagger\Phi K^{\text{tr}}\Phi, \quad (12.14)$$

where  $K$  is an operator of the form

$$K = \sum_{\alpha} Z\left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_s}\right) A_{\alpha_1}(x_1) \dots A_{\alpha_s}(x_s),$$

$Z$  is a polynomial function, and

$$K^{\text{tr}} = \sum_{\alpha} Z\left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_s}\right) A_{\alpha_1}^{\text{tr}}(x_1) \dots A_{\alpha_s}^{\text{tr}}(x_s).$$

Indeed by representing  $A_n$  in the form (12.13) and discarding the gradient terms, we see that we must carry out the commutations of terms of the form  $A^{\text{tr}}$  and  $F = a_0 + a_3$ . However,  $F$  and  $A^{\text{tr}}$  commute,  $F^{(+)}$  and  $F^{(-)}$  also commute, and  $\Phi_0^{\dagger} F^{(+)} = F^{(-)} \Phi_0 = 0$ . Thus only the commutators of  $A^{\text{tr}(-)}$  and  $A^{\text{tr}(+)}$  are left. Consequently, we have proved the validity of (12.14). We shall now show that

$$\dagger\Phi K^{\text{tr}}\Phi = \dagger\Phi_{\text{tr}} K^{\text{tr}}\Phi_{\text{tr}}, \quad (12.15)$$

where  $\Phi_{\text{tr}}$  is the amplitude of the pure photon state, i.e., of a state which does not contain longitudinal or time-like pseudophotons. Indeed, the amplitude of an arbitrary state may be represented in the form of a linear combination of a pure photon state and of terms containing various numbers of pseudophotons. In virtue of the subsidiary conditions the latter terms may contain the operators  $a_0$  and  $a_3$  only in the combination

$$a_3 + a_0,$$

i.e.,

$$\begin{aligned} \Phi &= \left\{ 1 + \sum_n c_n \prod_{1 \leq j \leq n} (a_3^+(k_j) + a_0^+(k_j)) \right\} \Phi_{\text{tr}}, \\ \dagger\Phi &= \dagger\Phi_{\text{tr}} \left\{ 1 + \sum_n c_n \prod_{1 \leq j \leq n} (a_3^-(k_j) + a_0^-(k_j)) \right\}. \end{aligned}$$

We shall refer to the state  $\Phi_{\text{tr}}$  as the *pseudophoton vacuum*. But since the combinations

$$\tilde{F}^+ = a_0^+ + a_3^+ \text{ and } \tilde{F}^- = a_0^- + a_3^-$$

commute with  $K^{\text{tr}}$  and with each other, we have

$$\begin{aligned}
\hat{\Phi} K^{\text{tr}} \Phi &= \hat{\Phi}_{\text{tr}} \left\{ 1 + \sum_n c_n \prod_j \tilde{F}^-(k_j) \right\} K^{\text{tr}} \left\{ 1 + \sum_m c_m \prod_i \tilde{F}^+(k_i) \right\} \Phi_{\text{tr}} = \\
&= \hat{\Phi}_{\text{tr}} \left\{ 1 + \sum_m c_m \prod_i \tilde{F}^+(k_i) \right\} K^{\text{tr}} \left\{ 1 + \sum_n c_n \prod_j \tilde{F}^-(k_j) \right\} \Phi_{\text{tr}} = \\
&= \hat{\Phi}_{\text{tr}} K^{\text{tr}} \Phi_{\text{tr}} = \hat{\Phi}_{\text{tr}} \eta K^{\text{tr}} \Phi_{\text{tr}} = \hat{\Phi}_{\text{tr}} K^{\text{tr}} \eta \Phi_{\text{tr}} = \hat{\Phi}_{\text{tr}} K^{\text{tr}} \Phi_{\text{tr}},
\end{aligned}$$

since

$$\hat{\Phi}_{\text{tr}} a_{3,0}^{\pm} = 0, \quad a_{3,0}^{-} \Phi_{\text{tr}} = 0$$

and

$$K^{\text{tr}} \eta = \eta K^{\text{tr}}, \quad \eta \Phi_{\text{tr}} = \Phi_{\text{tr}}.$$

Thus, we have established that the expectation value of  $K$  over the indefinite metric is equal to the ordinary expectation value of  $K^{\text{tr}}$ .

We conclude that the use of the indefinite metric in calculating real observable quantities cannot lead to any paradoxical results such as “negative probabilities.”

For the sake of convenience, we shall always use the usual  $\Phi^*$  in place of  $\Phi^{\dagger}$ .

**12.3. The Form of the Basic Quantities.** In conclusion, we list the basic expressions used in the theory of the quantized electromagnetic field:  
the Lagrangian

$$\mathcal{L}(x) = -\frac{1}{2} : \frac{\partial A_m}{\partial x_n} \frac{\partial A^m}{\partial x^n} :, \quad (12.16)$$

the field equations

$$\square A_n = 0, \quad (12.17)$$

the Lorentz subsidiary condition

$$\left( \frac{\partial A^-}{\partial x} \right) \Phi \equiv \frac{\partial A_n^-}{\partial x_n} \Phi = 0, \quad (12.8)$$

the general expression for the operator of the energy-momentum four-vector

$$P^n = \int d\mathbf{k} \cdot k^n (-a_n^+(\mathbf{k}) a^{-n}(\mathbf{k})) \quad (12.18)$$

and its expectation value over allowed states

$$\langle P^n \rangle = \hat{\Phi} \int d\mathbf{k} \cdot k^n \sum_{\mathbf{v}=1,2} a_{\mathbf{v}}^+(\mathbf{k}) a_{\mathbf{v}}^-(\mathbf{k}) \Phi, \quad (12.12)$$

the commutator of the field functions

$$[A_m(x), A_n(y)]_- = i g^{mn} D_0(x-y), \quad (12.4)$$

the spin-angular momentum vector

$$\mathbf{S} = i \int d\mathbf{k} [\mathbf{a}^+(k) \times \mathbf{a}^-(k)]. \quad (12.19)$$

### §13. The Spinor Field

*13.1. Fermi-Dirac Quantization and Commutation Functions.* It may be seen from the structure of the energy-momentum four-vector of the spinor field

$$P^n = \int d\mathbf{k} \cdot k^n \sum_{\nu=1,2} (a_{\nu}^*(k) a_{\nu}^-(k) - a_{\nu}^-(k) a_{\nu}^*(k)) \quad (7.34)$$

that the independent amplitudes  $a_{\nu}$  may be subjected to quantization.

As mentioned earlier, the requirement that the expectation value of the energy operator  $P^0$  should be positive leads to the demand that the spinor field must be quantized in accordance with Fermi-Dirac rules. Therefore, the Fourier amplitudes of the spinor field  $a_{\nu}(k)$  must be regarded as operators which satisfy the Fermi-Dirac commutation relations:

$$\left. \begin{aligned} [a_{\nu}^-(k), a_{\mu}^+(k')]_+ &= \delta_{\nu\mu} \delta(k-k'), \\ [a_{\nu}^-(k), a_{\mu}^+(k')]_+ &= \delta_{\nu\mu} \delta(k-k') \end{aligned} \right\} \quad (13.1)$$

(all the other anticommutators are equal to zero).

The commutation relations for the operators  $\psi(x)$  will be obtained from the above with the aid of (7.5), (7.13), (7.14), (7.20) and (7.21). We obtain in turn ( $\alpha, \beta$  are spinor indices):

$$\begin{aligned} [\psi_{\alpha}^-(x), \bar{\psi}_{\beta}^+(y)]_+ &= \frac{1}{(2\pi)^3} \int d\mathbf{k} \int d\mathbf{l} e^{i(\mathbf{l}y - \mathbf{k}x)} [\psi_{\alpha}^-(k), \bar{\psi}_{\beta}^+(l)]_+ = \\ &= \frac{1}{(2\pi)^3} \int d\mathbf{k} \int d\mathbf{l} e^{i(\mathbf{l}y - \mathbf{k}x)} \sum_{\nu, \mu} v_{\alpha}^{\nu,-}(k) \bar{v}_{\beta}^{\mu,+}(l) [a_{\nu}^-(k), a_{\mu}^+(l)]_+ = \\ &= \frac{1}{(2\pi)^3} \int d\mathbf{k} e^{i\mathbf{k}(y-x)} \sum_{\nu} v_{\alpha}^{\nu,-}(k) \bar{v}_{\beta}^{\nu,+}(k) = \frac{1}{(2\pi)^3} \int d\mathbf{k} e^{i\mathbf{k}(y-x)} \frac{(\hat{k} + m)_{\alpha\beta}}{2k^0} = \\ &= \frac{1}{(2\pi)^3} \int d\mathbf{k} (\hat{k} + m)_{\alpha\beta} \delta(k^2 - m^2) \theta(k^0) e^{i\mathbf{k}(y-x)} = \left( i\gamma^n \frac{\partial}{\partial x^n} + m \right)_{\alpha\beta} iD^+(y-x) \end{aligned}$$

or by virtue of the symmetry of the  $D$ -functions (11.2a)

$$[\psi^-(x), \bar{\psi}^+(y)]_+ = \left( i\gamma^n \frac{\partial}{\partial x^n} + m \right) \frac{1}{i} D^-(x-y). \quad (13.2)$$

By methods similar to the above it may be shown that

$$[\psi^+(x), \bar{\psi}^-(y)]_+ = \left( i\gamma^n \frac{\partial}{\partial x^n} + m \right) \frac{1}{i} D^+(x-y). \quad (13.3)$$

Introducing the notation

$$S_{\alpha\beta}^{\pm}(x) = \left( i\gamma^n \frac{\partial}{\partial x^n} + m \right)_{\alpha\beta} D^{\pm}(x),$$

we obtain from (13.2) and (13.3)

$$[\psi(x), \bar{\psi}(y)]_+ = \frac{1}{i} S(x-y), \quad (13.4)$$

where

$$S(x) = \left( i\gamma^n \frac{\partial}{\partial x^n} + m \right) D(x) = \frac{i}{(2\pi)^3} \int e^{ikx} \delta(k^2 - m^2) \varepsilon(k^0) (\hat{k} - m) dk. \quad (13.5)$$

The commutation relations (13.2)–(13.4) are compatible with the field equations since

$$\left( i\gamma^n \frac{\partial}{\partial x^n} - m \right) S(x-y) = \left( i\gamma^n \frac{\partial}{\partial x^n} - m \right) \left( i\gamma^k \frac{\partial}{\partial x^k} + m \right) D(x-y) = (\square - m^2) D(x-y) = 0.$$

In order to write the commutation relations (13.1) in a manifestly covariant form, we shall transform from  $\psi^{\pm}(k)$  to  $\psi(\pm k)$ , where

$$\psi_{\alpha}(\pm k) = \sum_{\mathbf{v}} a_{\mathbf{v}}(\pm k) v_{\alpha}^{\mathbf{v}}(\pm k) \quad (k^0 = \sqrt{\mathbf{k}^2 + m^2}).$$

To do this, we use (7.13), (7.14), and the definitions

$$a_{\mathbf{v}}^{\pm}(\mathbf{k}) = \frac{a_{\mathbf{v}}^{\pm}(k)}{\sqrt{2k_0}} = \frac{\theta(k^0) a_{\mathbf{v}}(\pm k)}{\sqrt{2k^0}}, \quad v_{\alpha}^{\mathbf{v}, \pm}(\mathbf{k}) = \frac{v_{\alpha}^{\mathbf{v}, \pm}(k)}{\sqrt{2k^0}} = \theta(k^0) \frac{v_{\alpha}^{\mathbf{v}}(\pm k)}{\sqrt{2k^0}}, \quad k^0 = \sqrt{\mathbf{k}^2 + m^2}.$$

The operators  $a_{\mathbf{v}}(k)$  are relativistically covariant and satisfy commutation relations which are analogous to the relations that hold for the scalar field (10.24):

$$\begin{aligned} \delta(k^2 - m^2) \delta(k'^2 - m^2) [a_{\mathbf{v}}^*(k), a_{\mu}^{\dagger}(k')]_+ &= \delta_{\mu\mathbf{v}} \delta(k - k') \delta(k^2 - m^2), \\ \delta(k^2 - m^2) \delta(k'^2 - m^2) [a_{\mathbf{v}}^*(k), \hat{a}_{\mu}^{\dagger}(k')]_+ &= \delta_{\mu\mathbf{v}} \delta(k - k') \delta(k^2 - m^2). \end{aligned}$$

By going over to  $a_{\mathbf{v}}(\pm k)$  in these expressions, we obtain

$$\delta(k^2 - m^2) \delta(k'^2 - m^2) [a_{\mathbf{v}}^*(k), a_{\mu}(k')]_+ = \delta_{\mu\mathbf{v}} \delta(k + k') \delta(k^2 - m^2). \quad (13.6)$$

*13.2. Dynamic Variables.* If we define the Lagrangian of the spinor field in terms of the normal product

$$\mathcal{L} = \frac{i}{2} : \left( \bar{\psi}(x) \gamma^n \frac{\partial \psi}{\partial x^n} - \frac{\partial \bar{\psi}}{\partial x^n} \gamma^n \psi(x) \right) : - m : \bar{\psi}(x) \psi(x) :, \quad (13.7)$$

we obtain the following expressions for the dynamic variables:  
the energy-momentum tensor

$$T^{kl} = \frac{i}{2} : \left( \bar{\psi} \gamma^k \frac{\partial \psi}{\partial x_l} - \frac{\partial \bar{\psi}}{\partial x_l} \gamma^k \psi \right) :,$$

the spin density tensor

$$S^{lm,k} = \frac{1}{4} : \bar{\psi}(x) \gamma^k \sigma^{ml} \psi(x) : + \frac{1}{4} : \bar{\psi}(x) \sigma^{ml} \gamma^k \psi(x) :,$$

the current vector

$$J^k(x) = : \bar{\psi}(x) \gamma^k \psi(x) :.$$

Transforming to the momentum representation in accordance with the formulas

$$\psi^\pm(x) = \frac{1}{(2\pi)^{3/4}} \int dke^{\pm ikx} \sum_{\mathbf{v}} a_{\mathbf{v}}^\pm(k) v^{\mathbf{v}, \pm}(k) \quad (7.32)$$

$$\bar{\psi}^\pm(x) = \frac{1}{(2\pi)^{3/4}} \int dke^{\pm ikx} \sum_{\mathbf{v}} a_{\mathbf{v}}^{\pm*}(k) \bar{v}^{\mathbf{v}, \pm}(k) \quad (7.33)$$

and using (7.34), (7.42), and (7.43), we find that the corresponding integrals of motion assume the following form:  
energy-momentum four-vector

$$P^n = \int d\mathbf{k} \cdot k^n \sum_{\mathbf{v}} (a_{\mathbf{v}}^+(k) a_{\mathbf{v}}^-(k) + a_{\mathbf{v}}^-(k) a_{\mathbf{v}}^{+*}(k)); \quad (13.8)$$

charge

$$Q = \int d\mathbf{k} \sum_{\mathbf{v}} (a_{\mathbf{v}}^+(k) a_{\mathbf{v}}^-(k) - a_{\mathbf{v}}^-(k) a_{\mathbf{v}}^{+*}(k)); \quad (13.9)$$

spin component along direction of motion

$$S_3 \sim \frac{1}{2} (a_1^+(k) a_1^-(k) - a_2^+(k) a_2^-(k) - a_1^+(k) a_1^{+*}(k) + a_2^+(k) a_2^{+*}(k)). \quad (13.10)$$

From (13.8)-(13.10), it follows that the operators  $a_{\mathbf{v}}^{+*}(k)$  and  $a_{\mathbf{v}}^-(k)$  are respectively the creation and annihilation operators for particles of momentum  $\mathbf{k}$ , mass  $m(k^2 = m^2)$ , charge  $+1$ , and component of spin along the  $z$  axis equal to  $\frac{1}{2}(\nu = 1)$  or  $-\frac{1}{2}(\nu = 2)$ .

The operators  $a_v^+(k)$  and  $a_v^-(k)$  correspond to particles which differ from those described above only by the sign of the charge  $(-1)$  and by the sign of the spin (i.e.,  $\frac{1}{2}(\nu = 2)$  and  $-\frac{1}{2}(\nu = 1)$ ).

**13.3. The Quantized Neutrino Field.** We write the Lagrangian for the neutrino field in the form

$$\mathcal{L}(x) = : \frac{i}{2} \left( \bar{\psi}(x) \gamma^\mu \frac{\partial \psi}{\partial x^\mu} - \frac{\partial \bar{\psi}}{\partial x^\mu} \gamma^\mu \psi(x) \right) : = \quad (13.11)$$

$$= \frac{i}{4} : \left( \bar{\psi}(x) (1 + \gamma^5) \gamma^\mu \frac{\partial \psi}{\partial x^\mu} - \frac{\partial \bar{\psi}}{\partial x^\mu} \gamma^\mu (1 - \gamma^5) \psi(x) \right) :. \quad (13.12)$$

In this expression, we have used the relation (§7.4)

$$\psi(x) = P_- \psi(x) = \frac{1 - \gamma^5}{2} \psi(x). \quad (13.13)$$

The expressions for the energy-momentum tensor  $T^{lk}$  and the spin tensor  $S^{lm, k}$  in the configuration representation can be obtained from (7.28) and (7.31) by replacing  $\psi$  by  $\psi$ . In the momentum representation, the formula for  $P^l$  is obtained from (7.34) and (12.10) by omitting summation over the spin index:

$$P^l = \int d\mathbf{p} \cdot p^l (\hat{a}_{-1}^*(\mathbf{p}) \hat{a}_{-1}(\mathbf{p}) + \hat{a}_{+1}^*(\mathbf{p}) \hat{a}_{+1}(\mathbf{p})). \quad (13.14)$$

The lower index  $(-1)$  corresponds to negative helicity.

Let us consider the spin of the neutrino field separately. We start with (7.40) for the density of the vector spin:

$$\frac{1}{2} : \{ \hat{a}_{-1}^*(\mathbf{p}) \hat{a}_{-1}(\mathbf{p}) \hat{v}^+(p) \sigma^3 v^-(p) + \hat{a}_{+1}^*(\mathbf{p}) \hat{a}_{+1}(\mathbf{p}) \hat{v}^-(p) \sigma^3 v^+(p) \} :. \quad (13.15)$$

This is valid for the reference frame in which  $p_1 = p_2 = 0$ , i.e., when the  $x^3$  axis lies along the momentum. In this frame, the normalized spinors satisfying

$$\hat{p} v^\pm(p) = \hat{v}^\pm(p) \hat{p} = 0 \quad (13.16)$$

and the subsidiary conditions

$$(1 + \gamma^5) v^\pm(p) = \hat{v}^\pm(p) (1 + \gamma^5) = 0, \quad (13.17)$$

can be taken in the form [we have in mind here the representation of the Dirac matrices given by (6.18)]

$$v^+(\mathbf{p}) = v^-(\mathbf{p}) = \frac{1}{\sqrt{2}} \begin{pmatrix} w(\mathbf{p}) \\ w(\mathbf{p}) \end{pmatrix} = v(\mathbf{p}), \quad \bar{v}^*(\mathbf{p}) = \frac{1}{\sqrt{2}} \begin{pmatrix} \bar{w}^*(\mathbf{p}) \\ \bar{w}^*(\mathbf{p}) \end{pmatrix}, \quad (13.18)$$

where  $w(\mathbf{p}) = \Phi_2^\pm(\mathbf{p})$  are two-component spinors satisfying the Weyl equation (7.50)

$$(p^0 + \sigma_3 p^3) w(\mathbf{p}) = 0, \quad (13.19)$$

of the form

$$w(\mathbf{p}) = \frac{1}{2} \begin{pmatrix} 0 \\ 1 + p^3/p^0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \quad (13.20)$$

We have used the fact that, for a massless neutrino,  $p^3 = p^0$ . It is also obvious that

$$\bar{w}^*(\mathbf{p}) = (0, 1),$$

so that

$$\bar{v}^{\pm*}(\mathbf{p}) \sigma_3 v^\mp(\mathbf{p}) = 1.$$

Substituting this in (13.15), we find that

$$S_3 \sim -\frac{1}{2} \bar{a}_{-1}^*(\mathbf{p}) a_{-1}(\mathbf{p}) + \frac{1}{2} \bar{a}_{-1}^*(\mathbf{p}) \bar{a}_{-1}^*(\mathbf{p}). \quad (13.21)$$

The first term in this expression corresponds to the left-handed neutrino (negative helicity) and the second corresponds to the right-handed antineutrino. The helicity of the antineutrino is +1.

The expression given by (13.21) corresponds to (13.10) for  $a_1 = 0, a_2 = a_{-1}$ . If instead of the function  $v = \psi_-$  we were to consider the function  $\psi_+$ , we would obtain the analog of (13.10) for  $a_2 = 0$ , which describes particles with positive helicity and antiparticles with negative helicity.

**13.4. Charge Conjugation.** We now introduce the operation of charge conjugation of a spinor field. If we consider the fields describing charged particles which we discussed earlier, i.e., the complex scalar and the complex vector fields, then it follows from the structure of the dynamic variables and the commutation relations that the transformation

$$\varphi(x) \rightarrow \varphi'(x) = \bar{\varphi}^*(x), \quad \bar{\varphi}(x) \rightarrow \bar{\varphi}'(x) = \varphi(x) \quad (13.22)$$

to the new wave functions  $\varphi'(x)$ ,  $\bar{\varphi}'(x)$  leaves all the physical quantities, with the exception of the current, unchanged, and changes the sign of the current four-vector. Thus the transformation (13.22) corresponds to a transition to particles having charges of opposite sign as compared with the original particles.



A transformation such as (13.22) is therefore called the transformation of *charge conjugation*. The charge conjugation of a spinor field is more complicated than (13.22) because of the matrix character (multicomponent nature) of spinor wave functions; it is a matrix transformation of the following form:

$$\psi(x) = C\bar{\psi}'(x), \quad \bar{\psi}(x) = C^{-1}\psi'(x) = \psi'(x) C^{\text{T}-1}. \quad (13.23)$$

For consistency in (13.23), we must have

$$C^{\text{T}}\gamma_0 C^{\dagger}\gamma_0 = 1. \quad (13.24)$$

The abbreviated method of writing it is equivalent to the following statement in terms of components:

$$\psi_{\alpha} = \sum_{\beta} C_{\alpha\beta} \bar{\psi}'_{\beta}, \quad \bar{\psi}_{\alpha} = \sum_{\beta} C_{\alpha\beta}^{\dagger} \psi'_{\beta} = \sum_{\beta} \psi'_{\beta} (C^{-1})_{\beta\alpha},$$

since in virtue of the definition of a transposed matrix we always have

$$(C)_{\alpha\beta} = (C^{\text{T}})_{\beta\alpha}.$$

The transformation inverse to (13.23) is given similarly by

$$\psi'(x) = C\bar{\psi}(x), \quad \bar{\psi}'(x) = C^{-1}\psi(x) = \psi(x) C^{\text{T}-1}. \quad (13.25)$$

We thus see that charge conjugation defined by (13.23) in addition to the obvious property that the twice-iterated transformation is the identity transformation, also has the property of “mirror” symmetry, i.e., its form coincides with that of the inverse transformation.

We shall determine the explicit form of the matrices  $C$  by requiring that the free-field Lagrangian and, consequently, the energy-momentum four-vector, should not change their form, while the current four-vector should change its sign, i.e., that the following relations should hold:

$$\mathcal{L}(\psi) = \mathcal{L}(\psi'), \quad T(\psi) = T(\psi') \text{ and } J^k(\psi) = -J^k(\psi').$$

For this it will be sufficient to demand that the following two relations should hold:

$$:\bar{\psi}_1(x) \gamma^k \psi_2(x): = -\bar{\psi}'_2(x) \gamma^k \psi'_1(x): \quad (13.26)$$

and

$$:\bar{\psi}(x)\psi(x): = :\bar{\psi}'(x)\psi'(x):, \quad (13.27)$$

where  $\psi_1(x)$  and  $\psi_2(x)$  are either equal to the function  $\psi$  itself, or to its derivatives  $\partial\psi/\partial x^k$ . In any case

$$\psi_i = C\bar{\psi}'_i(x), \quad \bar{\psi}_i = \psi'_i C^{-1} \quad (i = 1, 2). \quad (13.28)$$

Indeed, if we first set

$$\psi_1(x) = \psi(x), \quad \psi_2(x) = \frac{\partial\psi}{\partial x},$$

and then

$$\psi_1 = \frac{\partial\psi}{\partial x}, \quad \psi_2 = \psi,$$

we obtain from (12.26)

$$\begin{aligned} :\bar{\psi}(x)\gamma^k\frac{\partial\psi}{\partial x^k}: &= -:\frac{\partial\bar{\psi}'(x)}{\partial x^k}\gamma^k\psi'(x):, \\ :\frac{\partial\bar{\psi}}{\partial x^k}\gamma^k\psi(x): &= -:\bar{\psi}'(x)\gamma^k\frac{\partial\psi'}{\partial x^k}:, \end{aligned}$$

which together with (13.27) give:

$$\mathcal{L}(\psi) = \mathcal{L}(\psi') \quad \text{and} \quad T(\psi) = T(\psi').$$

Setting in (13.26)

$$\psi_1 = \psi \quad \text{and} \quad \bar{\psi}_2 = \bar{\psi},$$

we also obtain

$$J(\psi) = -J(\psi').$$

Let us now examine the restrictions imposed on the matrix  $C$  by (13.26) and (13.27). Substituting (13.28) into (13.26) we obtain:

$$:\bar{\psi}_1(x)\gamma^k\psi_2(x): = :\psi'_1(x)C^{-1}\gamma^k C\bar{\psi}'_2(x):.$$

Since the quantized spinors  $\psi'$  and  $\bar{\psi}'$  anticommute, we obtain from the preceding the expression

$$:\bar{\psi}(x) \gamma^k \psi_2(x): = - : \bar{\psi}'_2(x) C \gamma^k C^{-1} \psi'_1(x) :,$$

the comparison of which with (13.26) yields the first condition for the matrix  $C$

$$C \gamma^k C^{-1} = \gamma^k$$

or, in transposed form,

$$C^{-1} \gamma^k C = \gamma^k. \quad (13.29)$$

Similarly, by substituting (13.28) into (13.27), we obtain

$$:\bar{\psi}(x) \psi(x): = : \psi'(x) C^{-1} C \bar{\psi}'(x) : = - : \bar{\psi}'(x) C C^{-1} \psi'(x) :,$$

from which we find the second condition for the matrix  $C$ :

$$C C^{-1} = -1 \quad (13.30)$$

or

$$C = -C. \quad (13.31)$$

Utilizing (13.31), we may also write (13.29) in the form

$$C^{-1} \gamma^k C = -\gamma^k. \quad (13.32)$$

From equation (13.32) we can determine the form of the matrix  $C$  in the representation of the Dirac matrices  $\gamma^k$  (6.18). In this representation

$$\gamma^0 = \gamma^0, \quad \gamma^1 = -\gamma^1, \quad \gamma^2 = \gamma^2, \quad \gamma^3 = -\gamma^3.$$

We now note that equation (13.32) on being written in the form

$$C^{-1} \gamma^k C = \pm \gamma^k \quad \begin{array}{ll} (+ \text{ for } & k = 1, 3, \\ - \text{ for } & k = 0, 2), \end{array}$$

coincides with equation (6.26), which defines the form of the matrix  $\Lambda$  of the transformation of reflection of the  $x^0$  and  $x^2$  axes. Therefore, the matrix  $C$  in the representation (6.18) coincides with the matrix  $\Lambda$  of the transformation indicated above, i.e. (see (6.33)):

$$C = \Lambda_{02}(-i\pi) = e^{\frac{i\pi}{2} \gamma^0 \gamma^2} = \gamma^0 \gamma^2 = \alpha_2. \quad (13.33)$$

In the representation given by (6.18),  $\alpha_2$  has the form

$$\alpha_2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}.$$

This means that (13.24) and (13.31) are also satisfied.

As previously indicated, the specific form of the matrix  $C$  depends on the representation of the Dirac-matrices. In going over from the representation (6.18) to some other  $\gamma'^k$  by means of the formulas

$$\gamma'^k = O \gamma^k O^{-1}, \quad (13.34)$$

we obtain the new matrix  $C'$  for the transformation of charge conjugation from the relation

$$C' = O C O^T, \quad (13.35)$$

whose validity may easily be verified by substituting (13.34), (13.35) into (13.29) and (13.31).

## §14. CPT Theorem

In conclusion of our examination of quantized free fields, we must establish one of the most important properties of local quantum field theory, namely, the property of invariance under the product of the three discrete symmetric transformations: charge conjugation  $C$ , space inversion  $P$ , and time reversal  $T$ . These transformations were discussed above (see, for example, §§6.4, 10.2, and 13.4) in a purely classical fashion. We shall now need their operator formulation.

**14.1. Charge Conjugation in the Quantized Case.** Similarly to the way transformations from the Lorentz group  $L$  are placed in correspondence with unitary operators  $U_L$  which transform state vectors into field operators in accordance with (9.9), the transformation  $C$  can also be associated with a unitary operator  $U_C$ . For the complex scalar field, the transformation law is

$$\begin{aligned}\varphi(x) &\rightarrow \varphi_C(x) = U_C^{-1} \varphi(x) U_C = \eta_\varphi(C) \varphi^*(x), \\ \varphi^*(x) &\rightarrow \varphi_C^*(x) = U_C^{-1} \varphi^*(x) U_C = \eta_\varphi^*(C) \varphi(x),\end{aligned}\quad (14.1)$$

where  $\eta_\varphi, \eta_\varphi^*$  are numerical factors.

Since repeated transformation is equivalent to identity

$$U_C^2 = 1, \quad (14.2)$$

the unitary operator  $U_C$  is Hermitian.

It also follows from (14.2) that

$$|\eta_\varphi(C)|^2 = 1.$$

If the field  $\varphi$  is a real, then

$$\varphi_C(x) = U_C^{-1} \varphi(x) U_C = \eta_\varphi(C) \varphi(x) \quad (14.3)$$

and the phase factor  $\eta_\varphi$  is real and

$$\eta_\varphi(C) = \pm 1.$$

This can also be represented as the eigenvalue of the operator  $U_C$  for the single-particle state

$$\Phi_1(x) = \varphi(x) \Phi_0.$$

In point of fact, applying the operator equation

$$U_C \varphi(x) = \eta_\varphi(C) \varphi(x) U_C$$

which follows from (14.3), to the vacuum amplitude  $\Phi_0$ , and using the fact that  $U_C \Phi_0 = \Phi_0$ , we find that

$$U_C \Phi_1(x) = \eta_\varphi(C) \Phi_1(x). \quad (14.4)$$

When  $\eta_\varphi(C) = 1$ , the field is called *charge even* and when  $\eta_\varphi(C) = -1$  it is said to be *charge odd*. The quantum number  $\eta_\varphi$  can be called the *charge* (or *C*) *parity* of neutral particles described by the field  $\varphi$ .

Analogous relationship can be written for the vector and the electromagnetic fields. It was shown above that the four-vector corresponding to the current of charged particles changes sign under charge conjugation. It follows that the requirement of invariance of the Lagrangian for the electromagnetic interaction under charge conjugation ensures that the electromagnetic field is *C*-odd.

The operator  $U_C$  for the spinor field is defined by

$$\begin{aligned}\psi(x) &\rightarrow \psi_C(x) = U_C^{-1} \psi(x) U_C = \eta_\psi(C) \bar{\psi}^T, \\ \bar{\psi}(x) &\rightarrow \bar{\psi}_C(x) = U_C^{-1} \bar{\psi}(x) U_C = -\eta_\psi(C) \psi^T C^{-1},\end{aligned}\quad (14.5)$$

Using (14.1)–(14.5) and the expressions for the field operators in terms of the particle annihilation and creation operators, we can readily find the transformation law for states with a given number of particles. It is clear that application of the transformation defined by (13.1) and (13.5) replaces particles by antiparticles, and the state vector is multiplied by a phase factor equal to the product of the phase factors  $\eta_i(C)$  for each of the particles in the particular state.

The only exception from this general rule is connected with the neutrino. The point is that the operator  $U_C$  defined by (13.5) takes the left-handed neutrino [i.e., a neutrino function  $\nu(x)$  satisfying the subsidiary condition (7.52)] to the left-handed state which does not satisfy (7.54). To obtain the right-handed antineutrino from the left-handed neutrino we must follow the  $C$  transformation by the  $P$  transformation.

**14.2.  $P$ -transformation.** We now define the unitary operators  $U_P$  corresponding to the inversion of all three space axes. For a spinless field, the operator  $U_P$  is defined by

$$\varphi(x^0, \mathbf{x}) \rightarrow \varphi_P(x^0, \mathbf{x}) = U_P^{-1} \varphi(x^0, \mathbf{x}) U_P = \eta_\varphi(P) \varphi(x^0, -\mathbf{x}). \quad (14.6)$$

Since  $U_P^2 = 1$ , the numerical factor  $\eta_\varphi(P)$  is equal to  $+1$  (the field  $\varphi$  is a scalar in this case) or  $-1$  (the field  $\varphi$  is a pseudoscalar). It is called the *internal parity*, or simply *parity*, of the particles described by the field  $\varphi$ . The application of the operator  $U_P$  to the state of one particle with momentum  $\mathbf{k}$  takes this particle to the state with momentum  $-\mathbf{k}$  and multiplies it by  $\eta_\varphi(P)$ . The internal parity of a state containing several particles is, obviously, equal to the product of the internal parities of each of the particles. The operators  $U_P$  for the vector and electromagnetic fields can be defined in an analogous fashion:

$$\begin{aligned}U^n(x^0, \mathbf{x}) &\rightarrow U_P^n(x^0, \mathbf{x}) = U_P^{-1} U^n(x^0, \mathbf{x}) U_P = \\ &= \eta_U(P) g^{nm} U_m(x^0, -\mathbf{x}) = \eta_U(P) U_n(x^0, -\mathbf{x}), \\ A^n(x^0, \mathbf{x}) &\rightarrow A_P^n(x) = U_P^{-1} A^n(x) U_P = \eta_A(P) A_n(x^0, -\mathbf{x}).\end{aligned}\quad (14.7)$$

From the invariance of the electromagnetic interaction under the  $P$ -transformation it obviously follows that

$$\eta_A(P) = 1.$$

It follows from (6.34) that the unitary operator  $U_P$  for the spinor field is defined by

$$\begin{aligned}\psi(x) &\rightarrow \psi_P(x) = U_P^{-1} \psi(x) U_P = \eta_\psi(P) \gamma^0 \psi(x^0, -\mathbf{x}), \\ \bar{\psi}(x) &\rightarrow \bar{\psi}_P(x) = U_P^{-1} \bar{\psi}(x) U_P = \eta_\psi^*(P) \bar{\psi}(x^0, -\mathbf{x}) \gamma^0.\end{aligned}\quad (14.8)$$

We note that, since the spinor representation is two-valued, a double inversion of the space axes may, in general, change the sign of the spinor  $\psi$ . In fact, a double  $P$ -transformation can be written not only in the form of the identity transformation but also, for example,

as rotation through the angle of  $2\pi$  of the  $x^1x^2$  plane around the  $x^3$  axis, so that, in accordance with (6.31), the spinor will change sign. In contrast to the case of the scalar and vector fields, for the spinor field  $\eta_\psi^2(P) = +1$  or  $-1$ , i.e.,  $\eta_\psi(P) = \pm 1$  or  $\pm 1$ . However, we shall agree henceforth to take  $\eta_\psi(P) = \pm 1$  for the spinor fields. When this choice is adopted, the operator  $U_P$  is not only unitary but also Hermitian, since  $U_P^2 = 1$ . It is readily verified that with this choice of  $\eta_\psi$ , the fields  $\psi$  and  $\psi_C$  will transform in different ways. In fact,

$$\begin{aligned}\psi_C \rightarrow (\psi_C)_P &= U_P^{-1} \psi_C U_P = \eta_\psi(C) \eta_\psi(P) \bar{\psi} \gamma^0 C^T = - \\ &= -\eta_\psi(P) \eta_\psi(C) \bar{\psi} C^T \gamma^0 = -\eta_\psi(P) \gamma^0 \psi_C,\end{aligned}$$

from which it follows that fermions and antifermions have opposite parities [to verify this it is sufficient to obtain from (14.4) the transformation law for the single-particle state].

As an exercise, the reader should verify that  $CP$  will transform a left-handed neutrino into a right-handed antineutrino. This is why the Lagrangian for the neutrino field is not invariant under the  $C$ -transformation or the  $P$ -transformations, but is invariant under the  $CP$ -transformation.

**14.3. Time Reversal.** In contrast to  $C$ ,  $P$ , and  $CP$ , time reversal in the space of states can be achieved only with the aid of the antiunitary operator  $U_T$  [Wigner (1959)]. The antiunitary time reversal operator is defined by [compare this with the definition of the unitary operator (9.9)]

$$(U_T \Phi_1)^* (U_T \Phi_2) = \dot{\Phi}_2^* \Phi_1. \quad (14.9)$$

From (14.9) it also follows that

$$U_T(a_1 \Phi_1 + a_2 \Phi_2) = \dot{a}_1^* U_T \Phi_1 + \dot{a}_2^* U_T \Phi_2 = \dot{a}_1^* \dot{\Phi}_1 + \dot{a}_2^* \dot{\Phi}_2 \quad (14.10)$$

and

$$\dot{U}_T U_T = 1. \quad (14.11)$$

Hence norm is conserved under both the antiunitary and the unitary operators. The compatibility condition (9.15) for the antiunitary transformation will now have a slightly different form. Thus, consider the matrix element of a certain operator  $A$  between states  $\Phi'_1 = U_T \Phi_1$  and  $\Phi'_2 = U_T \Phi_2$ , and let us express it in terms of the matrix element of the transformed operator  $A'$  between states  $\Phi_1$  and  $\Phi_2$ . Using (14.10) and (14.11), we obtain

$$\dot{\Phi}_2^* A \Phi'_1 = (U_T \Phi_2)^* A U_T \Phi_1 = (U_T^{-1} U_T \Phi_2)^* U_T^{-1} A U_T \Phi_1 = \Phi_2^* (U_T^{-1} A U_T) \Phi_1 \equiv \Phi_1^* A' \Phi_2.$$

The transformation therefore has the form

$$A' = (U_T^{-1} A U_T)^\dagger \quad (14.12)$$

and the initial and final states change places in the matrix elements. The invariance of

the operator  $A$  under the  $T$ -transformation therefore means that

$$U_T^\dagger A U_T = A^\dagger.$$

This reduces to the condition that the operators  $U_T$  and  $A$  commute only when  $A$  is Hermitian, i.e.,

$$A^\dagger = A.$$

Since  $U_T$  is unitary, the  $T$ -transformation for the scalar field can be written in the form

$$\varphi(x^0, \mathbf{x}) \rightarrow \varphi_T(x^0, \mathbf{x}) = [U_T^\dagger \varphi(x^0, \mathbf{x}) U_T]^\dagger = \eta_\varphi(T) \overset{\dagger}{\varphi}(-x^0, \mathbf{x}). \quad (14.13)$$

Similarly, for the vector field

$$U^n(x^0, \mathbf{x}) \rightarrow U_T^n(x^0, \mathbf{x}) = \eta_U(T) \overset{\dagger}{U}_n(-x^0, \mathbf{x}). \quad (14.14)$$

The  $T$ -transformation for the spinor field can be written in the form

$$\psi_T(x^0, \mathbf{x}) = [U_T^\dagger \psi(x) U_T]^\dagger = \eta_\psi(T) \bar{\psi}(-x^0, \mathbf{x}) T, \quad (14.15)$$

whereas in the case of the  $P$ - and  $C$ -transformations we set

$$\overset{\dagger}{T} = T^{-1}, \quad |\eta_\psi(T)|^2 = 1.$$

Without loss of generality, we may assume that  $\gamma^0 T \gamma^0 = \lambda T$ . The field  $\bar{\psi}$  then transforms as follows:

$$\bar{\psi}_T(x^0, \mathbf{x}) = [U_T^\dagger \bar{\psi}(x^0, \mathbf{x}) U_T]^\dagger = \lambda \overset{*}{\eta}_\psi(T) T^{-1} \psi(-x^0, \mathbf{x}).$$

The properties of the matrix  $T$  can be established from the invariance of the Lagrangian for the free spinor field  $[x' = (-x^0, \mathbf{x})]$ :

$$\begin{aligned} \mathcal{L}(x') \rightarrow \mathcal{L}_T(x') &= (U_T^\dagger \mathcal{L}(x') U_T)^\dagger = \\ &= \frac{i}{2} \left( U_T^\dagger \bar{\psi}(x') \gamma^\alpha \frac{\partial \psi}{\partial x'^\alpha} U_T \right)^\dagger - \frac{i}{2} \left( U_T^\dagger \frac{\partial \bar{\psi}}{\partial x'_\alpha} \gamma_\alpha \psi(x') U_T \right)^\dagger - m (U_T^\dagger \bar{\psi}(x') \psi(x') U_T)^\dagger = \\ &= \left( U_T^\dagger \frac{\partial \psi}{\partial x'_\alpha} U_T \right)^\dagger \frac{i}{2} \gamma_\alpha^\top (U_T^\dagger \bar{\psi}(x') U_T)^\dagger - \frac{i}{2} (U_T^\dagger \bar{\psi}(x') U_T)^\dagger \gamma_\alpha^\top \left( U_T^\dagger \frac{\partial \bar{\psi}}{\partial x'_\alpha} U_T \right)^\dagger - \\ &\quad - m (U_T^\dagger \bar{\psi}(x') U_T)^\dagger (U_T^\dagger \bar{\psi}(x') U_T)^\dagger = \mathcal{L}(x). \end{aligned}$$

It is clear that the last term is equal to  $-m \bar{\psi} T \lambda T^{-1} \psi$  from which it follows that  $\lambda = 1$ . Since  $x'^n = -x_n$ , the first two terms can be written in the form



$$- \frac{i}{2} \frac{\partial \bar{\psi}}{\partial x^n} T g^{nm} \gamma^m T^{-1} \psi(x) + \frac{i}{2} \bar{\psi}(x) T g^{nm} \gamma^m T^{-1} \frac{\partial \psi}{\partial x^n}.$$

To ensure the invariance of these terms under the  $T$ -transformation, it is sufficient to demand that

$$T g^{nm} \gamma^m T^{-1} = \gamma^n.$$

(When  $n = 0$ , this condition again yields  $\lambda = 1$ .) The matrix  $T$  is usually expected to satisfy the further condition

$$T^\dagger = -T.$$

We emphasize that when the above transformations are applied, it may be supposed that the Hermitian conjugation operation (indicated by the symbol  $(\quad)^*$ ) does not act on the complex numbers  $i$ ,  $\gamma_{\alpha\beta}^n$ ,  $\dots$ , and so on. This is so because the antiunitary operator  $U_T$  satisfies the equation

$$(U_T^\dagger \lambda A U_T)^* = (\lambda^* U_T^\dagger A U_T)^* = \lambda (U_T^\dagger A U_T)$$

for any complex  $\lambda$ .

**14.4. CPT Theorem.** We shall now show how the above rules can be used to construct Lagrangians that are invariant under the  $C$ ,  $P$ , and  $T$  transformations [Grawert, Lüders, and Rollnik (1959)]. The reader will readily verify that the free-field Lagrangians are invariant under each of these operations separately (excluding the case of the free neutrino field mentioned above). As an illustration, we shall therefore consider the Lagrangian for the interaction between two spinor fields and one spinless field

$$\begin{aligned} \mathcal{L} = & g \bar{\psi}_1 \psi_2 \varphi + g' \bar{\psi}_1 \gamma_5 \psi_2 \varphi + f \bar{\psi}_1 \gamma^n \psi_2 \varphi_{;n} + f' \bar{\psi}_1 \gamma^n \gamma^5 \psi_2 \varphi_{;n} + \\ & + g^* \bar{\psi}_2 \psi_1 \varphi + g'^* \bar{\psi}_2 \gamma_5 \psi_1 \varphi + f^* \bar{\psi}_2 \gamma^n \psi_1 \varphi_{;n} + f'^* \bar{\psi}_2 \gamma^n \gamma^5 \psi_1 \varphi_{;n}. \end{aligned} \quad (14.16)$$

To simplify our presentation, we write the Lagrangian  $\mathcal{L}$  in the form  $\mathcal{L}(g, g', f, f')$ , and assume that all the phase factors  $\eta$  are real so that

$$\eta_\varphi(\dots) = \eta_0(\dots), \quad \eta_{\psi_i}(\dots) = \eta_i(\dots).$$

Using the above relationships for the  $P$ -transformation, we can readily verify that

$$\mathcal{L}(x') \rightarrow \mathcal{L}_P(x') = \eta(P) \mathcal{L}(g, -g', f, -f'), \quad \eta(P) = \eta_0(P) \eta_1(P) \eta_2(P).$$

Lagrangian is therefore invariant under  $P$  if  $g' = f' = 0$  for  $\eta(P) = +1$  or, if  $g = f = 0$ , for  $\eta(P) = -1$ . If the  $P$ -parity of the fields  $\psi_1$  and  $\psi_2$  is the same, i.e.,  $\eta_1(P) = \eta_2(P)$ , the

Lagrangian is invariant for the pseudoscalar field  $\varphi$  ( $\eta_0(P) = -1$ ) when  $g = f = 0$ , and for the scalar field ( $\eta_0(P) = 1$ ) when  $g' = f' = 0$ .

To find the correct transformation law for the Lagrangian  $\mathcal{L}$  under the  $C$ -transformation, we must take into account the anticommutation property of the spinor fields under the normal-product sign which is always implied when the Lagrangians are written down in quantum field theory. For example,

$$\bar{\psi}_1 \psi_2 \varphi \rightarrow U_C^{-1} \bar{\psi}_1 \psi_2 \varphi U_C = -\eta(C) \psi_1^T C^{-1} C \bar{\psi}_2^* \varphi = \eta(C) \bar{\psi}_2 \psi_1 \varphi, \quad \eta(C) = \eta_0(C) \eta_1(C) \eta_2(C).$$

To obtain the last equation, we use the anticommutation property of the spinors  $\psi_1$  and  $\psi_2$ . Applying the analogous transformations to all the terms in the Lagrangian, we obtain

$$\mathcal{L} \rightarrow \mathcal{L}_C = \eta(C) \mathcal{L}(g^*, -g', f^*, -f'),$$

and hence we readily obtain the invariance conditions under charge conjugation.

Finally, consider the  $T$ -transformation. Using the formulas given above, we find that

$$\mathcal{L}(x') \rightarrow \mathcal{L}_T(x') = \eta(T) \mathcal{L}(g^*, g', f^*, f').$$

If we now apply all three transformations in succession, we obtain

$$\mathcal{L}(x') \rightarrow \mathcal{L}_\Theta(x') = \eta(\Theta) \mathcal{L}(g, g', f, f'), \quad \eta(\Theta) = \eta(P) \eta(C) \eta(T),$$

where  $\Theta = P \cdot C \cdot T$ . Since the phase factor  $\eta(T)$  is not uniquely defined, we can always choose it so that  $\eta(\Theta) = 1$ . Hence it follows that the Lagrangian (14.16) from which we have demanded only that it must be Hermitian and invariant under the proper Lorentz transformations, is also invariant under  $PCT$  ( $CPT$ ,  $TCP$ , and so on). This is the essence of the Lüders-Pauli  $CPT$  theorem [for further details see Pauli (1955) and Grawert, Lüders and Rollnik (1959)]. The above discussion can readily be generalized to the case of an arbitrary Hermitian Lagrangian, written in the form of a polynomial of finite degree in the fields and their derivatives (of finite order) which transform in accordance with the irreducible representations of the proper Lorentz group.

The requirement that the interaction is local has played an essential role in the above discussion. In the axiomatic formulation of quantum field theory, this requirement can be made less stringent. The proof of the  $CPT$  theorem in the axiomatic approach has been given by Jost (1965), Streater and Wightman (1964), and Bogolyubov, Logunov, and Todorov (1969). In this approach, it is also assumed that the Lagrangian is written in the form of the normal product and there is a connection between spin and statistics: fields with integer spin commute with one another and with other fields, whereas fields with half-integer spin anticommute with one another but commute with integer-spin fields.



# Chapter 3

## SINGULAR FUNCTIONS AND REGULARIZATION

### §15. Green's Functions

In the course of quantization of free wave fields, we encountered the scalar-field Pauli-Jordan commutation function  $D$ , its frequency parts  $D^\pm$ , and the corresponding functions for the electromagnetic ( $D_0$ ), spinor ( $S$ ), and vector ( $D_{In}$ ) fields, related to  $D$  by the differential relations

$$\begin{aligned} D_0(x) &= D(x)|_{m=0}, & S_{\alpha\beta}(x) &= (i\vec{\partial} + m)_{\alpha\beta} D(x), \\ D_{In}(x) &= \left( g_{In} + \frac{1}{m^2} \frac{\partial^2}{\partial x^i \partial x^n} \right) D(x). \end{aligned} \quad (15.1)$$

These functions are the solutions of the corresponding homogeneous field equations and decompose in an invariant fashion into the sum of their frequency parts, each of which satisfies separately the corresponding homogeneous equation.

The solutions of the corresponding inhomogeneous field equations with point sources, i.e., Green's functions, play an important role in the theory of interacting fields. Here we have in mind the retarded and advanced Green's functions, well known from classical theory of interacting fields, and also the so-called causal Green's function that arises in the quantum theory of interacting fields.

These Green's functions are similar to the commutation functions in that they can be expressed in terms of the corresponding scalar-field Green's functions through differential relationships. We therefore begin by considering Green's functions for the scalar field.

15.1. *Green's functions for the scalar field.* The scalar-field Green's function  $G$  will be defined as the solution of the inhomogeneous Klein-Gordon equation

$$(\square_x - m^2) G(x) = -\delta(x). \quad (15.2)$$

To be specific, we shall take the sign in front of the  $\delta$ -function on the right-hand side of this equation to be the same as the sign in front of the mass term on the left-hand side.

The Fourier transformation then yields the following formal expression for  $G$ :

$$G(x) = \frac{1}{(2\pi)^4} \int \frac{e^{-ikx}}{m^2 - k^2} dk; \quad \tilde{G}(k) = \frac{1}{m^2 - k^2}. \quad (15.3)$$

This expression is not properly defined because we have not specified the rules for going round the poles  $k^2 = m^2$ . This is connected with the fact that the complete solution of (15.2) is written as the sum of a particular solution of the inhomogeneous equation and the solutions  $D^+$  and  $D^-$  of the homogeneous equation, taken with arbitrary coefficients. These coefficients are uniquely determined once we specify the rules for bypassing the two poles at  $k^2 = m^2$ , or if we impose boundary conditions on  $G(x)$ .

We shall show this for the retarded Green's functions satisfying the boundary condition

$$D^{\text{ret}}(x) = 0 \quad \text{for} \quad x^0 < 0. \quad (15.4)$$

To obtain  $D^{\text{ret}}$  in a form similar to (15.3), we note that when this function is multiplied by  $\exp(-\epsilon x^0)$ , where  $\epsilon > 0$ , the boundary condition (15.4) ensures that it does not acquire any additional singularities

$$D^{\text{ret}}(x) e^{-\epsilon x^0} = G_\epsilon(x), \quad (15.5)$$

so that it can be written as the limit

$$D^{\text{ret}}(x) = \lim_{\epsilon \rightarrow +0} G_\epsilon(x). \quad (15.6)$$

The definition given by (15.5) then ensures that the function  $G_\epsilon$  satisfies the equation

$$\left\{ \Delta - \left( \frac{\partial}{\partial t} + \epsilon \right)^2 - m^2 \right\} G_\epsilon(x) = -\delta(x)$$

and, therefore, in the limit as  $\epsilon \rightarrow 0$ , it assumes the following form in the momentum representation:

$$\frac{1}{m^2 - (k^0 + i\epsilon)^2 + \mathbf{k}^2} \rightarrow \frac{1}{m^2 - k^2 - 2i\epsilon k^0}.$$

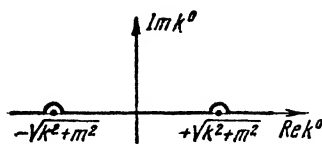


Fig. 2. Integral path for function  $D^{\text{ret}}$  in the plane of complex  $k^0$ .

In accordance with (15.6), the retarded Green's function can therefore be written in the form

$$D^{\text{ret}}(x) = \frac{1}{(2\pi)^4} \int \frac{e^{-ikx}}{m^2 - k^2 - 2i\epsilon k^0} dk. \quad (15.7)$$

It is not difficult to verify that (15.7) satisfies (15.4). All that needs to be done is to evaluate the integral with the aid of the theory of residues. The infinitesimal component  $2i\epsilon k^0$  in the denominator of (15.7) shows that both poles in the plane of complex  $k^0$  must be bypassed from above as shown in Fig. 1. For  $x^0 < 0$  therefore, if we can close the contour of integration by a large circle in the upper half plane, there will be no poles in the interior of this contour, and we obtain (15.4). For  $x^0 > 0$ , the contour of integration is closed in the lower half-plane. Evaluating the residues, we have for this case,

$$D^{\text{ret}}(x) = \frac{1}{(2\pi)^3 i} \int \frac{e^{ik^0 x^0} - e^{-ik^0 x^0}}{2k^0} \Big|_{k^0 = \sqrt{k^2 + m^2}} e^{-ikx} dk = D(x).$$

Therefore

$$D^{\text{ret}}(x) = \theta(x^0) D(x). \quad (15.8)$$

Similarly, we can show that the advanced Green's function defined by

$$D^{\text{adv}}(x) = 0 \quad \text{for} \quad x^0 > 0$$

and equation (15.2) has the form

$$D^{\text{adv}}(x) = \frac{1}{(2\pi)^4} \int \frac{e^{-ikx} dk}{m^2 - k^2 + 2i\epsilon k^0} = -\theta(-x^0) D(x). \quad (15.9)$$

**15.2. Causal Green's function for a scalar field.** A particularly important role in quantum field theory is played by the *causal Green's function*\*  $D^c(x - y)$  which describes, as is commonly stated, the *causal relationship* between the processes of creation and annihilation of particles at different space-time points  $x$  and  $y$ . We shall establish the explicit form of the function  $D^c$ , restricting ourselves at first to the case of the scalar field.

\*This was apparently introduced for the first time by Stueckelberg and Rivier (1948).

The process of creation of a scalar particle at the point  $x$  and of its subsequent annihilation at the point  $y$  is described by the matrix element

$$\hat{\Phi}_1(y) \Phi_1(x) = \hat{\Phi}_0 \varphi^-(y) \varphi^+(x) \Phi_0 = \frac{1}{i} D^-(y-x) = i D^+(x-y). \quad (15.10)$$

In the preceding equation one should obviously assume that  $y^0 > x^0$ . Conversely, if  $x^0 > y^0$ , the particle is created at the point  $y$  and annihilated at the point  $x$ . The expression

$$\hat{\Phi}_1(x) \Phi_1(y) = \frac{1}{i} D^-(x-y) \quad (15.11)$$

corresponds to this latter process.

Thus, the causal function  $D^c(x-y)$  must be proportional to the function  $D^{(-)}(x-y)$  for  $x^0 > y^0$ , while for  $x^0 < y^0$  it must be proportional to the function  $D^{(+)}(x-y)$ . In order to obtain the explicit form of this function we shall make use of the fact already noted earlier that an arbitrary solution of equation (15.2) may be represented as a sum of its particular solution and of a linear combination of the solutions of the homogeneous equation. For example, taking the retarded Green's function as such a particular solution, we shall obtain the formula for the arbitrary solution of equation (15.2) in the following form:

$$G(x) = D^{\text{ret}}(x) + a_1 D^+(x) + a_2 D^-(x),$$

where  $a_1$  and  $a_2$  are certain arbitrary numerical coefficients. Setting  $a_1 = -1$ ,  $a_2 = 0$  in this expression, we obtain

$$D^{\text{ret}}(x-y) - D^+(x-y) = \theta(x^0 - y^0) D^-(x-y) - \theta(y^0 - x^0) D^+(x-y),$$

which satisfies all the conditions imposed on  $D^c$ .

In order to obtain an expression for the causal function

$$D^c(x) = \theta(x^0) D^-(x) - \theta(-x^0) D^+(x) \quad (15.12)$$

in the momentum representation, we note that the difference  $D^{\text{ret}} - D^{(+)}$  may be represented in the form

$$\begin{aligned} \frac{1}{m^2 - k^2 - 2i\epsilon k^0} + 2\pi i \theta(-k^0) \delta(k^2 - m^2) &= \frac{\mathcal{P}}{m^2 - k^2} + (2\theta(-k^0) + \epsilon(k^0)) \pi i \delta(k^2 - m^2) = \\ &= \frac{\mathcal{P}}{m^2 - k^2} + i\pi \delta(k^2 - m^2) = \frac{1}{m^2 - k^2 - i\epsilon} \end{aligned}$$

which gives

$$D^c(x) = \frac{1}{(2\pi)^4} \int \frac{e^{-ikx}}{m^2 - k^2 - i\varepsilon} dk. \quad (15.13)$$

The causal function  $D^c$  may be expressed directly in terms of matrix elements of the form of (15.10), (15.11). In order to do this, we introduce the chronological product (or the *T-product*) of two field operators  $T(u(x)u(y))$ , which is equal to the product of these operators taken in the order from right to left, which corresponds to an increase in the time arguments, with due regard being paid to the overall sign which may change if the operators are quantized in accordance with Fermi-Dirac rules, i.e.,\*

$$T(u(x)u(y)) = \begin{cases} u(x)u(y) & \text{for } x^0 > y^0, \\ \pm u(y)u(x) & \text{for } x^0 < y^0 \end{cases} \quad (15.14)$$

(+ for Bose-operators, - for Fermi-operators).

Calculating the matrix element of (15.14) in the vacuum state for the scalar field under consideration we shall obtain with the aid of (15.10) and (15.11):

$$\Phi_0^*(T(\varphi(x)\varphi(y))\Phi_0 = \begin{cases} \Phi_0^*\varphi(x)\varphi(y)\Phi_0 = \frac{1}{i}D^-(x-y) & \text{for } x^0 > y^0, \\ \Phi_0^*\varphi(y)\varphi(x)\Phi_0 = iD^+(x-y) & \text{for } x^0 < y^0. \end{cases}$$

Thus

$$D^c(x-y) = i\Phi_0^*T(\varphi(x)\varphi(y))\Phi_0 = i\langle T\varphi(x)\varphi(y) \rangle_0. \quad (15.15)$$

**15.3. Causal Green's functions for different fields.** From (15.13) it follows that in order that a function of the form (15.3) should have the property of "causality" it is sufficient to regard the square of the mass in the denominator of its momentum representation as containing an infinitesimal negative imaginary additional term.

Recalling that in accordance with (15.1) the commutation functions for the electromagnetic, the spinor, and the vector fields differ from the Pauli-Jordan function only by differential operators, we shall suppose that the causal functions of the above fields are so *defined* that differ from  $D^c$  by the same operators, i.e.,

$$D_\alpha^c(x) = D^c(x)|_{m=0} = -\frac{1}{(2\pi)^4} \int \frac{e^{-ikx}}{k^2 + i\varepsilon} dk, \quad (15.16)$$

$$S_{\alpha\beta}^c(x) = (i\hat{\partial} + m)_{\alpha\beta}D^c(x) = \frac{1}{(2\pi)^4} \int \frac{(m + \hat{p})_{\alpha\beta}}{m^2 - p^2 - i\varepsilon} e^{-ipx} dp, \quad (15.17)$$

\*For a more detailed investigation of *T*-products and of their various properties, refer to the theory of the scattering matrix (Chapter 4).



$$D_{nl}^c(x) = \left( g_{nl} + \frac{1}{m^2} \frac{\partial^2}{\partial x^n \partial x^l} \right) D^c(x) = \frac{1}{(2\pi)^4} \int \frac{\left( g_{nl} - \frac{k_n k_l}{m^2} \right)}{m^2 - k^2 - i\varepsilon} e^{-ikx} dk. \quad (15.18)$$

The causal functions defined in this way may be expressed by analogy with the function  $D^c$  in terms of the expectation values over the vacuum state of the chronological products of the operators of the corresponding fields:

$$-D_{mn}^c(x-y) = i\Phi_0^* T(U_m(x) U_n(y)) \Phi_0, \quad (15.19)$$

$$-g^{mn} D_0^c(x-y) = i\Phi_0^* T(A_m(x) A_n(y)) \Phi_0, \quad (15.20)$$

$$S_{\alpha\beta}^c(x-y) = i\Phi_0^* T(\psi_\alpha(x) \bar{\psi}_\beta(y)) \Phi_0. \quad (15.21)$$

The formulas given by (15.19) and (15.20) are obvious consequences of (15.1). To prove (15.21), we use the spinor-field commutators (12.2) and (12.3). This yields

$$i(\Phi_0^* T(\psi(x) \bar{\psi}(y)) \Phi_0) = \begin{cases} i(\Phi_0^* \psi^-(x) \bar{\psi}^+(y) \Phi_0) = (i\hat{\partial} + m) \frac{1}{i} D^-(x-y) & \text{for } x_0 > y_0, \\ i(\Phi_0^* \bar{\psi}^-(y) \psi^+(x) \Phi_0) = (i\hat{\partial} + m) i D^+(x-y) & \text{for } x_0 < y_0, \end{cases}$$

which in view of (15.17) is equivalent to (15.21).

It is also clear that if in the denominators of (15.16)–(15.18) we replace  $-i\varepsilon$  by  $2i\varepsilon k^0$ , we obtain expressions which reduce to zero for  $x^0 < y^0$  and which consequently represent the retarded Green's functions for the corresponding fields.

We note that one should exercise a certain degree of caution in representing Green's functions of the various fields in terms of the commutation functions by relations of the type (15.8), (15.9), and (15.12). Thus, for example, the causal function of the vector field might have been written down in analogy with (15.12) in the form

$$\theta(x^0) D_{mn}^-(x) - \theta(-x^0) D_{mn}^+(x), \quad (15.22)$$

where

$$D_{ln}^\pm(x) = \frac{\pm 1}{(2\pi)^3 i} \int e^{ikx} (\pm k^0) \delta(k^2 - m^2) \left( g_{ln} - \frac{k_l k_n}{m^2} \right) dk.$$

However, a direct evaluation of expression (15.22), which may easily be carried out with the aid of the integral representation of the  $\theta$ -function:

$$\theta(\pm x^0) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{e^{ix^0\tau}}{\varepsilon \pm i\tau} d\tau,$$

leads to the expression

$$D_{ln}^c(x) - \frac{\delta_{l0}\delta_{n0}}{m^2} \delta(x),$$

which differs from  $D_{ln}^c(x)$  in the neighborhood of the point  $x = 0$ . Such a result is a consequence of an actual indefiniteness of expression (15.22) in an infinitesimal neighborhood of the point  $x = 0$ .

This indefiniteness of the causal function may be seen particularly simply if the function is written in the form

$$i\Phi_0^* T(U_l(x) U_n(y)) \Phi_0. \quad (15.23)$$

Indeed, expression (15.23) in accordance with (15.14) is uniquely defined for  $x^0 < y^0$ , and also for  $y^0 < x^0$ . Its meaning is also clear for  $(x - y)^2 = -(x - y)^2 \neq 0$ , since (15.23) may be written either in the form

$$i\Phi_0^* U_l(x) U_n(y) \Phi_0,$$

or in the form

$$i\Phi_0^* U_n(y) U_l(x) \Phi_0$$

since, for  $(x - y)^2 < 0$  (refer to the discussion of the properties of the  $D$  and  $D^c$ -functions in §16) of the operators  $U(x)$  and  $U(y)$  commute:

$$[U_l(x), U_n(y)]_- = \frac{1}{i} D_{ln}(x - y) |_{(x-y)^2 < 0} = 0,$$

and the two forms are equivalent. However, the meaning of (15.4) is entirely unclear when the arguments are equal ( $x = y$ ). From this it follows that the  $T$ -product is not defined at the point  $x = y$ , and that using different methods for its actual construction one can obtain expressions which differ from each other by terms proportional to  $\delta(x - y)$  and its derivatives.

We shall encounter this circumstance once again in the theory of interacting fields (Chapter 4), where it will be investigated in greater detail. To avoid any uncertainty, we shall agree for the time being to define chronological products of field operators not by (15.22), but by (15.16)–(15.18), in the form of Green's functions of inhomogeneous equations of the corresponding fields:

$$\begin{aligned} \square D_0^c(x) &= -\delta(x), \\ (i\hat{\sigma} - m) S^c(x) &= -\delta(x), \\ (\square - m^2) D_{nl}^c(x) &= -\left(g^{nl} + \frac{1}{m^2} \frac{\partial^2}{\partial x^n \partial x^l}\right) \delta(x). \end{aligned}$$

The form of the right-hand side of the last equation is determined by the requirement of compatibility of the causal function of the vector field with the subsidiary condition

$$\frac{\partial U_n}{\partial x_n} = 0.$$

Analogous relations between the commutation and the causal functions may be established for an arbitrary field. In order to derive them it is convenient to start with the field equations written in Kemmer's form (see §4.4):

$$\left(i\Gamma^n \frac{\partial}{\partial x^n} - m\right) u(x) = 0.$$

Setting up the Lagrangian in the usual manner, and then with its aid the energy-momentum four-vector, we obtain by using the method indicated in §10 the commutation functions for the field  $\Delta_{\alpha\beta}$ . The result for  $\Delta_{\alpha\beta}(x)$  is:

$$\Delta_{\alpha\beta}(x) = \frac{i}{(2\pi)^3} \int P_{\alpha\beta}(k) e^{-ikx} \delta(k^2 - m^2) \varepsilon(k^0) dk = P_{\alpha\beta} \left(i \frac{\partial}{\partial x}\right) D(x), \quad (15.24)$$

where  $P_{\alpha\beta}(k)$  is a certain polynomial in  $k$ .

Repeating the arguments of that section we shall obtain the expression for the causal function  $\Delta_{\alpha\beta}^c$  in the form

$$\Delta_{\alpha\beta}^c(x) = \frac{1}{(2\pi)^4} \int \frac{P_{\alpha\beta}(k) e^{-ikx} dk}{m^2 - k^2 - i\varepsilon} = P_{\alpha\beta} \left(i \frac{\partial}{\partial x}\right) D^c(x) \quad (15.25)$$

with the same polynomial  $P_{\alpha\beta}$ .

Because of the indeterminacy that arises when the arguments are equal, the operation of chronological ordering of field derivatives can be defined in various ways. First, the derivatives can be transferred to the coefficient functions and the field functions can be ordered prior to differentiation. Second, the derivatives can first be taken to the fields and then the product of the field derivatives can be ordered. The first operation is often called the *Wick chronological product* and is indicated by the symbol  $T_W$ . The second operation is called the *Dyson chronological product* and is denoted by  $T_D$ . The difference between  $T_W$  and  $T_D$  appears explicitly when the chronological product is reduced to the normal form, and affects the chronological pairings in the coefficient functions. In the case of  $T_W$ , the "basic" pairing is the function  $-iD^c(x-y)$  which is differentiated as a whole, i.e., for example,

$$\begin{aligned} \langle T_W(\varphi; n(x) \varphi; m(y)) \rangle_0 &\equiv \frac{\partial^2}{\partial x^n \partial y^m} \frac{1}{i} D^c(x-y) = \\ &= \frac{1}{(2\pi)^4 i} \int \frac{k^n k^m}{m^2 - k^2 - i\varepsilon} e^{-ik(x-y)} dk. \end{aligned} \quad (15.26)$$

In the second case, we use the representation (15.12) for  $D^c$  in terms of the frequency  $D^\pm$  which are in fact differentiated:

$$\begin{aligned}
\langle T_D(\varphi; n(x) \varphi; m(y)) \rangle_0 &\equiv \theta(x^0 - y^0) \langle \varphi; n(x) \varphi; m(y) \rangle_0 + \theta(y^0 - x^0) \langle \varphi; m(y) \varphi; n(x) \rangle_0 = \\
&= \theta(x^0 - y^0) \frac{\partial^2}{\partial x^n \partial y^m} \frac{1}{i} D^-(x - y) + \theta(y^0 - x^0) \frac{\partial^2}{\partial x^n \partial y^m} i D^+(x - y) = \\
&= \langle T_W(\varphi; n(x) \varphi; m(y)) \rangle_0 - i \delta_{n0} \delta_{m0} \delta(x - y). \quad (15.27)
\end{aligned}$$

## § 16. Properties of Green's Functions and Regularization

*16.1. Evaluation of  $D^+$  and  $D^-$  Functions.* We shall now investigate the commutation functions and Green's functions. In the preceding section it was shown that all of these functions may be expressed in terms of the positive- and negative-frequency parts of the Pauli-Jordan function. Therefore we begin by evaluating the integrals

$$D^+(x) = \frac{1}{(2\pi)^3 i} \int e^{ikx} \delta(k^2 - m^2) \theta(k^0) dk, \quad (16.1)$$

$$D^-(x) = \frac{i}{(2\pi)^3} \int e^{-ikx} \delta(k^2 - m^2) \theta(k^0) dk. \quad (16.2)$$

Carrying out the integration in (16.1) and (16.2) with respect to  $k^0$  and with respect to the angle variables in  $\mathbf{k}$ -space, we can rewrite these expressions in the form

$$D^+(x) = \frac{1}{4\pi r} \frac{\partial}{\partial r} f(x), \quad D^-(x) = \frac{1}{4\pi r} \frac{\partial}{\partial r} f^*(x), \quad (16.3)$$

where

$$f(x) = f(x^0, r) = \frac{i}{2\pi} \int_{-\infty}^{+\infty} \frac{e^{i(k^0 x^0 + kr)}}{k^0} dk, \quad k^0 = \sqrt{k^2 + m^2}, \quad k = \sqrt{k^2}, \quad r = \sqrt{x^2}, \quad (16.4)$$

while  $f^*(x)$  is the complex conjugate of  $f(x)$ . Thus the problem of determining  $D^{(+)}$  and  $D^{(-)}$  has been reduced to the evaluation of the one integral (16.4). By introducing the new variables

$$k = m \sinh \varphi, \quad k^0 = m \cosh \varphi, \quad (16.5)$$

in the above expression we obtain:

$$f(x) = \frac{i}{2\pi} \int_{-\infty}^{+\infty} d\varphi e^{im(x^0 \cosh \varphi + r \sinh \varphi)}. \quad (16.6)$$

Here four cases should be distinguished:

$$\begin{array}{ll} 1) & x^0 > 0, \quad x^0 > r, \quad 2) & x^0 > 0, \quad x^0 < r, \\ 3) & x^0 < 0, \quad |x^0| > r, \quad 4) & x^0 < 0, \quad |x^0| < r. \end{array}$$

Carrying out the respective substitutions, we have

$$\begin{array}{ll} 1) & x^0 = \sqrt{\lambda} \cosh \varphi_0, \quad r = \sqrt{\lambda} \sinh \varphi_0, \quad 2) & x^0 = \sqrt{-\lambda} \sinh \varphi_0, \quad r = \sqrt{-\lambda} \cosh \varphi_0, \\ 3) & x^0 = -\sqrt{\lambda} \cosh \varphi_0, \quad r = \sqrt{\lambda} \sinh \varphi_0, \quad 4) & x^0 = -\sqrt{-\lambda} \sinh \varphi_0, \quad r = \sqrt{-\lambda} \cosh \varphi_0 \end{array}$$

(here the notation  $\lambda = x^2 = (x^0)^2 - r^2$  has been introduced), and using the well-known integral representation for the cylindrical functions, we find that

$$\begin{array}{ll} 1) & \frac{1}{2\pi i} \int_{-\infty}^{+\infty} d\varphi e^{im\sqrt{\lambda} \cosh(\varphi + \varphi_0)} = \frac{1}{2} H_0^{(1)}(m\sqrt{\lambda}) = \frac{1}{2} J_0(m\sqrt{\lambda}) + \frac{i}{2} N_0(m\sqrt{\lambda}), \\ 2) & \frac{i}{2\pi} \int_{-\infty}^{+\infty} d\varphi e^{im\sqrt{-\lambda} \sinh(\varphi + \varphi_0)} = \frac{i}{\pi} K_0(m\sqrt{-\lambda}), \\ 3) & \frac{i}{2\pi} \int_{-\infty}^{+\infty} d\varphi e^{-im\sqrt{\lambda} \cosh(\varphi - \varphi_0)} = \frac{1}{2} H_0^{(3)}(m\sqrt{\lambda}) = \frac{1}{2} J_0(m\sqrt{\lambda}) - \frac{i}{2} N_0(m\sqrt{\lambda}), \\ 4) & \frac{i}{2\pi} \int_{-\infty}^{+\infty} d\varphi e^{-im\sqrt{-\lambda} \sinh(\varphi - \varphi_0)} = \frac{i}{\pi} K_0(m\sqrt{-\lambda}), \end{array}$$

from which it follows that

$$f(x) = \varphi(x^0, \lambda) = \begin{cases} \frac{1}{2i} N_0(m\sqrt{\lambda}) - \frac{1}{2} \varepsilon(x^0) J_0(m\sqrt{\lambda}) & \text{for } \lambda > 0, \\ \frac{i}{\pi} K_0(m\sqrt{-\lambda}) & \text{for } \lambda < 0. \end{cases} \quad (16.7)$$

Here  $J_0(z)$  is the Bessel function of order zero,  $N_0(z)$  is the Neumann function of order zero, and  $K_0(z)$  is the Hankel function of an imaginary argument of order zero, all defined in the usual way:

$$\begin{aligned} J_0(z) &= \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k}}{2^{2k} (k!)^2}, \\ \pi N_0(z) &= 2J_0(z) \left( \ln \frac{z}{2} + C \right) - 2 \sum_{k=1}^{\infty} \frac{(-1)^k z^{2k}}{2^{2k} (k!)^2} \left[ \sum_{m=1}^k \frac{1}{m} \right], \\ K_0(z) &= \frac{\pi i}{2} H_0^{(1)}(iz) = \frac{\pi i}{2} [J_0(iz) + iN_0(iz)] \end{aligned}$$

( $C$  is Euler's constant, equal to  $0.577215 \dots$ ). We note that in the neighborhood of the point  $z = 0$ , these functions may be represented in the form

$$\left. \begin{aligned} J_0(z) &= 1 - \left(\frac{z}{2}\right)^2 + O(z^4), \\ N_0(z) &= \frac{2}{\pi} \left[ 1 - \left(\frac{z}{2}\right)^2 \right] \ln \frac{z}{2} + \frac{2}{\pi} C + O(z^2), \\ K_0(z) &= - \left[ 1 + \left(\frac{z}{2}\right)^2 \right] \ln \frac{z}{2} - C + O(z^2) \end{aligned} \right\} \quad (16.8)$$

and that for large  $z$  the following asymptotic formulas hold:

$$\left. \begin{aligned} J_0(z) &= \sqrt{\frac{2}{\pi z}} \cos\left(z - \frac{\pi}{4}\right), \\ N_0(z) &= \sqrt{\frac{2}{\pi z}} \sin\left(z - \frac{\pi}{4}\right), \\ K_0(z) &= \sqrt{\frac{\pi}{2z}} e^{-z} \end{aligned} \right\} (z \gg 1). \quad (16.9)$$

Replacing differentiation with respect to  $r'$  by differentiation with respect to  $\lambda$  in (16.3), and taking into account the discontinuity of the function  $f$  at the point  $\lambda = 0$  (compare (16.8)), we obtain from (16.7) the following expressions for  $D^{(+)}$  and  $D^{(-)}$ :

$$\begin{aligned} D^+(x) &= \frac{1}{4\pi} \varepsilon(x^0) \delta(\lambda) - \frac{mi}{8\pi \sqrt{\lambda}} \theta(\lambda) [N_1(m\sqrt{\lambda}) - i\varepsilon(x^0) J_1(m\sqrt{\lambda})] - \\ &\quad - \theta(-\lambda) \frac{mi}{4\pi^2 \sqrt{-\lambda}} K_1(m\sqrt{-\lambda}), \end{aligned} \quad (16.10)$$

$$\begin{aligned} D^-(x) &= \frac{1}{4\pi} \varepsilon(x^0) \delta(\lambda) - \theta(\lambda) \varepsilon(x^0) \frac{m}{8\pi \sqrt{\lambda}} J_1(m\sqrt{\lambda}) + \\ &\quad + \frac{mi}{8\pi \sqrt{\lambda}} \theta(\lambda) N_1(m\sqrt{\lambda}) + \theta(-\lambda) \frac{mi}{4\pi^2 \sqrt{-\lambda}} K_1(m\sqrt{-\lambda}). \end{aligned} \quad (16.11)$$

Formulas (16.10) and (16.11) will be our starting point for the investigation of the singularities of all the functions of interest to us.

**16.2. Explicit Form and Singularities of the Functions  $D(x)$  and  $D^c(x)$ .** Adding (16.10) and (16.11), we obtain the Pauli-Jordan function in the form

$$D(x) = \frac{1}{2\pi} \varepsilon(x^0) \delta(\lambda) - \frac{m}{4\pi \sqrt{\lambda}} \theta(\lambda) \varepsilon(x^0) J_1(m\sqrt{\lambda}). \quad (16.12)$$

From this formula it follows that the Pauli-Jordan function has an extremely important property: *it vanishes outside the light cone* (for  $\lambda < 0$ ). Therefore, all the (anti) commutators of the field operators whose arguments differ by a space-like interval also vanish. Physically, this corresponds to the fact that events separated by a space-like interval are independent, since the speed of propagation of a signal cannot exceed the speed of light.

To obtain the causal function, we shall use (16.10) and (16.11) to construct the combination

$$\theta(x^0) D^-(x) - \theta(-x^0) D^+(x).$$

This gives

$$D^c(x) = \frac{1}{4\pi} \delta(\lambda) - \frac{m}{8\pi\sqrt{\lambda}} \theta(\lambda) \{J_1(m\sqrt{\lambda}) - iN_1(m\sqrt{\lambda})\} + \frac{mi}{4\pi^2\sqrt{-\lambda}} \theta(-\lambda) K_1(m\sqrt{-\lambda}). \quad (16.13)$$

This formula can also be written in the more compact form

$$D^c(x) = \frac{1}{4\pi} \delta(\lambda) + \frac{mi}{4\pi^2} \cdot \frac{K_1(m\sqrt{-\lambda+i\varepsilon})}{\sqrt{-\lambda+i\varepsilon}}, \quad (16.14)$$

where

$$\sqrt{-\lambda+i\varepsilon} = i\sqrt{\lambda} \quad \text{for} \quad \lambda > 0.$$

Finally for the retarded Green's function we obtain with the aid of (15.8) and (16.12) the expression:

$$D^{\text{ret}}(x) = \frac{\theta(x^0)}{2\pi} \left\{ \delta(\lambda) - \frac{m}{2\sqrt{\lambda}} \theta(\lambda) J_1(m\sqrt{\lambda}) \right\}. \quad (16.15)$$

It is now evident that  $D^{\text{ret}}$  vanishes everywhere, except for the light cone directed into the future.

We now turn to the investigation of the singularities of commutation functions and Green's functions. From (16.10), (16.11), (16.13), (16.14), and (16.15) it follows that all the singularities of the functions under investigation are situated only on the light cones (for  $\lambda = 0$ ), since when the argument tends to infinity in either the space-like or the time-like directions, these functions in accordance with (16.9) fall off like

$$|\lambda|^{-3/4} \exp(-m\sqrt{|\lambda|})$$

and  $\lambda^{-3/4}$ , respectively.

To investigate the regularity properties, we shall therefore examine the behavior of the above functions in the neighborhood of the light cone. Utilizing (16.16)-(16.19) for this purpose, we find that

$$D^\pm(x) = \frac{1}{4\pi} \varepsilon(x^0) \delta(\lambda) \pm \frac{i}{4\pi^2\lambda} \mp \frac{im^2}{8\pi^2} \ln \frac{m|\lambda|^{1/2}}{2} - \frac{m^2}{16\pi} \varepsilon(x^0) \theta(\lambda) + O(\sqrt{|\lambda|} \ln |\lambda|), \quad (16.16)$$

$$D(x) = \frac{1}{2\pi} \varepsilon(x^0) \delta(\lambda) - \frac{m^2}{8\pi} \varepsilon(x^0) \theta(\lambda) + O(\lambda), \quad (16.17)$$

$$D^c(x) = \frac{1}{4\pi} \delta(\lambda) + \frac{1}{4\pi^2 i} \frac{1}{\lambda} - \frac{m^2}{16\pi} \theta(\lambda) + \frac{im^2}{8\pi^2} \ln \frac{m|\lambda|^{1/2}}{2} + O(\sqrt{|\lambda|} \ln|\lambda|), \quad (16.18)$$

$$D^{\text{ret}}(x) = \frac{1}{2\pi} \theta(x^0) \delta(\lambda) - \frac{m^2}{8\pi} \theta(x^0) \theta(\lambda) + \theta(x^0) \theta(\lambda) O(\lambda). \quad (16.19)$$

Expressions (16.16)–(16.19) have four types of singularities on the light cone: a pole  $1/\lambda$ ,  $\ln|\lambda|$ , a  $\delta$ -function  $\delta(\lambda)$ , and a discontinuity  $\theta(\lambda)$ . The factors  $\varepsilon(x^0)$  and  $\theta(x^0)$  in the individual terms of (16.16), (16.17), and (16.19) do not introduce any additional singularities outside the light cone since the adjoining factors  $\delta(\lambda)$  and  $\theta(\lambda)$  ensure that the discontinuities  $\varepsilon(x^0)$  and  $\theta(x^0)$  appear only at the origin of coordinates, i.e., again on the light cone.

The above commutation functions and Green's functions for quantum wave fields are thus *singular functions with relatively strong singularities on the light cone*.

**16.3. Pauli-Villars Regularization.** It will be shown later (Chapter 4) in connection with the theory of interacting fields that we shall have to deal with expressions containing products of different numbers of the above singular functions. In certain cases (see also §18), the singularities of the individual functions occurring in such products will be superimposed on one another, leading to meaningless nonintegrable expressions. Later (Chapters 5 and 6), a method will be outlined for the removal of individual singularities from such expressions. In order to avoid infinities at intermediate stages, we shall find it convenient at such stages to replace the singular functions with certain regular approximations, and to pass to the limit (in which regularity is destroyed) only in the final expressions.

Using as our basis the preceding analysis of the singularities of commutation functions and of Green's functions, we shall now formulate a method of constructing such approximations to show these functions which, together with all their derivatives up to any given order, will have no singularities on the light cone. An example of such a procedure is afforded by the so-called Feynman regularization (Feynman (1948)) of the causal photon Green's function which consists of replacing

$$D_0^c(k) = -\frac{1}{k^2}$$

with

$$\text{reg}_F D_0^c(k) = D_0^c(k) - D_M^c(k) = -\frac{1}{k^2} - \frac{1}{M^2 - k^2} = -\frac{M^2}{k^2(M^2 - k^2)},$$



where  $M^2$  is the square of an "auxiliary mass," which is evidently equivalent to the introduction of the factor

$$\frac{M^2}{M^2 - k^2}.$$

It is easy to show with the aid of (16.18) that the function  $\text{reg}_F D_0^c$  differs from  $D_0^c$  in that it does not have on the light cone the strongest singularities  $\delta(\lambda)$  and  $\lambda^{-1}$ . On the other hand,

$$\lim_{M \rightarrow \infty} \text{reg}_F D_0^c(k) = D_0^c(k).$$

We note now that, in accordance with (16.3) and (16.7), the above singular functions may be represented in the form

$$\frac{\partial}{\partial \lambda} f_1(m^2 \lambda) = m^2 f_2(m^2 \lambda). \quad (16.20)$$

The last assertion is correct to within the factors  $\epsilon(x^0)$  and  $\theta(x^0)$  in the individual terms of the above functions. We emphasize, however, that our aim is to form combinations of commutation functions and Green's functions that have a sufficient number of regular derivatives with respect to  $\lambda$  on the light cone (or, what is the same, with respect to the components  $x^k$ ). As has been noted earlier, the singularities corresponding to  $\epsilon(x^0)$  and  $\theta(x^0)$  make themselves felt, because of the factors  $\delta(\lambda)$  and  $\theta(\lambda)$ , only on the light cone, and will therefore be cancelled by a zero with respect to  $\lambda$  of sufficiently high order together with the singularities with respect to the variable  $\lambda$ . We are therefore justified in the analysis of the singularities of the above singular functions to start with (16.20).

From this formula it follows directly that the singularities  $1/\lambda$  and  $\delta(\lambda)$  always appear with coefficients that do not depend on the mass, while the singularities in  $|\lambda|$  and  $\theta(\lambda)$  appear with coefficients proportional to  $m^2$ . Therefore, taking a linear combination of several functions  $\Delta$  (we shall here denote by the symbol  $\Delta$  any of the functions under discussion:  $D^{(+)}$ ,  $D^{(-)}$ ,  $D$ ,  $D^c$ ,  $D^{\text{ret}}$  . . .) which correspond to fields characterized by different masses  $M_i$ :

$$c_1 \Delta_{M_1}(x) + c_2 \Delta_{M_2}(x) + \dots + c_k \Delta_{M_k}(x),$$

the coefficients of which satisfy the conditions

$$\begin{aligned} c_1 + c_2 + \dots + c_k &= 0, \\ c_1 M_1^2 + c_2 M_2^2 + c_k M_k^2 &= 0, \end{aligned}$$

we shall obtain an expression which contains no singularities on the light cone.

To obtain an expression which is continuous together with all its derivatives up to order  $n - 1$  inclusive, we subject the coefficients  $c_i$  to the  $(n + 1)$ th condition:

$$\sum_i c_i = 0, \sum_i c_i M_i^2 = 0, \dots, \sum_i c_i M_i^{2n} = 0.$$

Clearly, the minimum required number of masses in the above does not exceed  $n + 2$ .

The result is an expression which in the neighborhood of  $\lambda = 0$  is a polynomial in  $\lambda$ , with coefficients containing  $\ln M_i$ , and is continuous together with its  $n - 1$  derivatives.

We note that in practical calculations (see Chapter 5) in place of a complete elimination of the singularities of singular functions, it turns out to be sufficient to reduce these singularities to ones of integrable type; this leads to a reduction in the number of required auxiliary masses.

In the above procedure, one of the auxiliary masses  $M$  is chosen equal to the fundamental mass  $m$  of the field, while the corresponding coefficient  $c$  is set equal to unity. In this way, for any given singular field function  $\Delta(x)$  with mass  $m$ , we obtain a corresponding function which is continuous together with all its derivatives up to order  $n - 2$  inclusive

$$\text{reg} \{\Delta_m(x)\} = \Delta_m(x) + \sum_{i=1}^n c_i \Delta_{M_i}(x). \quad (16.21)$$

This function contains  $n$  auxiliary masses  $M_1, \dots, M_n$ , while its  $n$  coefficients are subjected to the  $n$  conditions

$$1 + \sum_{i=1}^n c_i = 0, \quad m^2 + \sum_i c_i M_i^2 = 0, \quad \dots, \quad m^{2n-2} + \sum_i c_i M_i^{2n-2} = 0. \quad (16.22)$$

It will be shown in §18 that the compensating masses  $M_i$  may be so chosen that, when their magnitudes are made to approach infinity in a certain prescribed way, the coefficients  $c_i$  remain finite. For large but finite masses we shall then have the situation in which, by virtue of the asymptotic properties (16.9) of cylindrical functions mentioned before, the values of the auxiliary functions  $\Delta_{M_i}$  will turn out to be vanishingly small everywhere except for a small region in the neighborhood of the light cone.

Since the coefficients  $c_i$  are finite, the regularized function  $\text{reg}\{\Delta\}$  will in practice differ from the function  $\Delta$  only in a small region in the neighborhood of the light cones where, in contrast to  $\Delta(x)$ , the function  $\text{reg}\{\Delta(x)\}$  will be continuous together with a certain number of its derivatives. As the values of the auxiliary masses  $M_i$  increase, the region in the neighborhood of the light cone in which the difference  $\text{reg}\{\Delta\} - \Delta$  differs from zero becomes smaller, while the finite values of  $\text{reg}\{\Delta(x)\}$  and of its derivatives become larger. In the limit as  $M_i \rightarrow \infty$ , we reach the situation in which the function  $\text{reg}\{\Delta\}$  ceases to differ from  $\Delta$ .

The above formal method of removing singularities from singular field functions by introducing auxiliary masses is a variant of a method known as *Pauli-Villars regularization* (Pauli and Villars (1949)). This method is equivalent to the introduction of additional fields with masses which compensate the singularities in the functions of the main field.

However, we must emphasize that, *physically, the concept of "compensating" fields*

is *completely inconsistent*, since some of the functions  $\Delta_{M_i}$  will necessarily have negative coefficients in (16.21). (As was shown in §13 a negative sign in front of a commutation function leads to a contradiction with the basic assumptions of the theory, and is therefore inadmissible.) We shall therefore regard the Pauli-Villars regularization of singular field functions as a purely *formal procedure*.

Of course, the above approximations is not the only one. One may, for example, obtain regularized approximations to the  $D$ -functions by limiting the region of integration in momentum-space, or by introducing a cut-off factor (form factor).

However, we shall prefer the above method of obtaining a regularized approximation to all the others, because of the following advantages: manifest relativistic invariance, absence of mixing of positive- and negative-frequency parts of singular functions, and universality in the sense of simultaneous regularization of the singular functions  $D^{(+)}$ ,  $D^{(-)}$ , and  $D^c$ .

## §17. Reduction to Normal Form

*17.1. The Coefficient Functions of Operator Expressions.* Before we present the theory of interacting fields, we shall have to familiarize ourselves with a number of properties, largely of algebraic character, of operator expressions consisting of quantized field functions of free fields.

Consider a typical operator expression which depends on the values of the positive- and negative-frequency parts of such functions at a number of space-time points  $x_1, \dots, x_n$  and is given in the normal form:

$$A(x_1, \dots, x_n) = \sum K_{\dots\alpha\dots\beta\dots}(x_1, \dots, x_n) \dots u_{\alpha}^{+}(x_r) \dots u_{\beta}^{-}(x_s) \dots \quad (17.1)$$

Here  $\dots u_{\alpha}(x) \dots$  are the components of wave functions or their partial derivatives and  $K_{\dots\alpha\dots\beta\dots}(x_1, \dots, x_n)$  are certain  $c$ -functions of the variables  $x_1, \dots, x_n$  which, in view of the homogeneity of space-time, are invariant under translations:

$$K_{\dots\alpha\dots\beta\dots}(x_1 + a, \dots, x_n + a) = K_{\dots\alpha\dots\beta\dots}(x_1, \dots, x_n). \quad (17.2)$$

We shall call these functions the *coefficient functions* of the given operator expression (17.1). Due to the singularity of the commutation relations, these functions will, in general, be singular. Their mathematical nature will be discussed specifically later. For the time being, we note that it is not difficult to express the matrix elements of the operator  $A(x_1, \dots, x_n)$  for the various states directly in terms of the coefficient functions:

$$\Phi_{\dots\nu\dots\rho\dots} = \dots u_{\nu}^{+}(p) \dots \Phi_0, \quad (17.3)$$

which correspond to the presence of given kinds of particles with given momenta  $p_1, \dots, p_j$ . To do this, it will only be necessary to form matrix elements of the form

$$\begin{aligned}
& \Phi_{\dots \gamma \dots \rho \dots}^* u_{\alpha}^+(x_r) \dots u_{\beta}^-(x_s) \dots \Phi_{\dots \gamma' \dots \rho' \dots} = \\
& = \int e^i (\sum x_r k_r - \sum x_s k_s) \Phi_0 \dots u_{\gamma}^-(p) \dots u_{\alpha}^+(k_r) \dots u_{\beta}^-(k_s) \dots u_{\gamma'}^+(p') \dots \Phi_0 \times \\
& \quad \times \left( \prod \delta(k_r^2 - m_{\alpha}^2) dk_r \right) \left( \prod \delta(k_s^2 - m_{\beta}^2) dk_s \right), \quad (17.4)
\end{aligned}$$

and this can readily be carried out with the aid of the commutation relations written in the form

$$\delta(k^2 - m^2) \{u_{\alpha}^+(k), u_{\beta}^-(p)\} = \delta(k - p) P_{\alpha\beta}(k) \quad (p^0 = \sqrt{\mathbf{p}^2 + m^2}), \quad (17.5)$$

where  $P_{\alpha,\beta}(k)$  are polynomials in  $k$ .

Indeed, we shall commute the creation operators  $u_{\alpha}^{(+)}(k_r)$  to the left and the annihilation operators  $u_{\beta}^{(-)}(k_s)$  to the right until they either "cancel" with  $\dots u_{\gamma}^{(-)}(p) \dots, \dots u_{\gamma'}^{(+)}(p') \dots$ , respectively, or give zero by operating on the vacuum state amplitude. It may be seen that a result different from zero will thus be obtained only when all the  $u_{\alpha}^{(+)}(k_r)$  cancel with  $u_{\gamma}^{(-)}(p)$ , and all the  $u_{\beta}^{(-)}(k_s)$  cancel with  $u_{\gamma'}^{(+)}(p')$ .

The  $\delta$ -functions which appear in the course of this "cancellation," remove the integration with respect to the  $k_r$  making them equal to  $\dots p' \dots$ . The functions  $u_{\gamma}^{(-)}(p_{\alpha}) \dots, u_{\gamma'}^{(+)}(p'_{\alpha})$  which remain free must also mutually cancel as a result of which factors of the form

$$\theta(p_{\omega}^0) \delta(p_{\omega}^2 - m_{\gamma}^2) \delta(p_{\omega} - p'_{\omega}) \quad (17.6)$$

will appear.

As a result of these elementary operations we obtain the matrix elements (17.4) in the following form:

$$\sum P_{\dots \gamma \dots \gamma' \dots}^{\alpha \dots \beta \dots} (\dots p \dots p' \dots) e^i (\sum p_{\nu} x_{\nu} - \sum p'_{\lambda} x_{\lambda}) \quad (17.7)$$

or

$$\sum Z_{\dots \gamma \dots \gamma' \dots}^{\alpha \dots \beta \dots} (\dots p' \dots p \dots) e^i (\sum p_{\nu} x_{\nu} - \sum p'_{\lambda} x_{\lambda}) \prod \{\theta(p_{\omega}^0) \delta(p_{\omega}^2 - m_{\gamma}^2) \delta(p_{\omega} - p'_{\omega})\}, \quad (17.8)$$

in which  $P_{\dots \gamma \dots \gamma' \dots}^{\alpha \dots \beta \dots}, Z_{\dots \gamma \dots \gamma' \dots}^{\alpha \dots \beta \dots}$  are polynomials in the components of  $\dots p \dots p' \dots$ . We obtain (17.7) when all the  $u_{\gamma}^{(-)}, u_{\gamma'}^{(+)}$  are compensated by  $u_{\alpha}^{(+)}, u_{\beta}^{(-)}$ ; in the opposite case, we obtain (17.8).

Thus on the basis of (17.1) we finally obtain

$$\begin{aligned}
& \Phi_{\dots \gamma \dots \rho \dots}^* A(x_1, \dots, x_n) \Phi_{\dots \gamma' \dots \rho' \dots} = \\
& = \sum P_{\dots \gamma \dots \gamma' \dots}^{\alpha \dots \beta \dots} (\dots p \dots p' \dots) K_{\dots \alpha \dots \beta \dots}(x_1, \dots, x_n) e^i (\sum p_{\nu} x_{\nu} - \sum p'_{\lambda} x_{\lambda}) + \\
& + \sum Z_{\dots \gamma \dots \gamma' \dots}^{\alpha \dots \beta \dots} (\dots p \dots p' \dots) K'_{\dots \alpha \dots \beta \dots}(x_1, \dots, x_n) e^i (\sum p_{\nu} x_{\nu} - \sum p'_{\lambda} x_{\lambda}) \times \\
& \quad \times \prod \{\theta(p_{\omega}^0) \delta(p_{\omega}^2 - m_{\gamma}^2) \delta(p_{\omega} - p'_{\omega})\}. \quad (17.9)
\end{aligned}$$

In view of the obvious convenience of the normal form of representation of operator expressions, for example, for the determination of their matrix elements, the problem of the appropriate *method of reduction* becomes of interest. It is clear that, for the reduction to the normal form of operator expressions which are polynomials in wave functions, it is sufficient to be able to reduce products of the form  $A_1(x_1) \dots A_n(x_n)$  in which  $A_j(x)$  are "linear operators," i.e., linear combinations of the corresponding  $u_j^{(+)}(x)$ ,  $u_j^{(-)}(x)$ .

In any given case, direct transformation of such a product may be carried out without any fundamental difficulty. It is quite sufficient, in order to achieve this, to move in turn  $u^{(+)}$  to the left and  $u^{(-)}$  to the right, and to make use of the commutation relations in each such "displacement." Nevertheless, because of the large number of terms obtained in the above procedure, even for relatively small values of  $n$ , it is useful to have a well-developed set of prescriptions for carrying out the operation of reduction to the normal form in as automatic a manner as possible.

Such a set of prescriptions follows from an important theorem due to Wick (1950) which we shall now formulate.

**17.2. Wick's Theorem for Normal Products.** Consider first of all the case  $n = 2$  in which we have the product  $A_1(x_1)A_2(x_2)$  of two linear operators. To reduce the product to the normal form, it is obviously sufficient to carry out not more than one displacement of the operators  $u^{(-)}(x_1)$ ,  $u^{(+)}(x_2)$ , so that the result contains the terms in the "proper order" in the sequence of positive- and negative-frequency parts of wave functions, and a term in the commutation functions which no longer includes any operator expressions.

The product under discussion may therefore differ from the normal product

$$:A_1(x_1)A_2(x_2):$$

only by a  $c$ -expression which we shall call a *pairing* and which we shall indicate by bracketing below the line:

$$A_1(x_1)A_2(x_2) = :A_1(x_1)A_2(x_2): + \underbrace{A_1(x_1)A_2(x_2)}. \quad (17.10)$$

Since, for the vacuum state, the expectation value of the normal product is always equal to zero, we may define the pairing as the vacuum expectation value of the ordinary product:

$$\underbrace{A_1(x_1)A_2(x_2)} = \Phi_0^* A_1(x_1)A_2(x_2) \Phi_0 \equiv \langle A_1(x_1)A_2(x_2) \rangle_0. \quad (17.11)$$

As an example, consider the real scalar field. Starting with the commutation relations

$$\varphi(x)\varphi(y) - \varphi(y)\varphi(x) = -iD(x-y),$$

we have, as usual,

$$\varphi^-(x)\varphi^+(y) = \varphi^+(y)\varphi^-(x) - iD^-(x-y),$$

in view of which we obtain

$$\begin{aligned}\varphi(x)\varphi(y) &= (\varphi^+(x) + \varphi^-(x))(\varphi^+(y) + \varphi^-(y)) = \\ &= \varphi^+(x)\varphi^+(y) + \varphi^+(y)\varphi^-(x) + \varphi^+(x)\varphi^-(y) + \varphi^-(x)\varphi^-(y) - iD^-(x-y)\end{aligned}$$

or, in accordance with the definition of the normal product,

$$\varphi(x)\varphi(y) = :\varphi(x)\varphi(y): - iD^-(x-y).$$

Therefore

$$\left. \begin{aligned}\underline{\varphi(x)\varphi(y)} &= \langle \varphi(x)\varphi(y) \rangle_0 = -iD^-(x-y), \\ \underline{\varphi^+(x)\varphi^-(y)} &= 0, \quad \underline{\varphi^-(x)\varphi^+(y)} = -iD^-(x-y).\end{aligned}\right\} \quad (17.12)$$

Similarly, in the case of the electromagnetic field, we obtain

$$\underline{A_n(x)A_m(y)} = \langle A_n(x)A_m(y) \rangle_0 = ig^{nm}D_0^-(x-y). \quad (17.13)$$

Let us also consider a fermion field for which

$$\begin{aligned}\psi_\alpha(x)\bar{\psi}_\beta(y) + \bar{\psi}_\beta(y)\psi_\alpha(x) &= -iS_{\alpha\beta}(x-y), \\ \psi_\alpha(x)\bar{\psi}_\beta^\dagger(y) &= -\bar{\psi}_\beta^\dagger(y)\psi_\alpha(x) - iS_{\alpha\beta}^-(x-y).\end{aligned}$$

Then

$$\psi(x)\bar{\psi}(y) = \psi^+(x)\bar{\psi}^+(y) - \bar{\psi}^+(y)\psi^-(x) + \psi^+(x)\bar{\psi}^-(y) + \psi^-(x)\bar{\psi}^-(y) - iS^-(x-y)$$

or

$$\psi_\alpha(x)\bar{\psi}_\beta(y) = :\psi_\alpha(x)\bar{\psi}_\beta(y): - iS_{\alpha\beta}^-(x-y),$$

and therefore

$$\underline{\psi_\alpha(x)\bar{\psi}_\beta(y)} = -iS_{\alpha\beta}^-(x-y). \quad (17.14)$$

In a similar way we may show that

$$\underline{\bar{\psi}_\alpha(x)\psi_\beta(y)} = -iS_{\beta\alpha}^+(y-x). \quad (17.15)$$

Since  $\psi(x)$  always anticommutes with  $\psi(y)$ , and  $\bar{\psi}(x)$  with  $\bar{\psi}(y)$ , we also have

$$\underline{\psi_\alpha(x)\psi_\beta(y)} = 0, \quad \underline{\bar{\psi}_\alpha(x)\bar{\psi}_\beta(y)} = 0. \quad (17.16)$$

In order to be able to formulate Wick's theorem, it is also necessary to introduce the concept of the *normal product with pairing*.

As we have already mentioned in §9, the normal product  $:A_1(x_1) \dots A_n(x_n):$  can be defined as the result of reducing the ordinary product  $A_1(x_1) \dots A_n(x_n)$  to the normal form, provided that during the *process of reduction* the quantized field functions are regarded as being strictly commuting or anticommuting and the value zero is assigned to all the commutation functions that appear. From this it follows directly that when the order of factors is altered in a normal product, only its sign may change:

$$:A_{i_1}(x_{i_1}) \dots A_{i_n}(x_{i_n}): = \eta :A_1(x_1) \dots A_n(x_n): \quad (\eta = (-1)^p),$$

where  $p$  is the parity of the permutation to which the Fermi operators are subjected in the transition from the initial sequence of the factors  $1, \dots, n$  to the sequence  $i_1, \dots, i_n$ .

It is evident that the normal product is linear in each of its factors:

$$:A_1(x_1) \dots (aA'_j(x_j) + bA''_j(x_j)) \dots A_n(x_n): = a :A_1(x_1) \dots A'_j(x_j) \dots A_n(x_n): + \\ + b :A_1(x_1) \dots A''_j(x_j) \dots A_n(x_n):,$$

where  $a$  and  $b$  are arbitrary complex  $c$ -numbers.

We now introduce the normal product with pairing, where, by definition,

$$:A_1(x_1) \dots A_j(x_j) \dots A_k(x_k) \dots A_n(x_n): = \\ = \eta \underbrace{A_j(x_j) A_k(x_k)} :A_1(x_1) \dots A_{j-1}(x_{j-1}) A_{j+1}(x_{j+1}) \dots A_{k-1}(x_{k-1}) A_{k+1}(x_{k+1}) \dots A_n(x_n): \\ (\eta = (-1)^p),$$

in which  $p$  is the parity of the Fermi permutations in going over from the sequence

$$1, \dots, j-1, j, j+1, \dots, k-1, k, k+1, \dots, n$$

to the sequence

$$1, k, 1, \dots, j-1, j+1, \dots, k-1, k+1, \dots, n.$$

The normal product with an arbitrary number of pairings can be defined in a similar way: for example,

$$:A_1(x_1) A_2(x_2) A_3(x_3) A_4(x_4) \dots A_{n-1}(x_{n-1}) A_n(x_n):$$

$\underbrace{\hspace{10em}}$

will be taken to be equal to the product of all the pairings with the normal product of the remaining unpaired operators and with the number  $\eta = (-1)^p$ :

$$\eta \underbrace{A_1 A_3} \underbrace{A_2 A_{n-1}} \dots :A_4 \dots A_n:,$$

where  $p$  is the parity of the permutations to which the Fermi operators are subjected in

the process of taking the pairings outside the normal product sign.

Thus, for example:

$$\begin{aligned} :\bar{\psi}_\alpha(x_1)\bar{\psi}_\beta(x_2)\psi_\gamma(x_3)\psi_\delta(x_4): &= -\underbrace{\bar{\psi}_\alpha(x_1)\bar{\psi}_\beta(x_2)}_{\text{pairing}}\underbrace{\psi_\gamma(x_3)\psi_\delta(x_4)}_{\text{pairing}} = \\ &= S_{\gamma\alpha}^+(x_3-x_1)S_{\delta\beta}^+(x_4-x_2). \end{aligned}$$

It follows immediately from the above definition that the normal product with pairing is linear in its factors, and that when they are transposed within such a product without changing the pairings and without changing the order of paired operators in a given pair, this product is multiplied by  $\eta = (-1)^p$ .

We can now give a ~~simple~~ formulation of Wick's theorem: *the ordinary product of linear operators is equal to the sum of all the corresponding normal products with all possible pairings, including the normal product without pairings:*

$$\begin{aligned} A_1 \dots A_n =: A_1 \dots A_n : &+ :\underbrace{A_1 A_2}_{\text{pairing}} \dots A_n : + \dots + :\underbrace{A_1 \dots A_{n-1}}_{\text{pairing}} A_n : + \\ &+ :\underbrace{A_1 \dots A_n}_{\text{pairing}} : + :\underbrace{A_1 A_2 A_3 A_4}_{\text{pairing}} \dots A_n : + \dots \end{aligned} \quad (17.17)$$

For the proof of the foregoing we shall need the following lemma: if  $A_1, \dots, A_n, B$  are linear operators, then

$$:A_1 \dots A_n : B =: A_1 \dots A_n B : + \sum_{(1 \leq k \leq n)} :A_1 \dots \underbrace{A_k \dots A_n}_{\text{pairing}} B :. \quad (17.18)$$

We note that the validity of this lemma will be established as soon as we have proved it for the special case in which each of the operators  $A_j, B$  is either a creation or an annihilation operator. Indeed, in the general case we may represent each such operator by the sum of creation and annihilation operators. Making use of the linearity of the products under discussion with respect to their factors, we shall represent them in the form of sums of products of creation and annihilation operators, i.e., of products which correspond to the special case mentioned above. Therefore, the validity of the lemma in the general case will follow from its validity in this special case.

Let us also note that the case when  $B$  is an annihilation operator is trivial. Indeed, the expression

$$:A_1 \dots A_n : B$$

will then contain the operator  $B$  in its "proper" place, so that this expression will be equal to

$$:A_1 \dots A_n B :.$$

Moreover, since  $B$  is an annihilation operator, it follows that  $\underbrace{A_k B}_{\text{pairing}} = 0$ .

Thus, we only have to consider the case when  $B$  is a creation operator. But then if some of the operators  $A_j$  are creation operators we may take them to the left outside the normal product symbols; their pairings with  $B$  are equal to zero, so that among the  $A_j$  inside the



normal product symbols only the annihilation operators remain. Thus in order to prove the lemma quoted before we only have to prove it for the case when all the  $A_j$  are annihilation operators while  $B$  is a creation operator.

Since for  $n = 1$  the validity of this lemma follows directly from the definition of pairing

$$A_1 B = : A_1 B : + \underbrace{A_1 B},$$

we may prove the lemma by induction.

We assume that the lemma holds for a certain number  $n$  of operators  $A_j$ . Then by multiplying equation (17.18) on the left by certain annihilation operator  $A_0$  we shall obtain:

$$A_0 : A_1 \dots A_n : B = A_0 : A_1 \dots A_n B : + \sum_{(1 \leq k \leq n)} A_0 : A_1 \dots \underbrace{A_k \dots A_n B} :. \quad (17.19)$$

But the only creation operator  $B$  which occurs in the expression

$$: A_1 \dots \underbrace{A_k \dots A_n B} :$$

is paired, and therefore this expression is equal to a  $c$ -number multiplied into the normal product of annihilation operators only. On the other hand, the normal product of annihilation operators coincides with their ordinary product. Therefore

$$A_0 : A_1 \dots \underbrace{A_k \dots A_n B} : = : A_0 A_1 \dots \underbrace{A_k \dots A_n B} :. \quad (17.20)$$

We now turn to the analysis of the first term in the right-hand side of equation (17.19). We have

$$: A_1 \dots A_n B : = (-1)^p : B A_1 \dots A_n : = (-1)^p B : A_1 \dots A_n :,$$

where  $p$  is the parity of the Fermi permutations required to bring  $B$  from the extreme right-hand side to the extreme left-hand side position. From this we obtain

$$\begin{aligned} A_0 : A_1 \dots A_n B : &= (-1)^p A_0 B : A_1 \dots A_n : = \\ &= (-1)^p : A_0 B : : A_1 \dots A_n : + (-1)^p \underbrace{A_0 B} : A_1 \dots A_n : = \\ &= (-1)^p : A_0 B : : A_1 \dots A_n : + (-1)^p : \underbrace{A_0 B A_1} \dots A_n : = \\ &= (-1)^{p+p'} : B A_0 : : A_1 \dots A_n : + : \underbrace{A_0 A_1} \dots A_n B :, \end{aligned}$$

where  $p'$  is the parity of the Fermi permutations needed for  $A_0$  to exchange places with  $B$ .

On the other hand, since  $B$  is a creation operator while all the  $A_j$  are annihilation operators, it follows that:

$$\begin{aligned} (-1)^{p+p'} : B A_0 : : A_1 \dots A_n : &= (-1)^{p+p'} : B A_0 A_1 \dots A_n : = (-1)^p : A_0 B A_1 \dots A_n : = \\ &= : A_0 A_1 \dots A_n B :. \end{aligned}$$

We have therefore

$$A_0 : A_1 \dots A_n B : = : A_0 A_1 \dots A_n B : + : \underbrace{A_0 A_1 \dots A_n B} : ,$$

from which, by virtue of (17.20) and (17.19)

$$: A_0 A_1 \dots A_n : B = : A_0 A_1 \dots A_n B : + \sum_{(0 \leq k \leq n)} : A_0 A_1 \dots \underbrace{A_k \dots A_n B} : .$$

Thus, having assumed the validity of equation (17.18) for the case when the number of operators  $A$  was  $n$  we have now established its validity also for the case when this number is  $n + 1$ . This concludes the proof of the lemma formulated above.

We note further that this lemma may also be generalized directly to the case of normal products with an arbitrary number of pairings. Indeed, since the normal product with pairings is always equal to the product of pairings by  $(-1)^P$  and by the normal product of operators remaining unpaired, we see that in (17.18) we may take in place of the "pure" normal product of the operators  $A$  their normal product with an arbitrary number of pairings.

We now begin the proof of Wick's theorem itself. It is evident that the theorem is trivial for  $n = 2$ . We shall therefore use the method of induction. We assume that equation (17.17) which expresses Wick's theorem holds for the case of the product of  $n$  operators  $A$ , and we shall attempt to prove it for the case of the product of  $n + 1$  linear operators. In order to do this, we multiply equation (17.17) on the right by a certain linear operator  $A_{n+1}$ . We then express the ordinary product  $A_1, \dots, A_{n+1}$  in the form of a sum of all the normal products of operators  $A_1, \dots, A_n$  with all possible pairings between them multiplied on the right, in the ordinary sense, by  $A_{n+1}$ . (We emphasize that among the terms with all possible pairings, the term with zero number of pairings is also included.)

But, in accordance with the generalized lemma proved above, the normal product of the operators  $A_1, \dots, A_n$  with an arbitrary given number of pairings among them multiplied on the right, in the ordinary sense, by  $A_{n+1}$  is equal to the sum of normal products of the operators  $A_1, \dots, A_{n+1}$  in which in addition to the pairings already occurring among  $A_1, \dots, A_n$  we also take into account all possible (including zero)\* pairings of the free operators  $A_1, \dots, A_n$  with  $A_{n+1}$ . In this way, we show that the ordinary product of  $n + 1$  linear operators  $A_1, \dots, A_{n+1}$  may be represented in the form of a sum of their normal products with all possible pairings, which completes the proof by induction. The proof of Wick's theorem is thus completed.

It may easily be seen that this theorem is also applicable to the case when some of the factors themselves appear as normal products:

$$: A_1 \dots A_{k_1} : : A_{k_1+1} \dots A_{k_2} : \dots : A_{k_s} \dots A_n : . \quad (17.21)$$

In this case, Wick's theorem is proved in exactly the same way as for the "pure" product  $A_1 \dots A_n$  with the one obvious difference, that we now need not take into account pairings between those factors in (17.21) that belong to the same normal product. For example, the pairings between  $A_1, \dots, A_{k_1}$ , the pairings between  $A_{k_1+1}, \dots, A_{k_2}$ , and so on.

In order to illustrate the application of Wick's theorem in its general form we consider the products

\*I.e., the term containing no pairings of  $A_1, \dots, A_n$  with  $A_{n+1}$ .

$$:\bar{\psi}(x) \gamma^k \psi(x) : : \bar{\psi}(y) \gamma^l \psi(y) : = \sum_{(\sigma, \sigma', \tau, \tau')} : \bar{\psi}_\sigma(x) \gamma_{\sigma\sigma'}^k \psi_{\sigma'}(x) : : \bar{\psi}_\tau(y) \gamma_{\tau\tau'}^l \psi_{\tau'}(y) :. \quad (17.22)$$

We have:

$$\begin{aligned} &:\bar{\psi}_\sigma(x) \gamma_{\sigma\sigma'}^k \psi_{\sigma'}(x) : : \bar{\psi}_\tau(y) \gamma_{\tau\tau'}^l \psi_{\tau'}(y) : = \\ &= : \bar{\psi}_\sigma(x) \gamma_{\sigma\sigma'}^k \psi_{\sigma'}(x) \bar{\psi}_\tau(y) \gamma_{\tau\tau'}^l \psi_{\tau'}(y) : + \underbrace{\bar{\psi}_\sigma(x) \psi_{\tau'}(y)}_{\text{contract}} \gamma_{\sigma\sigma'}^k \gamma_{\tau\tau'}^l : \psi_{\sigma'}(x) \bar{\psi}_\tau(y) : + \\ &+ \underbrace{\psi_{\sigma'}(x) \bar{\psi}_\tau(y)}_{\text{contract}} \gamma_{\sigma\sigma'}^k \gamma_{\tau\tau'}^l : \bar{\psi}_\sigma(x) \psi_{\tau'}(y) : + \underbrace{\psi_{\sigma'}(x) \bar{\psi}_\tau(y)}_{\text{contract}} \underbrace{\bar{\psi}_\sigma(x) \psi_{\tau'}(y)}_{\text{contract}} \gamma_{\sigma\sigma'}^k \gamma_{\tau\tau'}^l; \end{aligned}$$

from which on the basis of the pairing formulas (17.14), (17.15), we obtain:

$$\begin{aligned} &:\bar{\psi}_\sigma(x) \gamma_{\sigma\sigma'}^k \psi_{\sigma'}(x) : : \bar{\psi}_\tau(y) \gamma_{\tau\tau'}^l \psi_{\tau'}(y) : = \\ &= : \bar{\psi}_\sigma(x) \gamma_{\sigma\sigma'}^k \psi_{\sigma'}(x) \bar{\psi}_\tau(y) \gamma_{\tau\tau'}^l \psi_{\tau'}(y) : + i : \bar{\psi}_\tau(y) \gamma_{\tau\tau'}^l S_{\tau'\sigma}^+(y-x) \gamma_{\sigma\sigma'}^k \psi_{\sigma'}(x) : - \\ &- i : \bar{\psi}_\sigma(x) \gamma_{\sigma\sigma'}^k S_{\sigma'\tau}^-(x-y) \gamma_{\tau\tau'}^l \psi_{\tau'}(y) : + \gamma_{\sigma\sigma'}^k S_{\sigma'\tau}^-(x-y) \gamma_{\tau\tau'}^l S_{\tau'\sigma}^+(y-x). \end{aligned}$$

Substituting this expression in (17.22) and carrying out the summation over the spinor indices, we finally obtain

$$\begin{aligned} &:\bar{\psi}(x) \gamma^k \psi(x) : : \bar{\psi}(y) \gamma^l \psi(y) : = \\ &= : \bar{\psi}(x) \gamma^k \psi(x) \bar{\psi}(y) \gamma^l \psi(y) : + i : \bar{\psi}(y) \gamma^l S^+(y-x) \gamma^k \psi(x) : - \\ &- i : \bar{\psi}(x) \gamma^k S^-(x-y) \gamma^l \psi(y) : + \text{Sp} \{ \gamma^k S^-(x-y) \gamma^l S^+(y-x) \}. \end{aligned}$$

This expression also leads to the following formula for the commutator of free fields:

$$\begin{aligned} [J^k(x), J^l(y)]_- &= i \{ : \bar{\psi}(y) \gamma^l S(y-x) \gamma^k \psi(x) : - : \bar{\psi}(x) \gamma^k S(x-y) \gamma^l \psi(y) : \} + \\ &+ \text{Sp} \{ \gamma^k S^-(x-y) \gamma^l S^+(y-x) - \gamma^k S^+(x-y) \gamma^l S^-(y-x) \}. \quad (17.23) \end{aligned}$$

With the aid of Wick's theorem we are able to reduce, in an almost automatic manner, ordinary products of operators to sums of normal products multiplied by  $c$ -numbers. One should terminate the reduction to the normal form at this stage, since, generally speaking, it is not useful to express the normal products in the explicit normal form. The representation of operators in terms of linear combinations of normal products may in fact be regarded as the most convenient way of writing their representations in the normal form. For example, it is shorter to write

$$:\bar{\psi}(x) \psi(x) :,$$

than the same expression in the expanded form

$$\bar{\psi}^+(x) \psi^+(x) + \bar{\psi}^+(x) \psi^-(x) - \psi^+(x) \bar{\psi}^-(x) + \bar{\psi}^-(x) \psi^-(x),$$

and this is even more true in the case of products containing a larger number of factors.

As we have seen, it is very easy to deal with normal products. For example, suppose that we have to evaluate a matrix element of the type:

$$\begin{aligned} \Phi_{\dots \gamma \dots \rho \dots}^* : A_1(x_1) \dots A_n(x_n) : \Phi_{\dots \gamma' \dots \rho' \dots} &= \\ = \Phi_0^* \dots u_{\bar{\gamma}}^-(p) \dots : A_1(x_1) \dots A_n(x_n) : \dots u_{\gamma'}^+(p') \dots \Phi_0. \end{aligned} \quad (17.24)$$

We note that, in accordance with Wick's theorem, the product

$$\begin{aligned} \dots u_{\bar{\gamma}}^-(p) \dots : A_1(x_1) \dots A_n(x_n) : \dots u_{\gamma'}^+(p') \dots &= \\ = \dots u_{\bar{\gamma}}^-(p) \dots : A_1(x_1) \dots A_n(x_n) : \dots u_{\gamma'}^+(p') \dots &: \end{aligned}$$

is equal to the sum of the corresponding normal products with all possible pairings:  $u_{\bar{\gamma}}^-(p)$  with  $A_j(x_j)$ ,  $A_j(x_j)$  with  $u_{\gamma'}^+(p')$ , and  $u_{\bar{\gamma}}^-(p)$  with  $u_{\gamma'}^+(p')$ . On the other hand, the vacuum expectation value of a normal product which contains at least one unpaired operator is equal to zero. We therefore see that the matrix element (17.24) consists of the sum of only those normal products for which all the operators are paired. It will therefore be given by a sum of terms of the type:

$$(-1)^p \prod \{ \underbrace{u_{\bar{\alpha}}^-(p_{\alpha}) A_j(x_j)} \} \prod \{ \underbrace{A_j(x_j) u_{\bar{\gamma}'}^-(p_{\beta}')} \} \prod \{ \underbrace{u_{\bar{\sigma}}^-(p_{\sigma}) u_{\bar{\gamma}'}^-(p_{\tau}')} \}.$$

*17.3. Some Definitions.* Up until now we have been considering operator functions of points  $x_1, \dots, x_n$  which have been expressed in terms of positive- and negative-frequency parts of quantized field functions. An important role is played by a more specialized type of operator functions in which the quantized field functions appear, so to speak, "entirely" within the symbols for the normal product:

$$A(x_1, \dots, x_n) = \sum K_{\dots \alpha \dots}(x_1, \dots, x_n) : \dots u_{\alpha}(x_j) \dots :. \quad (17.25)$$

If the Fermi field functions occur in the preceding expression only in even combinations, then we call such a sum a *polylocal operator*. From the definitions given before and from Wick's theorem, it follows immediately that the multiplication of several polylocal operators results again in a polylocal operator.

Polylocal operators have the following important property:

$$[A_1(x_1, \dots, x_n); A_2(y_1, \dots, y_m)] = 0, \quad (17.26)$$

if each of the  $x_j$  is space-like with respect to any arbitrary  $y_i$ .

Indeed, since the Pauli-Jordan function vanishes inside the light cone, we have

$$\{u_{\alpha}(x_j), u_{\beta}(y_i)\}_{\pm} = 0 \quad (x_j \sim y_i),$$

Therefore, in view of Wick's theorem we obtain

$$:\dots u_{\alpha}(x_j)\dots::\dots u_{\beta}(y_i)\dots:=:\dots u_{\alpha}(x_j)\dots u_{\beta}(y_i)\dots: \quad (17.27)$$

But since the field functions occur only in even combinations, as mentioned before, we may bring  $\dots u_{\beta}(y_i)\dots$  to the left of  $\dots u_{\alpha}(x_j)\dots$  in the normal product on the left-hand side of (17.27) without introducing a change of sign. We therefore obtain

$$[:\dots u_{\alpha}(x_j)\dots:, \dots u_{\beta}(y_i)\dots:] = 0$$

for

$$(x_1, \dots, x_n) \sim (y_1, \dots, y_m),$$

from which follows the validity of (17.26).

We now consider  $n = 1$ . In this case, the polylocal operator depends only on the behavior of the field functions (their derivatives may, of course, be included among them) at the single point  $x$ , so that we call it a *local operator*.

In view of (17.26), we have for two local operators:

$$[\mathcal{L}(x), \mathcal{L}(y)] = 0 \quad (x \sim y). \quad (17.28)$$

We note that for  $n > 1$  too, the polylocal operator may depend on the behavior of the field functions at only one point. Suppose that in fact, all the coefficient functions in the sum (17.25) vanish for all  $x_1, \dots, x_n$  with the exception of those which satisfy the equation

$$x_1 = x_2 = \dots = x_n.$$

It is clear that in this case,  $K$  may be constructed only from the expression

$$\delta(x_1 - x_2) \dots \delta(x_1 - x_n)$$

and its partial derivatives. Since, by definition, the coefficient functions must have the property of translational invariance, we see that the general expression for them in the above case must be

$$z\left(\dots \frac{\partial}{\partial x} \dots\right) \delta(x_1 - x_2) \dots \delta(x_1 - x_n),$$

where  $z(\dots \partial/\partial x \dots)$  is a certain polynomial in  $\partial/\partial x_j^{\alpha}$  with *constant* coefficients. We shall call a polylocal operator with coefficient functions of this type a *quasilocal operator*. It is readily seen that integration of a quasilocal operator over all the points  $x_1, \dots, x_n$  except one will lead to an ordinary local operator.

## §18. Coefficient Functions

*18.1. Improper Nature of Singular Functions.* We shall now discuss the question of the structure of the coefficient functions which appear when the operation of multiplication of operator expressions is carried out. We note that in accordance with Wick's theorem the coefficient functions which are obtained in evaluating products of linear operators  $A_1(x_1) \dots A_n(x_n)$  have the form

$$\prod_{(r < s)} D_{\alpha\beta}^-(x_r - x_s), \quad (18.1)$$

where  $D_{\alpha\beta}$  are commutation functions.

The product

$$A_1(x_1, \dots, x_n) A_2(y_1, \dots, y_m)$$

of any two operators belonging to type (17.1) under investigation

$$\begin{aligned} A_1(x_1, \dots, x_n) &= \sum K_{\dots\alpha\dots\beta\dots}(x_1, \dots, x_n) \dots u_{\alpha}^+(x_r) \dots u_{\beta}^-(x_s) \dots, \\ A_2(y_1, \dots, y_m) &= \sum Q_{\dots\sigma\dots\tau\dots}(y_1, \dots, y_m) \dots u_{\sigma}^+(y_i) \dots u_{\tau}^-(y_j) \dots, \end{aligned}$$

belongs to the same type and its coefficient functions may be expressed in the form

$$K(x_1, \dots, x_n) Q(y_1, \dots, y_m) \prod D_{\beta\sigma}^-(x_s - y_i). \quad (18.2)$$

It should be noted, however, that the function  $D^{(-)}(x)$  and, to an even greater extent the functions

$$D_{\beta\sigma}^-(x) = P_{\beta\sigma} \left( \frac{\partial}{\partial x} \right) D^-(x),$$

have a high degree of singularity on the light cone, and therefore serious doubts may arise as to whether expressions such as (18.1) and (18.2) which contain products of arbitrary numbers of these functions have any real meaning.

In this connection it is now useful to consider the more general question as to the conditions which should be imposed on the coefficient functions appearing in the operator expressions (17.1) in order that a definite meaning may be given to these expressions, or as is usually stated, that they should contain no "divergences."

First of all it is clear that we cannot demand that the coefficient functions  $K(x_1, \dots, x_n)$  should be functions in the generally accepted mathematical sense, since in that case we ought to exclude from consideration all "singular" or "improper" functions such as  $\delta(x)$  and  $D(x)$  which we continually encounter in relativistic quantum mechanics. It

is therefore natural to regard the  $K(x_1, \dots, x_n)$  as being defined specifically as *improper functions*.\*

We shall now attempt to present the ideas which are usually implied in this concept in publications on quantum mechanics, but which as a rule are not clearly formulated. It may easily be seen that, in contrast to ordinary functions, singular or improper functions are defined not by giving their values for all the values of their arguments (for a certain set of values of their arguments they may be either infinite or, in general, not defined at all), but by prescribing rules for the integration of their products with sufficiently regular functions. For example, the  $\delta$ -function is characterized by the rules of integration of its products with continuous functions; the derivatives of the  $\delta$ -function are characterized by the rules of integration of their products with appropriately differentiable functions, and so on.

In other words, an improper function is defined by specifying an appropriate linear functional in a suitable "linear space" of sufficiently regular functions. In recent years, this functional point of view has been developed in the work of Sobolev (1936) and of Schwartz (1950) on a new mathematical theory—the so-called "theory of distributions."

Of course, we cannot undertake here a presentation of the general theory of improper functions in any detail, and we will therefore restrict ourselves to the principal formulations which refer only to those properties of specific coefficient functions that we shall need for direct use in subsequent sections.

We now formulate the definition of the coefficient functions  $K(x_1, \dots, x_n)$ . In accordance with the commonly accepted point of view we shall consider  $K(x_1, \dots, x_n)$  as a given and integrable function if in a certain given linear space  $L$  of functions  $F(x_1, \dots, x_n)$ , which are sufficiently smooth and which fall off sufficiently rapidly at infinity, a linear functional can be defined which we shall agree to represent symbolically in the form

$$\int K(x_1, \dots, x_n) F(x_1, \dots, x_n) dx_1 \dots dx_n. \quad (18.3)$$

Let us make this definition more precise. We introduce the class  $C(q, r, n)$  of functions  $F(x_1, \dots, x_n)$  which are continuous together with all their partial derivatives up to the  $q$ th order inclusive, and for which all the products

$$\left. x_{i_1}^{\alpha_1} \dots x_{i_s}^{\alpha_s} \frac{\partial^p F(x_1, \dots, x_n)}{\partial x_{j_1}^{\beta_1} \dots \partial x_{j_p}^{\beta_p}} \right\} \begin{array}{l} (s=0, 1, \dots, r), \\ (p=0, 1, \dots, q), \\ (\beta, \alpha=0, 1, 2, 3) \end{array} \quad (18.4)$$

are bounded. The linear space formed by the functions of this class for any arbitrary given  $q, r$  will be taken as the space  $L$ .

Thus the main requirement which will always be imposed on the coefficient functions is that the integral (18.3) should be defined as a linear functional in the linear space

\*The name *generalized functions* has become commonly accepted in Soviet mathematical literature for such functions. In foreign literature they are referred to as *distributions*.

$C(q, r, n)$ , in any case for sufficiently large  $q, r$ . We shall in future refer to this requirement as the *condition of integrability*.

Usually it is convenient to carry out the actual construction of the coefficient functions by means of an improper limiting process. We shall say that the sequence

$$K_M(x_1, \dots, x_n) \quad (M \rightarrow \infty), \quad (18.5)$$

is convergent in an improper sense if the corresponding sequence of integrals

$$\int K_M(x_1, \dots, x_n) F(x_1, \dots, x_n) dx_1 \dots dx_n$$

converges in the ordinary sense for each function  $F(x_1, \dots, x_n)$  from a certain fixed class  $C(q, r, n)$ . It is of course assumed in the above that all the functions (18.5) are integrable in the given  $C(q, r, n)$ . The usual notation  $\rightarrow, \lim$  will be used to represent improper convergence.

Since

$$\lim_{M \rightarrow \infty} \int K_M(x_1, \dots, x_n) F(x_1, \dots, x_n) dx_1 \dots dx_n$$

will also be a linear functional in  $C(q, r, n)$ , then by agreeing to represent it symbolically in the form

$$\int \lim_{M \rightarrow \infty} K_M(x_1, \dots, x_n) F(x_1, \dots, x_n) dx_1 \dots dx_n,$$

we thereby define the improper limit

$$\lim_{M \rightarrow \infty} K_M(x_1, \dots, x_n)$$

as an integrable improper function. By virtue of this definition

$$\begin{aligned} \int \lim_{M \rightarrow \infty} K_M(x_1, \dots, x_n) F(x_1, \dots, x_n) dx_1 \dots dx_n &= \\ &= \lim_{M \rightarrow \infty} \int K_M(x_1, \dots, x_n) F(x_1, \dots, x_n) dx_1 \dots dx_n \end{aligned}$$

for any function  $F(x_1, \dots, x_n)$  from the class  $C(q, r, n)$  with sufficiently high indices  $q, r$ .

It should be emphasized that the improper limiting process is actually continually being used in the investigation of singular functions in quantum mechanics, although, as a rule, no attention is paid to the fact that it differs from the limiting process in the ordinary sense. Thus, for example, when it is stated that  $\delta(t)$  is the *limit* of the function

$$\delta_\varepsilon(t) = \begin{cases} 0 & \text{for } |t| > \varepsilon, \\ \frac{1}{2\varepsilon} & \text{for } |t| \leq \varepsilon \end{cases}$$



for  $\epsilon \rightarrow +0$  the notion of the limit in this case must evidently be interpreted in the improper sense. In an exactly similar way the limiting process will be an improper one when  $\epsilon \rightarrow +0$  in the expression

$$\frac{P_{\alpha\beta}(\rho)}{m^2 - \rho^2 - i\epsilon}$$

for the Fourier transform of the function  $D_{\alpha\beta}^c$ .

Let us now discuss, from the above point of view, the question of approximating the principal singular functions  $D_{\alpha\beta}^{(\pm)}$ ,  $D_{\alpha\beta}^c$  by means of regularized expressions. Let us take for example the function  $D_{\alpha\beta}^{(-)}(x)$ , defined by the formal relation

$$D_{\alpha\beta}^{(-)}(x) = P_{\alpha\beta} \left( i \frac{\partial}{\partial x} \right) D^{(-)}(x) = \frac{i}{(2\pi)^3} \int e^{-ikx} P_{\alpha\beta}(k) \theta(k^0) \delta(k^2 - m^2) dk. \quad (18.6)$$

Let us first of all determine the meaning which should be given to integrals with infinite limits which occur in the above. As is well known, an ordinary integral taken over an infinite region is defined as the limit of integrals with finite regions of integration which are expanded indefinitely and in the limit include the whole given infinite region. It is therefore natural to define integrals of the form

$$K(x) = \int e^{-ikx} P(k) \theta(k^0) \delta(k^2 - m^2) dk \quad (18.7)$$

with a polynomial  $P(k)$  as the improper limit of the integral

$$K_G(x) = \int_G e^{-ikx} P(k) \theta(k^0) \delta(k^2 - m^2) dk$$

as the region  $G$  is expanded indefinitely.

It may be shown that such a limit actually exists. We take a sequence of regions  $G_n$  such that the size of the four-dimensional cube

$$-L_n \leq k^\alpha \leq L_n \quad (\alpha = 0, 1, 2, 3),$$

which is entirely contained within  $G_n$  tends to  $\infty$  as  $n$  does, and consider the integral

$$\int K_G(x) F(x) dx$$

with the function  $F(x)$  chosen from a certain class  $C(q, r, 1)$ . We then have

$$\begin{aligned} \int K_G(x) F(x) dx &= \int_G \tilde{F}(k) P(k) \theta(k^0) \delta(k^2 - m^2) dk = \int \theta(k; G) \tilde{F}(k) P(k) \theta(k^0) \delta(k^2 - m^2) dk, \\ \tilde{F}(k) &= \int F(x) e^{-ikx} dx, \end{aligned}$$

where

$$\theta(k; G) = \begin{cases} 1 & \text{for } k \in G, \\ 0 & \text{for } k \notin G. \end{cases}$$

In going over to the three-dimensional integral we obtain

$$\int K_G(x) F(x) dx = \int_{(k^0 = \sqrt{k^2 + m^2})} \theta(k^0, \mathbf{k}, G) F(k^0, \mathbf{k}) P(k^0, \mathbf{k}) \frac{d\mathbf{k}}{2k^0} \quad (18.8)$$

We now evaluate the rate at which  $\tilde{F}(k)$  falls off at infinity. We assume that  $F(x)$  belongs to the class  $C(q, 5, 1)$ . The expressions

$$|F(x)|, \quad \left| \frac{\partial^p F(x)}{(\partial x^\alpha)^p} \right|, \\ \left( \sum_\alpha |x^\alpha| \right)^5 |F(x)|, \quad \left( \sum_\alpha |x^\alpha| \right)^5 \left| \frac{\partial^p F(x)}{(\partial x^\beta)^p} \right| \quad (p = 1, \dots, q)$$

are then bounded, so that

$$\int |F(x)| dx < \infty, \quad \int \left| \frac{\partial^p F(x)}{(\partial x^\beta)^p} \right| dx < \infty.$$

It is now clear that  $\tilde{F}(k)$  is continuous and bounded:

$$|\tilde{F}(k)| \leq \int |F(x)| dx.$$

Integrating by parts we then obtain

$$\tilde{F}(k) = \left( \frac{1}{ik_\alpha} \right)^q \int \frac{\partial^q F(x)}{(\partial x^\alpha)^q} e^{-ikx} dx,$$

from which it follows that

$$|k^\alpha|^q |\tilde{F}(k)| \leq \int \left| \frac{\partial^q F(x)}{(\partial x^\alpha)^q} \right| dx \leq \sum_\alpha \int \left| \frac{\partial^q F(x)}{(\partial x^\alpha)^q} \right| dx = \text{const} = C$$

and

$$\left( \sum_\alpha |k^\alpha| \right)^q |\tilde{F}(k)| \leq 4^{q-1} \sum |k^\alpha|^q |\tilde{F}(k)| \leq 4^q C. \quad (18.9)$$

Let  $\nu$  denote the degree of the polynomial  $P(k)$ . We see that the expression  $\tilde{F}(k)P(k)$ , which is a continuous function of  $k$ , falls off as  $|k| \rightarrow \infty$  not slower than

$$\frac{\text{const}}{|k|^{q-\nu}}.$$

Therefore

$$\int |\tilde{F}(k^0, \mathbf{k}) P(k^0, \mathbf{k})| \frac{d\mathbf{k}}{k^0} < \infty$$

for any arbitrary function  $F(x)$  from the class  $C(q, 5, 1)$  with  $q = \nu + 3$ . Consequently we have

$$\begin{aligned} & \left| \int \theta(k^0, \mathbf{k}; G_n) \tilde{F}(k^0, \mathbf{k}) P(k^0, \mathbf{k}) \frac{d\mathbf{k}}{k^0} - \int \tilde{F}(k^0, \mathbf{k}) P(k^0, \mathbf{k}) \frac{d\mathbf{k}}{k^0} \right| = \\ & = \left| \int \{\theta(k^0, \mathbf{k}; G_n) - 1\} \tilde{F}(k^0, \mathbf{k}) P(k^0, \mathbf{k}) \frac{d\mathbf{k}}{k^0} \right| \leq \int \left| \tilde{F}(k^0, \mathbf{k}) P(k^0, \mathbf{k}) \frac{d\mathbf{k}}{k^0} \right|_{n \rightarrow \infty} \rightarrow 0 \\ & \quad (\dots |k^\alpha| \leq L_n \dots). \end{aligned}$$

We can now assert on the basis of (18.8) that for every function  $F(x)$  from the class  $C(\nu + 3, 5, 1)$

$$\int K_{G_n}(x) F(x) dx \rightarrow \int \tilde{F}(k^0, \mathbf{k}) P(k^0, \mathbf{k}) \frac{d\mathbf{k}}{2k^0}.$$

This proves the existence of the improper limit

$$\lim_{n \rightarrow \infty} K_{G_n}(x) = K(x),$$

for which

$$\int K(x) F(x) dx = \int \tilde{F}(k^0, \mathbf{k}) P(k^0, \mathbf{k}) \frac{d\mathbf{k}}{2k^0}.$$

We shall adopt this improper limit, which evidently is independent of the particular choice of the sequence of expanding regions  $G_n$ , as the definition of the integral (18.7).

Thus relation (18.6) actually defines an integrable improper function  $D_{\alpha\beta}^{(-)}(x)$  with

$$\int D_{\alpha\beta}^{(-)}(x) F(x) dx = \frac{i}{(2\pi)^3} \int \tilde{F}(k^0, \mathbf{k}) P_{\alpha\beta}(-k^0, -\mathbf{k}) \frac{d\mathbf{k}}{2k^0}. \quad (18.10)$$

As has just been established,  $D_{\alpha\beta}^{(-)}(x)$  is an improper limit of the sequence of functions

$$\frac{i}{(2\pi)^3} \int_{G_n} e^{-ikx} P_{\alpha\beta}(-\mathbf{k}) \theta(k^0) \delta(k^2 - m^2) d\mathbf{k}, \quad (18.11)$$

which, as may readily be seen, are regular and analytic.

However, this approximation may turn out in a number of cases not to be sufficiently convenient. The point is that expressions (18.11) are not covariant, since the region of integration changes under Lorentz transformations.

**18.2. Some Properties of Pauli-Villars Regularization.** In order to be able to approximate  $D_{\alpha\beta}^{(-)}(x)$ , naturally in the improper sense, by means of continuous and at the same

time convariant functions, we shall make use of Pauli-Villars regularization which was mentioned in §16. We set

$$\left. \begin{aligned} \text{reg } (D_{\alpha\beta}^-(x)) &= P_{\alpha\beta} \left( i \frac{\partial}{\partial x} \right) \text{reg } (D^-(x)), \\ \text{reg } (D^-(x)) &= D^-(x) + \sum_{(1 \leq j \leq l)} c_j D_{M_j}^-(x), \end{aligned} \right\} \quad (18.12)$$

where  $D_M^{(-)}(x)$  is the negative frequency part of the Pauli function corresponding to mass  $M$ , while the numbers  $c_j$  are determined by

$$\left. \begin{aligned} 1 + \sum_{j=1}^l c_j &= 0, \\ m^2 + \sum_j M_j^2 c_j &= 0, \\ \dots & \\ m^{2(l-1)} + \sum_j M_j^{2(l-1)} c_j &= 0. \end{aligned} \right\} \quad (18.13)$$

Since  $\text{reg}(D^{(-)}(x))$  is continuous, and has continuous partial derivatives up to the  $(l-2)$ th order inclusive, we see that for  $l > \nu + 2$  ( $\nu$  is the degree of the polynomial  $P_{\alpha\beta}$ ) the functions  $\text{reg}(D_{\alpha\beta}^{(-)}(x))$  will be continuous.

We now show that the masses  $M$  can always be chosen in such a way that, as  $M \rightarrow \infty$ , the coefficients  $c_j$  remain bounded. We emphasize that in the limiting transition  $M \rightarrow \infty$ , the number  $l$  is always regarded as fixed. We take any arbitrary fixed *different* masses  $m_j$ . The determinant

$$\begin{vmatrix} 1 & \dots & 1 \\ m_1^2 & \dots & m_l^2 \\ \cdot & \cdot & \cdot \\ m_1^{2(l-1)} & \dots & m_l^{2(l-1)} \end{vmatrix}$$

will then differ from zero and, therefore, the equations

$$\begin{aligned} & \sum_{(1 \leq i \leq l)} a_i^{(\nu)} = \delta_{0\nu}, \\ & \dots\dots\dots * \\ & \sum_{(1 \leq i \leq l)} a_i^{(\nu)} m_i^2(l-1) = \delta_{l-1,\nu} \quad (\nu=0, \dots, l-1), \end{aligned}$$

in which  $\delta_{ij}$  denote the elements of the unit matrix, have finite solutions  $a_j^{(\nu)}$ . We set

$$M_l = M m_l. \quad (18.14)$$

For such a system of masses we find from equations (18.13)

$$c_l = - \sum_{(0 \leq \nu \leq l-1)} a_l^{(\nu)} \left( \frac{m}{M} \right)^{2\nu}.$$

These  $c_j$  will evidently be uniformly bounded as  $M \rightarrow \infty$ . The special choice (18.14) for the masses  $M_j$  is not essential; for the following analysis we require only that

$$|c_l| \underset{M \rightarrow \infty}{\leq} \text{const.}$$

To prove the validity of the relation

$$\text{reg} (D_{\alpha\beta}^-(x)) \underset{M \rightarrow \infty}{\rightarrow} D_{\alpha\beta}^-(x); \quad (18.15)$$

it is now only necessary to show that for any arbitrary function from the class  $C(\nu + 3, 5, 1)$ , the following limiting relation holds:

$$\int F(x) \text{reg} (D_{\alpha\beta}^-(x)) dx \rightarrow \int F(x) D_{\alpha\beta}^-(x) dx.$$

We have

$$\int F(x) \text{reg} (D_{\alpha\beta}^-(x)) dx - \int F(x) D_{\alpha\beta}^-(x) dx = \sum_{1 \leq l \leq l} c_l \int F(x) P_{\alpha\beta} \left( i \frac{\partial}{\partial x} \right) D_{Ml}^-(x) dx,$$

and therefore since the  $c_j$  are bounded, the proof of (18.15) will be complete as soon as we have shown that

$$\int F(x) \left\{ P_{\alpha\beta} \left( i \frac{\partial}{\partial x} \right) D_{Ml}^-(x) \right\} dx \underset{M \rightarrow \infty}{\rightarrow} 0.$$

But in virtue of (18.15) this integral is equal to

$$\frac{i}{(2\pi)^3} \int \tilde{F}(\sqrt{k^2 + M_l^2}, \mathbf{k}) P_{\alpha\beta}(\sqrt{k^2 + M_l^2}, \mathbf{k}) \frac{d\mathbf{k}}{2\sqrt{k^2 + M_l^2}}. \quad (18.16)$$

On the other hand, by virtue of (18.9), we obtain for  $q = \nu + 3$

$$|\tilde{F}(k) P_{\alpha\beta}(-k)| \leq \frac{Q}{\left(\sum_{\alpha} |k^{\alpha}| \right)^3} \leq \frac{Q}{(k^0)^3} \quad (Q = \text{const}).$$

Therefore in absolute value the integral (18.16) will be less than

$$\frac{Q}{(2\pi)^3} \int \frac{dk}{(k^2 + M_i^2)^2} = \frac{Q}{(2\pi)^3 M_i^3} \int \frac{dp}{(p^2 + 1)^2} \xrightarrow{M \rightarrow \infty} 0.$$

Thus the functions  $D_{\alpha\beta}^{(-)}(x)$  are, in fact, approximated in the improper sense by the continuous functions

$$\text{reg } \{D_{\alpha\beta}^-(x)\}, \quad M \rightarrow \infty,$$

which obviously have the required properties of covariance.

We have considered above only the functions  $D_{\alpha\beta}^{(-)}(x)$ . However, it may readily be seen that all the above arguments can be immediately carried over also to the functions  $D_{\alpha\beta}^{(+)}(x)$ ,  $D_{\alpha\beta}^c(x)$ .

## §19. Multiplication of Operator Functions

*19.1. Multiplication of Singular Functions.* We now consider the fundamental question of the definition of products of pairings (18.1) with the aid of Wick's theorem. We emphasize that the necessity of a separate definition of products is, generally speaking, typical for improper functions. The point is that an improper function is defined by setting up rules for integrating its products only with sufficiently regular functions, and from such rules a prescription does not directly follow for the integration of products of *several* singular functions. However, we may use the improper limiting process and define (18.1) by a convergent sequence of regular functions. To construct such a sequence in the most natural way possible, we first use a purely formal approach, typical of field theory. We take the representations (18.6) as our starting point and we set

$$\prod_{(r < s)} D_{\alpha_r \beta_s}^-(x_r - x_s) = \prod_{(r < s)} \left\{ \frac{i}{(2\pi)^3} \int e^{-ik(x_r - x_s)} P_{\alpha\beta}(k) \theta(k^0) \delta(k^2 - m_{\alpha\beta}^2) dk \right\}.$$

Formally carrying out the multiplication of these integrals we obtain

$$\begin{aligned} \prod_{(r < s)} D_{\alpha_r \beta_s}^-(x_r - x_s) &= \\ &= \left( \frac{i}{(2\pi)^3} \right)^N \int \exp \left[ i \sum_{(r < s)} \lambda_{\alpha_r \beta_s} (x_s - x_r) \right] \prod_{(r < s)} \{ P_{\alpha\beta}(\lambda_{\alpha\beta}) \theta(\lambda_{\alpha\beta}^0) \delta(\lambda_{\alpha\beta}^2 - m_{\alpha\beta}^2) d\lambda_{\alpha\beta} \} = \\ &= \int e^{i \sum k_r x_r} \Delta(k_1, \dots, k_n) dk_1 \dots dk_n, \quad (19.1) \end{aligned}$$

where  $N$  is the number of factors in the product under consideration and

$$\Delta(k_1, \dots, k_n) = \left(\frac{i}{(2\pi)^3}\right)^N \int \prod_r \delta\left(k_r + \sum_{(s < r)} \lambda_{\alpha_s \beta_r} - \sum_{(r < s)} \lambda_{\alpha_r \beta_s}\right) \times \\ \times \prod_{(r < s)} \{P_{\alpha\beta}(\lambda_{\alpha\beta}) \theta(\lambda_{\alpha\beta}^0) \delta(\lambda_{\alpha\beta}^2 - m_{\alpha\beta}^2) d\lambda_{\alpha\beta}\}. \quad (19.2)$$

It may be shown that the region of integration in (19.2) is in fact finite. Because of this, it turns out that a rigorous definition of the products (18.1) can be given in the form of improper limits of a sequence of corresponding regular analytic functions. Choosing for such analytic functions the products of reg  $\{D^{(-)}(x)\}$ , we obtain

$$\prod \{\text{reg}(D^-(x_r - x_s))\} \rightarrow \prod D^-(x_r - x_s). \quad (19.3)$$

Indeed, the integration in (19.2) is carried out only over the set of points for which

$$\left. \begin{aligned} k_r &= \sum_{(r < s)} \lambda_{\alpha_r \beta_s} - \sum_{(s < r)} \lambda_{\alpha_s \beta_r}, \\ \lambda_{\alpha\beta}^0 &> 0, \quad (\lambda_{\alpha\beta})^2 < (\lambda_{\alpha\beta}^0)^2. \end{aligned} \right\} \quad (19.4)$$

Therefore, in particular,

$$\begin{aligned} k_1^0 &= \sum_{(s > 1)} \lambda_{\alpha_1 \beta_s}^0, \\ k_2^0 &= \sum_{(s > 2)} \lambda_{\alpha_2 \beta_s}^0 - \sum_{(s < 2)} \lambda_{\alpha_s \beta_2}^0, \\ k_3^0 &= \sum_{(s > 3)} \lambda_{\alpha_3 \beta_s}^0 - \sum_{(s < 3)} \lambda_{\alpha_s \beta_3}^0, \\ &\dots \end{aligned}$$

from which, since  $\lambda_{\alpha\beta}^0$  are positive, we obtain in turn

$$\begin{aligned} \lambda_{\alpha_1 \beta_s}^0 &\leq k_1^0, \\ \sum_{(s > 2)} \lambda_{\alpha_2 \beta_s}^0 &= k_2^0 + \lambda_{\alpha_1 \beta_2}^0 \leq k_2^0 + k_1^0, \\ \sum_{(s > 3)} \lambda_{\alpha_3 \beta_s}^0 &= k_3^0 + \lambda_{\alpha_1 \beta_3}^0 + \lambda_{\alpha_2 \beta_3}^0 \leq k_3^0 + k_1^0 + (k_2^0 + k_1^0), \\ &\dots \\ \sum_{(s > r)} \lambda_{\alpha_r \beta_s}^0 &\leq k_r^0 + k_r^0 - 1 + 2k_r^0 - 2 + 2^2 k_r^0 - 3 + \dots + 2^{r-2} k_1^0, \\ &\dots \end{aligned}$$

so that in any case

$$0 \leq \lambda_{\alpha\beta}^0 \leq 2^{n-2} (|k_1^0| + \dots + |k_n^0|) \quad (19.5)$$

and, consequently,

$$|\lambda_{\alpha\beta}|^2 \leq 2^{2(n-2)} (|k_1^0| + \dots + |k_n^0|)^2.$$

The fact that the region of integration in (19.2) is actually finite leads to an absence of divergences and to the possibility of a rigorous definition of the above expression (18.1) as an improper limit of the sequence of regular analytic functions:

$$K(x_1, \dots, x_n | \Gamma_\omega) = \int_{\Gamma_\omega} e^{i \sum k_r x_r} \Delta(k_1, \dots, k_n) dk_1 \dots dk_n \quad (19.6)$$

as the region  $\Gamma_\omega$  of the  $4n$ -dimensional space of the points  $k_1, \dots, k_n$  is expanded indefinitely and, in the limit, includes all this space.

To prove this assertion, we shall prove the existence of such a limit and its independence of the particular choice of the sequence of regions  $\Gamma_\omega$ . To do this, we shall, as usual, consider the corresponding sequence of functionals

$$\int K(x_1, \dots, x_n | \Gamma_\omega) F(x_1, \dots, x_n) dx_1 \dots dx_n \quad (19.7)$$

in the linear space  $C(q, r, n)$  for fixed sufficiently large values of  $q, r$ . We shall have to evaluate the rate of falling off at infinity of the Fourier transform of the function  $F$

$$\tilde{F}(k_1, \dots, k_n) = \int e^{i \sum k_r x_r} F(x_1, \dots, x_n) dx_1 \dots dx_n.$$

Repeating in the present case  $n > 1$  in exact detail the arguments which were given earlier we can easily see that if  $F(x_1, \dots, x_n)$  belongs to the class  $C(q, 4n+1, n)$  then  $\tilde{F}(k_1, \dots, k_n)$  will be a continuous function which satisfies the inequality

$$|\tilde{F}(k_1, \dots, k_n)| \leq \frac{A}{(\sum |k_i^\alpha|)^q} \quad (A = \text{const}). \quad (19.8)$$

By substituting (19.6) into (19.7) and introducing the function

$$\theta(k_1, \dots, k_n | \Gamma_\omega) = \begin{cases} \text{inside the region } \Gamma_\omega, \\ \text{outside the region } \Gamma_\omega. \end{cases}$$

we obtain

$$\begin{aligned} \int K(x_1, \dots, x_n | \Gamma_\omega) F(x_1, \dots, x_n) dx_1 \dots dx_n &= \int_{\Gamma_\omega} \tilde{F}(k_1, \dots, k_n) \Delta(k_1, \dots, k_n) dk_1 \dots dk_n = \\ &= \int \theta(k_1, \dots, k_n | \Gamma_\omega) \tilde{F}(k_1, \dots, k_n) \Delta(k_1, \dots, k_n) dk_1 \dots dk_n = \\ &= \frac{i^N}{(2\pi)^{3N}} \int \theta(k_1(\lambda), \dots, k_n(\lambda) | \Gamma_\omega) \tilde{F}(k_1(\lambda), \dots, k_n(\lambda)) \prod_{(r < s)} \{P_{\alpha\beta}(\lambda_{\alpha\beta}) \theta(\lambda_{\alpha\beta}^0) \delta(\lambda_{\alpha\beta}^2 - m_{\alpha\beta}^2) d\lambda_{\alpha\beta}\}, \end{aligned}$$

where



$$k_r(\lambda) = \sum_{(r < s)} \lambda_{\alpha_r \beta_s} - \sum_{(s < r)} \lambda_{\alpha_s \beta_r}.$$

Since the region  $\Gamma_\omega$  is expanded indefinitely and in the limit includes the whole  $4n$ -dimensional space, we can assert that for points outside  $\Gamma_\omega$

$$\sum_{(\alpha, r)} |k_r^\alpha| \geq L_\omega \rightarrow \infty, \quad \omega \rightarrow \infty.$$

Thus for those  $\lambda$  for which

$$1 - \theta(k_1(\lambda), \dots, k_n(\lambda) | \Gamma_\omega) \neq 0,$$

we have

$$2 \sum |\lambda_{\alpha_r \beta_s}^\gamma| \geq \sum |k_r^\gamma| \geq L_\omega.$$

In view of this we obtain

$$\left| \int K(x_1, \dots, x_n | \Gamma_\omega) F(x_1, \dots, x_n) dx - \frac{i^n}{(2\pi)^{3N}} \int \tilde{F}(k_1(\lambda), \dots, k_n(\lambda)) \prod_{(r < s)} \{P_{\alpha\beta}(\lambda_{\alpha\beta}) \theta(\lambda_{\alpha\beta}^0) \delta(\lambda_{\alpha\beta}^2 - m_{\alpha\beta}^2) d\lambda_{\alpha\beta}\} \right| \leq \varepsilon_\omega \quad (19.9)$$

where

$$\varepsilon_\omega = (2\pi)^{-3N} \left| \int_{G_\omega} \tilde{F}(\dots k_j(\lambda) \dots) \prod_{(r < s)} |P_{\alpha\beta}(\lambda_{\alpha\beta})| \theta(\lambda_{\alpha\beta}^0) \delta(\lambda_{\alpha\beta}^2 - m_{\alpha\beta}^2) d\lambda_{\alpha\beta} \right|. \quad (19.10)$$

(Here  $G_\omega$  denotes a region limited by the condition  $\sum |\lambda_{\alpha\beta}^\gamma| \geq \frac{1}{2} L_\omega$ .)

Since the inequalities (19.5) follow from (19.4), we see that under the integral sign in (19.10),

$$\begin{aligned} (\lambda_{\alpha\beta}^0)^2 &\leq 2^{2(n-2)} \left( \sum |k_j^0(\lambda)| \right)^2, \\ \sum (\lambda_{\alpha\beta}^0)^2 &\leq 2^{2(n-2)} N \left( \sum |k_j^0(\lambda)| \right)^2, \end{aligned}$$

and, consequently,

$$\sum |k_j^0(\lambda)| \geq 2^{-(n-2)} N^{-1/2} \sqrt{\sum |\lambda_{\alpha\beta}^0|^2}.$$

From this by virtue of (19.8) we obtain

$$|\tilde{F}(\dots k_j(\lambda) \dots)| \leq \frac{A [2^{n-2} \sqrt{N}]^q}{(\sum |\lambda_{\alpha\beta}^0|^2)^{q/2}},$$

and, therefore, under the integral sign in (19.10)

$$|\tilde{F}(\dots k_j(\lambda) \dots) \prod P_{\alpha\beta}(-\lambda_{\alpha\beta})| \leq \frac{B}{(\sum |\lambda_{\alpha\beta}^0|^2)^{\frac{q-\nu}{2}}}, \quad (19.11)$$

where  $\nu$  is the sum of the degrees of all the polynomials  $P_{\alpha\beta}$  and  $B = \text{const}$ . It is now clear that

$$\begin{aligned} \epsilon_\omega &\leq (2\pi)^{-3N} \int_{\Gamma_\omega} \frac{\prod \{ \theta(\lambda_{\alpha\beta}^0) \delta(\lambda_{\alpha\beta}^2 - m_{\alpha\beta}^2) d\lambda_{\alpha\beta} \}}{(\sum |\lambda_{\alpha\beta}^0|^2)^{\frac{q-\nu}{2}}} = \\ &= \frac{(2\pi)^{-3N} B}{2^N} \int_{\Gamma_\omega} \frac{\dots d\lambda_{\alpha\beta} \dots}{\{ \prod \sqrt{m_{\alpha\beta}^2 + \lambda_{\alpha\beta}^2} \} \{ \sum (m_{\alpha\beta}^2 + \lambda_{\alpha\beta}^2) \}^{\frac{q-\nu}{2}}}. \end{aligned}$$

Therefore, if we take

$$q = \nu + 2N + 1,$$

the integral on the right-hand side of this inequality is absolutely convergent, and  $\epsilon_\omega \rightarrow 0$  as  $\omega \rightarrow \infty$ . Thus, for any arbitrary function  $F(x_1, \dots, x_n)$  from the class  $C(\nu + 2N + 1, 4n + 1, n)$ , the expression on the left-hand side of (19.9) tends to zero.

We may therefore assert that the sequence of regular analytic functions (19.6) is a convergent one (in the improper sense), and that

$$\begin{aligned} \int \left\{ \lim_{\omega \rightarrow \infty} \int_{\Gamma_\omega} e^{i \sum k_r x_r} \Delta(k_1, \dots, k_n) dk_1 \dots dk_n \right\} F(x_1, \dots, x_n) dx_1 \dots dx_n = \\ = \frac{i^N}{(2\pi)^{3N}} \int \tilde{F}(\dots k_j(\lambda) \dots) \prod_{(r < s)} \{ P_{\alpha\beta}(\lambda_{\alpha\beta}) \theta(\lambda_{\alpha\beta}^0) \delta(\lambda_{\alpha\beta}^2 - m_{\alpha\beta}^2) d\lambda_{\alpha\beta} \}. \end{aligned} \quad (19.12)$$

Since the right-hand side of this inequality is independent of the particular choice of the sequence of regions  $\Gamma_\omega$ , we see that the expression

$$\lim_{\omega \rightarrow \infty} \int_{\Gamma_\omega} e^{i \sum k_r x_r} \Delta(k_1, \dots, k_n) dk_1 \dots dk_n$$

is also independent of the choice of this sequence.

Thus (19.1) does in fact define the expression

$$\prod_{(r < s)} D_{\bar{\alpha}_r \beta_s}(x_r - x_s)$$

as an integrable improper function for which, in view of (19.12),

$$\begin{aligned} & \int \left\{ \prod_{(r < s)} D_{\bar{\alpha}_r \beta_s}(x_r - x_s) \right\} F(x_1, \dots, x_n) dx_1 \dots dx_n = \\ & = \frac{i^N}{(2\pi)^{3N}} \int \tilde{F}(\dots k_j(\lambda) \dots) \prod_{(r < s)} \{ P_{\alpha\beta}(\lambda_{\alpha\beta}) \theta(\lambda_{\alpha\beta}^0) \delta(\lambda_{\alpha\beta}^2 - m_{\alpha\beta}^2) d\lambda_{\alpha\beta} \}. \end{aligned} \quad (19.13)$$

The product of singular functions  $D_{\alpha\beta}^{(-)}$  under discussion may also be approximated by means of corresponding products of regularized functions  $\text{reg}\{D_{\alpha\beta}^{(-)}\}$

$$\prod_{(r < s)} \text{reg}(D_{\bar{\alpha}_r \beta_s}(x_r - x_s)) \rightarrow \prod_{(r < s)} D_{\bar{\alpha}_r \beta_s}(x_r - x_s). \quad (19.3)$$

We shall prove this property by establishing that for every function  $F(x_1, \dots, x_n)$  from the class  $C(\nu + 2N + 1, 4n + 1, n)$  the following limiting relation holds:

$$\begin{aligned} & \int F(x_1, \dots, x_n) \left\{ \prod_{(r < s)} \text{reg} D_{\bar{\alpha}_r \beta_s}(x_r - x_s) \right\} dx_1 \dots dx_n - \\ & - \int F(x_1, \dots, x_n) \left\{ \prod_{(r < s)} D_{\bar{\alpha}_r \beta_s}(x_r - x_s) \right\} dx_1 \dots dx_n \rightarrow 0 \quad \text{for } M \rightarrow \infty. \end{aligned} \quad (19.14)$$

In consequence of the definition of the regularized functions (18.12) the difference (19.14) is a sum of products of the coefficients  $c_j$  and integrals obtained from the integral on the right-hand side of (19.13) by replacing the masses  $m$  by the masses  $M$ . Since all the  $c_j$  remain bounded as  $M \rightarrow \infty$  the proof of (19.14) will be complete as soon as we have shown that such integrals tend to zero. Using (19.11), we find that in absolute value they will be less than

$$(2\pi)^{-3N} 2^{-N} \int \left\{ \dots + m_{\alpha\beta}^2 + \dots + M_j^2 + \dots + \sum \lambda_{\alpha\beta}^2 \right\}^{-\frac{2N+1}{2}} \prod \left( \frac{d\lambda_{\alpha\beta}}{|\lambda_{\alpha\beta}|} \right).$$

But due to the presence of the masses  $M_j$  in the denominator of the integrand, this integral tends to zero (as  $M \rightarrow \infty$ ) and this completes the proof of (19.3).

The arguments presented above may also be extended to the general case of coefficient functions obtained by multiplying two operator functions such as (17.1) with different arguments, and the corresponding expressions

$$K(x_1, \dots, x_n) Q(y_1, \dots, y_m) \prod D_{\bar{\rho}\sigma}(x_s - y_j)$$

(in which  $x_1, \dots, x_n, y_1, \dots, y_m$  are independent arguments, while  $K$  and  $Q$  are integrable coefficient functions invariant under translations) may be defined as integrable improper functions.

The following general theorem can be proved for expressions of this form:

If  $K_M(x_1, \dots, x_n)$ ,  $Q_M(y_1, \dots, y_m)$  are coefficient functions invariant under translations, and if in the improper sense:

$$\left. \begin{aligned} K_M(x_1, \dots, x_n) &\xrightarrow{M \rightarrow \infty} K(x_1, \dots, x_n); \\ Q_M(y_1, \dots, y_m) &\xrightarrow{M \rightarrow \infty} Q(y_1, \dots, y_m), \end{aligned} \right\}$$

while the arguments  $x_1, \dots, x_n, y_1, \dots, y_m$  are independent, then

$$K_M(x_1, \dots, x_n) Q_M(y_1, \dots, y_m) \prod \operatorname{reg} D_{\bar{\beta}\sigma}(x_s - y_j) \xrightarrow{M \rightarrow \infty} K(x_1, \dots, x_n) Q(y_1, \dots, y_m) \prod D_{\bar{\beta}\sigma}(x_s - y_j). \quad (19.15)$$

The proof of this theorem is based technically on the same principle as the proof of the limiting relation (19.14).

In essence, both these proofs are based on the simple fact that if the sum of the negative frequencies is bounded, then it follows that each individual frequency is bounded. We have examined at first the product of only the negative-frequency parts of the  $D$ -functions; our whole analysis may therefore be carried over in a trivial way also to the case when only the  $D^{(+)}$  occur in place of the  $D^{(-)}$ .

We leave it to the reader to derive the following formulas for the products of  $D$ - and  $S$ -functions of the same parity:

$$\begin{aligned} D^-(x) D^+(-x) &= -[D^-(x)]^2 = \frac{1}{(2\pi)^4} \int e^{-ipx} \frac{\theta(p^0)}{8\pi} \sqrt{1 - \frac{4m^2}{p^2}} \theta(p^2 - 4m^2) dp, \\ \operatorname{Sp} [\gamma^5 S^-(x) \gamma^5 S^+(-x)] &= \frac{1}{(2\pi)^4} \int e^{-ipx} \theta(p^0) f(p^2) dp, \\ f(p^2) &= \frac{\theta(p^2 - 4m^2)}{4\pi} \sqrt{1 - \frac{4m^2}{p^2}} (p^2 - 4m^2), \\ \operatorname{Sp} [\gamma^m S^-(x) \gamma^m S^+(-x)] &= \frac{1}{(2\pi)^4} \int e^{-ipx} \theta(p^0) f^{mn}(p) dp, \\ f^{mn}(p) &= \frac{\theta(p^2 - 4m^2)}{2\pi} \sqrt{1 - \frac{4m^2}{p^2}} \frac{p^2 + 2m^2}{3} \left( g^{mn} - \frac{p^m p^n}{p^2} \right). \end{aligned}$$

A totally different situation will arise if we consider in place of (18.1), for example, products of the type

$$\prod_{(r < s)} D_{\alpha_r \beta_s}^c(x_r - x_s), \quad (19.16)$$

since in  $D^c(x)$  both positive- and negative-frequency components are present. It can be shown that in spite of the weakness of our condition of integrability (§18.1), which requires the existence of the integral of the product of the singular function under discussion

only with functions from the class  $C(q, r, n)$  with arbitrarily high indices  $q, r$ , the product (19.16) is not a definite integrable function and an improper limiting relation such as (19.14) does not hold in this case.

*19.2. Some Properties of Singular Functions.* We shall now briefly consider the properties of continuity, differentiability, and so on, which the singular functions have for certain ranges of values of their arguments. Since we define a singular function merely as the symbolic "kernel" of a given linear functional, it may, in general, have no definite values anywhere. However, in a number of important cases singular functions have both definite finite values and various properties of regularity for certain ranges of values of their arguments. Thus, for example,  $\delta(x) = 0$  for  $x \neq 0$ , the functions  $D^{(\pm)}(x)$ ,  $D^c(x)$  are continuous outside the light cone, and so on.

It is therefore useful to introduce in this connection the following mathematical definition:

*Let  $K(x_1, \dots, x_n)$  be an improper function integrable over a certain class  $C(q, r, n)$ , and let  $f(x_1, \dots, x_n)$  be an ordinary function continuous within a certain region  $O$  (of the space of the points  $x_1, \dots, x_n$ ); we shall then say that*

$$K(x_1, \dots, x_n) = f(x_1, \dots, x_n)$$

*in the region  $O$ , and that  $K$  is continuous in this region if*

$$\int \{K'(x_1, \dots, x_n) - f(x_1, \dots, x_n)\} F(x_1, \dots, x_n) dx_1 \dots dx_n = 0$$

*for every function  $F$  of the class  $C(q, r, n)$  which vanishes outside  $O$  together with all its partial derivatives up to order  $q$ , inclusive.*

We shall also agree to interpret in exactly the same sense the other regularity properties of the improper functions in the given region. Similarly, we may also introduce the concept of improper convergence within a definite range of values of the arguments. We shall say that a sequence of improper functions  $K_M(x_1, \dots, x_n)$  integrable over a certain fixed class  $C(q, r, n)$  converges in the improper sense within the region  $O$  if the sequence of integrals

$$\int K_M(x_1, \dots, x_n) F(x_1, \dots, x_n) dx_1 \dots dx_n$$

converges for every function  $F$  in  $C(q, r, n)$  which vanishes outside  $O$  together with all its partial derivatives up to the  $q$ th order inclusive.

With the aid of the definitions given above, the methods of dealing with ordinary functions may to a large extent be extended also to improper functions. These methods may be used for the investigation of those cases frequently encountered in quantum field theory when the purely formal operations with improper functions lead to expressions which do not have a definite meaning, for example, to expressions containing "divergences."

*19.3. Multiplication of Operator Functions.* We now proceed to investigate operator expressions represented by finite sums of the form of (17.1). We note first of all that the existence of limiting relations (19.3) allows us to eliminate one possible objection to

the proof of Wick's theorem given above, which is based on the fact that in the course of the above proof, we carried out multiplication of singular functions. A proof of Wick's theorem quite free of such objections may be given very simply, for example, in accordance with the following outline. We at first examine a fictitious case where the field functions satisfy commutation relations in which  $D_{\alpha\beta}$  have been replaced by  $\text{reg } D_{\alpha\beta}$ , and for this case we carry out the proof of Wick's theorem in exactly the same form as was presented earlier. Since the regularized functions are continuous, such a proof will now be quite rigorous. For the transition to the actual case, it will now be sufficient to go to the limit  $M \rightarrow \infty$ , which removes the regularization.

Similarly, we can readily demonstrate with the aid of the more general relations (19.15) the validity of the following product of two integrable operator functions with independent arguments:

$$A_1(x_1, \dots, x_n) A_2(y_1, \dots, y_m). \quad (19.17)$$

We call those operator expressions integrable which may be represented by finite sums of the type of (17.1) in which all the coefficient functions satisfy the requirement of integrability.

To show that the product (19.17) is an integrable operator function of the arguments  $x_1, \dots, x_n, y_1, \dots, y_m$ , we again consider the fictitious case in which the commutation relations have been regularized, and in addition we shall replace all the coefficient functions  $K, Q$  of the operators  $A_1, A_2$  by some continuous functions  $K_M, Q_M$  making sure only that the following improper limiting relations hold:

$$K_M \rightarrow K, \quad Q_M \rightarrow Q \quad \text{for} \quad M \rightarrow \infty.$$

Then, having reduced to the normal form the product of the operator functions regularized in the manner indicated above, we shall obtain coefficient functions of the form:

$$\sum K_M(x_1, \dots, x_n) Q_M(y_1, \dots, y_m) \prod \text{reg } \{D_{\alpha\beta}(x_s - y_j)\},$$

which by virtue of (19.15) form (as  $M \rightarrow \infty$ ) a sequence which converges in the improper sense. We may therefore go to the limit which removes regularization, and determine the coefficient functions of the actual product (19.17) as integrable singular functions. The theorem which we have established above, stating that the product of two (and consequently of any arbitrary number of) integrable operator functions with independent arguments is again an integrable operator function, will turn out to be quite useful.

We now consider a certain integrable operator function  $A(x_1, \dots, x_n)$  and take the integral

$$\int A(x_1, \dots, x_n) F_1(x_1, \dots, x_n) dx_1 \dots dx_n, \quad (19.18)$$

in which  $F(x_1, \dots, x_n)$  is an arbitrary function of the class  $C(\infty, \infty, n)$ , and set up the matrix elements which characterize this operator integral

$$\tilde{\Phi}_{\dots\gamma\dots\rho\dots} \int A(x_1, \dots, x_n) F(x_1, \dots, x_n) dx_1 \dots dx_n \Phi_{\dots\gamma'\dots\rho'\dots} \quad (19.9)$$

over all the possible states

$$\Phi_{\dots\gamma\dots\rho\dots} = \dots u_{\gamma}^+(p) \dots \Phi_0,$$

which correspond to the presence of given kinds of particles with given momenta. In view of (17.9), we obtain for them the following expression:

$$\begin{aligned} \sum P_{\dots\gamma\dots\gamma'\dots}^{\alpha\dots\beta\dots} (\dots p \dots p' \dots) \int K_{\dots\alpha\dots\beta\dots}(x_1, \dots, x_n) \times \\ \times e^{i(\sum p_{\gamma} x_{\gamma} - \sum p'_{\lambda} x_{\lambda})} F(x_1, \dots, x_n) dx_1 \dots dx_n + \\ + \sum Z_{\dots\gamma\dots\gamma'\dots}^{\alpha\dots\beta\dots} (\dots p \dots p' \dots) \prod \{\theta(p_{\omega}^0) \delta(p_{\omega}^2 - m_{\gamma}^2) \delta(p_{\omega} - p'_{\omega})\} \times \\ \times \int K'_{\dots\alpha\dots\beta\dots}(x_1, \dots, x_n) e^{i(\sum p_{\gamma} x_{\gamma} - \sum p'_{\lambda} x_{\lambda})} F(x_1, \dots, x_n) dx_1 \dots dx_n, \end{aligned}$$

where, as mentioned earlier,  $P$  and  $Z$  are polynomials in the components of  $p$ . We also note that the  $\delta$ -functions in the second sum allow for the possibility of the momenta  $p, p'$  being the same for particles of the same kind. Consequently, the second sum will not occur if the matrix element is evaluated between states for which the momenta for particles of the same kind are different; in such a case

$$\begin{aligned} \tilde{\Phi}_{\dots\gamma\dots\rho\dots} \int A(x_1, \dots, x_n) F(x_1, \dots, x_n) dx_1 \dots dx_n \Phi_{\dots\gamma'\dots\rho'\dots} = \\ = \sum P_{\dots\gamma\dots\gamma'\dots}^{\alpha\dots\beta\dots} (\dots p \dots p' \dots) \int K_{\dots\alpha\dots\beta\dots}(x_1, \dots, x_n) \times \\ \times e^{i(\sum p_{\gamma} x_{\gamma} - \sum p'_{\lambda} x_{\lambda})} F(x_1, \dots, x_n) dx_1 \dots dx_n. \end{aligned}$$

Since the function  $F$  belongs to the class  $C(\infty, \infty, n)$ , the function

$$e^{i(\sum p_{\gamma} x_{\gamma} - \sum p'_{\lambda} x_{\lambda})} F(x_1, \dots, x_n)$$

also belongs to the same class. Therefore, since the coefficient functions  $K$  are integrable, the integrals

$$\int K_{\dots\alpha\dots\beta\dots}(x_1, \dots, x_n) e^{i(\sum p_{\gamma} x_{\gamma} - \sum p'_{\lambda} x_{\lambda})} F(x_1, \dots, x_n) dx_1 \dots dx_n \quad (19.20)$$

turn out to be finite. Moreover, since the differentiation of this integral with respect to  $p$  corresponds to multiplying  $F$  by the components of  $x$ , while the multiplication of  $F$  by an arbitrary polynomial in such components does not take it outside the class  $C(\infty, \infty, n)$ , we see that the integrals (19.20) are continuous functions of the momenta  $\dots p \dots p' \dots$  which have continuous partial derivatives of arbitrary order.

Thus we see that the matrix elements (19.19) under consideration are linear combinations of terms of the form

$$f(\dots p \dots p' \dots), f(\dots p \dots p' \dots) \prod \{\theta(p_\omega^0) \delta(p_\omega^2 - m_\gamma^2) \delta(p_\omega - p'_\omega)\},$$

in which  $f(\dots p \dots p' \dots)$  are continuous together with all their partial derivatives with respect to  $\dots p \dots p' \dots$ . In particular, in the region in which the momenta  $\dots p \dots p' \dots$  are not the same for identical particles the matrix elements themselves will possess such properties of "smoothness." The operator integrals (19.18) may therefore be considered to be convergent.

Integrals of the type

$$\int A(x_1, \dots, x_n) F(x_2, \dots, x_n) dx_2 \dots dx_n \quad (19.21)$$

with "one incomplete integration" will also be convergent in the same sense.

Indeed in the present case the only difference will consist of the fact that instead of the integrals (19.20) appearing in the matrix element the integrals

$$\begin{aligned} & \int K_{\dots \alpha \dots \beta \dots}(x, x_2, \dots, x_n) \varphi(x_2, \dots, x_n) dx_2 \dots dx_n, \\ & \varphi(x_2, \dots, x_n) = e^{i(\sum p_\nu x_\nu - \sum p'_\lambda x_\lambda)} F(x_2, \dots, x_n), \end{aligned} \quad (19.22)$$

will appear which, due to the invariance of the coefficient functions under translations, may be reduced in an elementary way to the form (19.20).

In fact, by virtue of this property we obtain

$$\begin{aligned} \int K(x, x_2, \dots, x_n) \varphi(x_2, \dots, x_n) dx_2 \dots dx_n &= \int K(x + \xi, x_2 + \xi, \dots, x_n + \xi) \times \\ & \times \varphi(x_2, \dots, x_n) dx_2 \dots dx_n. \end{aligned}$$

We multiply both sides of the above equality by some function  $\varphi(\xi)$  of the class  $C(\infty, \infty, 1)$  for which

$$\int \varphi(\xi) d\xi = 1,$$

and integrate over  $\xi$ . We then find that

$$\begin{aligned} \int K(x, x_2, \dots, x_n) \varphi(x_2, \dots, x_n) dx_2 \dots dx_n &= \\ &= \int K(x + \xi, x_2 + \xi, \dots, x_n + \xi) \varphi(\xi) \varphi(x_2, \dots, x_n) dx_2 \dots dx_n d\xi. \end{aligned}$$

If we introduce a change of variables

$$x + \xi \rightarrow x_1, \quad x_2 + \xi \rightarrow x_2, \dots, x_n + \xi \rightarrow x_n.$$

then

$$\xi \rightarrow x_1 - x, \quad x_2 \rightarrow x_2 - x_1 + x, \dots, x_n \rightarrow x_n - x_1 + x,$$

and we obtain



$$\int K(x_1, x_2, \dots, x_n) \varphi(x_2, \dots, x_n) dx_2 \dots dx_n = \int K(x_1, x_2, \dots, x_n) \varphi'(x_1, x_2, \dots, x_n) dx_1 \dots dx_n,$$

where

$$\varphi'(x_1, x_2, \dots, x_n) = \varphi(x_1 - x) \varphi(x_2 - x_1 + x, \dots, x_n - x_1 + x),$$

by means of which we reduce the integrals (19.22) to the form (19.20).

Having demonstrated the convergence of operator integrals of the form (19.18) and (19.21), we note, nevertheless, that due to the generality of our condition of integrability, the class of operator integrals that are guaranteed to converge turns out to be somewhat narrow. Thus, this class does not include the integrals

$$\int A(x_1, \dots, x_n) dx_1 \dots dx_n,$$

taken over the whole infinite space of the points  $x_1, \dots, x_n$ , since they correspond to the function  $F = 1$  which does not vanish at infinity.

The matrix elements of the operator integrals of the above form have to be defined separately by means of a limiting process starting with the corresponding sequence of functions of the class  $C(\infty, \infty, n)$ , which approaches 1 in an expanding region covering, in the limit, the entire range of integration.

# Chapter 4

## SCATTERING MATRIX

### §20. Basic Concepts of the Theory of Interacting Fields

#### 20.1. Introduction. The Schroedinger equation

$$i \frac{\partial \psi(t)}{\partial t} = H \psi(t), \quad (9.2)$$

is usually the starting point of the analysis of the interaction problem in quantum mechanics. The complete Hamiltonian  $H$  of the system is written as the sum of the free-motion Hamiltonian  $H_0$  and the interaction Hamiltonian  $H_1$  :

$$H = H_0 + H_1. \quad (20.1)$$

Perturbation theory is usually employed because an exact solution of the Schroedinger equation (19.2) cannot, as a rule, be obtained. The initial step then is to neglect the interaction term, and this yields the problem

$$i \frac{\partial \psi_0(t)}{\partial t} = H_0 \psi_0(t) \quad (20.2)$$

which can be solved *exactly*.

In the theory of quantized fields, as in nonrelativistic quantum mechanics, an *exact* solution can be obtained only for relatively simple models [see, for example, Schweber (1961), Chapter 7]. This is why it is again necessary to resort to perturbation theory based

on the initial approximation of noninteracting particles. One then has to introduce into the analysis the corresponding idealized free fields, and consider the interaction as an additional factor that has little effect on the properties of the dynamic system and can be “turned on” or “turned off.” At first sight, the problem presented in this form appears to be unobjectionable. In fact, elementary particles interact stringly only in sufficiently close encounters (in collision processes). It would therefore seem natural to suppose that the interaction between fields at large distances is unimportant and, to a particular approximation, can be neglected, and real particles can be looked upon as free.

However, even free particles interact with the fields produced by them (this is well known, for example, from classical electrodynamics). In the quantum case, this situation corresponds to interaction between the particles and vacuum which acts as a kind of physical medium in which they move (for example, see §27 for the phenomenon of vacuum polarization). Although the interaction may be weak, the fact that the particles have small dimensions ensures that the effects of this interaction turn out to be large (infinite, in the limit of point particles).

It follows that the use of the idea of “bare” (i.e., noninteracting, even with vacuum) particles turns out to be unsatisfactory, and it is very desirable to deal, right from the start, with real interacting particles without introducing the artificial concept of fictitious free fields. This program is, to some extent, realized in the so-called axiomatic development of the theory of quantized fields.

It is important to emphasize that recourse to perturbation theory implicitly assumes that one can use expansions into series in powers of the interaction constant (or constants). This, in fact, contains two hypotheses:

(a) The existence of the limit of weak coupling in physically meaningful solutions of the equations of quantum field theory. Solutions that do have this limit will be called *adiabatic solutions*.

(b) Analyticity, or sufficiently weak nonanalyticity of the adiabatic solution in the coupling constant in the region of the origin.

The use of these hypotheses is based historically on quantum electrodynamics, i.e., a quantum field-theory model describing the interaction between electrically charged particles and the electromagnetic field. In this model, the coupling constant is the electric charge  $e$ , and the perturbation-theory expansion parameter is proportional to the fine structure constant which, in our system of units, is  $\alpha = e^2/4\pi$ .

The actual expansion parameter is of the order of  $10^{-3}$  [ $\sim \alpha/\pi$ , see (41.40) and (50.10), below], so that the higher-order perturbation theory terms are, in quantum electrodynamics, numerically very small corrections to the main expressions obtained from the first non-vanishing perturbation-theory orders (similar to those considered in §26, below).

These expressions were derived in the late twenties and turned out to be in close agreement with experimental values, so that there seemed little doubt as to the validity of perturbation-theory expansions in quantum electrodynamics.

Hypothesis (a) has therefore always seemed completely natural in quantum electrodynamics. It was therefore actually adopted for strong interactions as well.

The validity of hypothesis (b) was unquestioned in quantum electrodynamics for the same reasons.

From the modern point of view, extension of hypotheses (a) and (b) to strong and weak interactions is *not* justified.

Thus, it is very probable that hypothesis (b) is not valid for renormalizable interactions.

At the same time, hypothesis (a) clearly contradicts some of the recently discovered nonpolynomial models of strong interactions [see, for example, Takhtadzhian and Faddeev (1974)].

Our immediate aim is to develop the perturbation-theory formalism for the constructive evaluation of matrix elements and transition probabilities. We shall adopt the procedure based on the initial approximation of noninteracting field.

The formalism obtained in this way will, in the first instance, be used in spinor electrodynamics (Chapters 5 and 6). Its application to renormalizable strong-interaction models (§36) is largely methodological in character. No reasonable quantitative results can as yet be obtained here.

*20.2. Interaction Representation.* In the absence of interaction, the system of wave fields is described by the analog of (20.2) for the state amplitude  $\Psi(t)$

$$i \frac{\partial \Psi(t)}{\partial t} = H_0 \Psi(t). \quad (20.3)$$

Formal integration yields

$$\Psi(t) = e^{-iH_0 t} \Phi. \quad (20.4)$$

The constant amplitude  $\Phi$  was introduced in §9. If there *is* an interaction, the expression given by (20.4) no longer satisfies the equation of motion

$$i \frac{\partial \Psi(t)}{\partial t} = (H_0 + H_1) \Psi(t) \quad (20.5)$$

but we can generalize it by assuming that  $\Phi$  is dependent:

$$\Psi(t) = e^{-iH_0 t} \Phi(t). \quad (20.6)$$

Substituting (20.6) in (20.5), we obtain

$$i \frac{\partial \Phi(t)}{\partial t} = e^{iH_0 t} H_1 e^{-iH_0 t} \Phi(t). \quad (20.7)$$

Let us now consider the meaning of the above equation. The density of the interaction Hamiltonian

$$H_1 = \int H_1(x) dx,$$

on the right-hand side of (20.5) and (20.7) is a polynomial function of the field operators in the Schroedinger representation:

$$H_1(\mathbf{x}) = u_\alpha(\mathbf{x}) \dots u_\omega(\mathbf{x}) = u_\alpha(0, \mathbf{x}) \dots u_\omega(0, \mathbf{x}) = u_\alpha(x) \dots u_\omega(x) |_{x^0=0}.$$

However, according to (9.28),

$$e^{iH_0 t} H_1(\mathbf{x}) e^{-iH_0 t} = (e^{iH_0 t} u_\alpha(0, \mathbf{x}) e^{-iH_0 t}) \dots (e^{iH_0 t} u_\omega(0, \mathbf{x}) e^{-iH_0 t}) = u_\alpha(x) \dots u_\omega(x)$$

where  $u_i(x) = u_i(x^0, \mathbf{x})$  are the field operators in the Heisenberg representation of the free-motion Hamiltonian.

The application of the operator  $e^{iH_0 t} \dots e^{-iH_0 t}$  is thus reduced to the replacement of the free-field operator functions of the Schroedinger representation  $u_i(x) \equiv u_i(0, \mathbf{x})$  by the free-field operator functions in the Heisenberg representation  $u_i(x)$ .

According to (20.7), the new state amplitude  $\Phi(t)$  is now the solution of

$$i \frac{\partial \Phi(t)}{\partial t} = H_1(t) \Phi(t), \quad (20.8)$$

where the operator space density

$$H_1(t) = \int H_1(t, \mathbf{x}) d\mathbf{x} = \int H_1(x) d\mathbf{x} \quad (20.9)$$

depends on free fields in the Heisenberg representation. This representation of the Schroedinger equation is called the *interaction representation*.

If we now consider the expectation value of the dynamic operator  $B$  in this representation, i.e.,

$$\bar{B}_t = \bar{\Psi}^*(t) B \Psi(t) = \bar{\Phi}^*(t) e^{iH_0 t} B e^{-iH_0 t} \Phi(t),$$

we see that this can be written as the average over the amplitudes  $\Phi(t)$  in the interaction representation of the operator expression

$$e^{iH_0 t} B e^{-iH_0 t} = \text{MMM} \quad (20.10)$$

which is naturally interpreted as the interaction representation for the dynamic variables  $B$ .

The above transformation of the interaction Hamiltonian can now be written in the form

$$e^{iH_0 t} H_1 e^{-iH_0 t} = H_{\text{int}}(t). \quad (20.11)$$

We thus see that in the interaction representation, the operator expressions for the dynamic variables must be regarded as functions of field operators in the Heisenberg

representation for the free field. In other words, the operators for the dynamic variables are expressed by forms which depend on the field functions

$$u(t, x) = u(x),$$

that satisfy the homogeneous equations of the free fields.

The principal defect of all the preceding representations of the Shroedinger equation, including the interaction representation, is the distinctive role played by the time, and the consequent manifest noncovariance of the formulation. This formal defect of the theory was eliminated in a special modification of the interaction representation developed by Tomonaga (1946) and by Schwinger (1948). In this modification, instead of dealing with the surfaces  $t = \text{const}$  in four-dimensional space-time, a more general class of space-like surfaces  $\sigma$  is introduced. We shall deal with this in greater detail in Chapter 7.

*20.3. The Scattering Matrix.* We note that even equation (20.8) can be used to introduce into our discussion a very important quantity characterizing the system, i.e., the so-called *scattering matrix*, or the *S-matrix*. Let us study a process at the beginning and at the end of which we have only particles that are widely separated from each other and may be considered to be free.

To calculate the probability amplitude for the scattering process and for the mutual transformations of particles in this process, we examine the situation in which the interaction  $H_1(t)$  is adiabatically switched on in the infinitely remote past and is adiabatically switched off in the infinitely remote future. By denoting the amplitude of the initial state by  $\Phi(-\infty)$  and the amplitude of the final state by  $\Phi(\infty)$ , we can establish a connection between them by means of the relation

$$\Phi(\infty) = S\Phi(-\infty), \quad (20.12)$$

in which the operator  $S$  is called the scattering operator or the scattering matrix. The squares of the corresponding matrix elements of the operator  $S$  determine the transition probabilities and the effective cross sections of the various possible scattering and mutual transformation processes involving the particles.

To obtain formulas suitable for calculations, one could, starting with (20.8), obtain its solution by the method of successive approximations in the form of an expansion in powers of the small interaction. One would then obtain a relation between  $\Phi(-\infty)$  and  $\Phi(\infty)$  of the form of (20.12), with the operator  $S$  expressed in the form of the expansion

$$S = 1 - i \int_{-\infty}^{+\infty} H_1(t) dt + (-i)^2 \int_{-\infty}^{+\infty} H_1(t) dt \int_{-\infty}^t H_1(t') dt' + \dots$$

It is just by such a method that the investigation of the  $S$ -matrix was carried out in most of the early fundamental work on quantum field theory [Tomonaga (1946), Schwinger (1948), Pauli and Villars (1949), Dyson (1949a, b), Salam (1951)].

In our opinion, however, it is more useful to employ the method proposed by Stueckelberg [Stueckelberg and Rivier (1949) and Stueckelberg and Green (1951)] in which the

generalized scattering matrix is introduced without reference to the Hamiltonian formalism or the Schroedinger equation. In their place, in order to obtain the explicit form of the  $S$ -matrix, one uses certain explicitly formulated physical conditions, among which the condition of causality plays an important role. Stueckelberg himself did not succeed in obtaining a sufficiently clear and general formulation of the causality condition as a result of which his ideas were not widely accepted.

We shall present below a new formulation of the condition of causality [Bogolyubov (1955)] and a method based on it of constructing the scattering matrix in the quantum theory of interacting fields.

In developing this theory, we shall, as usual, employ the operations of "switching on" and "switching off" of the interaction. To describe this operation mathematically, we introduce the function  $g(x)$  with values in the range  $(0, 1)$  which represents the extent of switching on the interaction. In those regions where  $g(x) = 0$ , the interaction is absent, in those regions where  $g(x) = 1$ , it is switched on completely, and for  $0 < g(x) < 1$ , the interaction is switched on only partially. By replacing the actual interaction Lagrangian  $\mathcal{L}(x)$  by the product  $\mathcal{L}(x)g(x)$ , we obtain an interaction "switched on with an intensity  $g(x)$ ."

Now let  $g(x)$  differ from zero only in a certain finite space-time region. In the sufficiently remote past and future the fields are then free, so that the initial and final states of the dynamic system may be characterized by the usual constant state amplitudes introduced in Chapter II. The two quantities  $\Phi(-\infty)$  and  $\Phi(\infty)$  are related by a certain operator  $S(g)$  which transforms  $\Phi(-\infty)$  into  $\Phi(\infty)$  and which depends on the behavior of the function  $g(x)$ . By fixing the amplitude of the initial state  $\Phi(-\infty) = \Phi$ , we may regard the final amplitude as a functional of  $g$ :

$$\Phi(\infty) = \Phi(g) = S(g)\Phi. \quad (20.13)$$

In accordance with this definition, it is natural to interpret  $S(g)$  as the scattering matrix for the case when the interaction is switched on with an intensity  $g$ . The actual case in which the interaction is switched on fully over the entire space-time must now be investigated within the given framework by means of a transition to the limit in which the region within which  $g = 1$  is extended indefinitely and in the limit includes all space-time. In this case, the usual scattering matrix  $S$  may be defined by

$$S = S(1). \quad (20.14)$$

**20.4. Relativistic Covariance and Unitary Nature of the  $S$ -Matrix.** Let us now formulate a number of fundamental physical conditions which must be satisfied by the matrix  $S$ . As always, an important physical requirement is the *condition of relativistic covariance*. To formulate it explicitly, let us consider a transformation  $L$  belonging to the inhomogeneous Lorentz group:

$$x \rightarrow x' = Lx. \quad (20.15)$$

In the absence of interaction in the free-field theory, the transformation law for the

state amplitude which corresponds to the transformation (20.15) had, in accordance with (9.12), the form

$$\Phi' = U_L \Phi. \quad (20.16)$$

In the present case, when  $\Phi = \Phi(g)$  it is also necessary to take into account the fact that the function  $g(x)$  itself, which may be regarded as a certain "classical field," undergoes the transformation (20.15); since the region of interaction described by the function  $g$  remains invariant under the transformation (20.15), the transition to new coordinates in the argument of  $g$  gives

$$g(x) \rightarrow Lg(x) = g(L^{-1}x). \quad (20.17)$$

The transformation law for the amplitude  $\Phi(g)$  will therefore now be of the form:

$$\Phi(g) \rightarrow \Phi'(Lg) = U_L \Phi(g). \quad (20.18)$$

Considerations of relativistic covariance also make us require that the transformation law (20.13) from the initial to the final amplitude should not depend on the frame of reference, i.e., that

$$\Phi'(g) = S(g) \Phi'. \quad (20.19)$$

Substituting (9.12) and formula (20.18) with shifted argument ( $Lg \rightarrow g$ ) into this equation relation, we find on taking (20.13) into account

$$U_L S(L^{-1}g) \Phi = S(g) U_L \Phi. \quad (20.20)$$

Since the amplitude of the initial state  $\Phi$  is arbitrary, this expression may be written in operator form

$$U_L S(L^{-1}g) = S(g) U_L,$$

or, after shifting the arguments by  $L$ , multiplying on the right by  $U_L^{-1}$ , and taking into account the unitary nature of  $U_L$ :

$$S(Lg) = U_L S(g) U_L^{-1} = U_L S(g) U_L^\dagger. \quad (20.21)$$

This formula expresses the *covariance of the operator*  $S(g)$ .

We shall now formulate another general requirement, i.e., requirement that the norm of the wave functions be conserved. In the present case, we must demand that

$$\Phi^*(g) \Phi(g) = \Phi^* \Phi,$$



from which it follows that

$$\hat{S}(g) S(g) = 1, \quad (20.22)$$

i.e., the operator  $S(g)$  must be *unitary*.

**20.5. The Condition of Causality.** We must also guarantee that the condition of causality is satisfied, according to which any event occurring in the system may exert an influence on the evolution of the system only in the future and cannot exert any influence on the behavior of the system in the past at times preceding the given event. We must therefore require that the change in the interaction law in any space-time region can influence the evolution of the system only at subsequent times.

To formulate the condition of causality explicitly, we shall first consider the case when the space-time region  $G$  in which the function  $g(x)$  differs from zero may be divided into two separate sub-regions  $G_1$  and  $G_2$  such that all points of one of them ( $G_1$ ) lie in the past with respect to a certain time instant  $t$ , while all points of the other sub-region ( $G_2$ ) lie in the future with respect to  $t$ . The function  $g(x)$  may in this case be represented as a sum of two functions

$$g(x) = g_1(x) + g_2(x), \quad (20.23)$$

one of which ( $g_1$ ) differs from zero only in  $G_1$ , while the second ( $g_2$ ) differs from zero only in  $G_2$ .

At time  $t$ , we may define a state characterized by the amplitude  $\Phi_t$  which, because of considerations of causality, should not depend on the interaction in the region  $G_2$  and which may therefore be written in the form

$$\Phi_t = S(g_1) \Phi, \quad (20.24)$$

where  $S(g_1)$  is the scattering matrix for the case when the interaction is switched on with an intensity  $g_1(x)$ . The final state  $\Phi(g)$  may now be obtained from  $\Phi_t$  with the aid of the operator  $S(g_2)$  which describes the interaction in the region  $G_2$

$$\Phi(g) = S(g_2) \Phi_t. \quad (20.25)$$

Comparing (20.23)–(20.25) with (20.13) we find that

$$S(g_1 + g_2) = S(g_2) S(g_1) \quad \text{for } G_2 > G_1. \quad (20.26)$$

(The inequality  $G_2 > G_1$  indicates that all points of the region  $G_2$  occur at times later than all points of the region  $G_1$ .) Relation (20.26) represents the formulation of the principle of causality for  $G_2 > G_1$ .

We now consider two cases which differ from each other by the form of the interaction in the region  $G_2$  and which are described by the same function in  $G_1$ , i.e.,

$$g'(x) = g'_2(x) + g_1(x), \quad g''(x) = g''_2(x) + g_1(x).$$

By constructing the expression  $S(g'')S(g')$  we see without difficulty that it does not depend on the behavior of the functions  $g''$  and  $g'$  in the region  $G_1$  since according to (20.26)

$$\dot{S}(g_1 + g_2) = \dot{S}(g_1) \dot{S}(g_2),$$

and taking into account the unitary properties of the matrix  $S$  we obtain:

$$S(g'') \dot{S}(g') = S(g'_2) S(g_1) \dot{S}(g_1) \dot{S}(g'_2) = \dot{S}(g'_2) \dot{S}(g'_2).$$

Thus the product  $S(g'')S^\dagger(g')$  does not in fact depend on the behavior of the function  $g$  in the region  $G_1$ . This is due to the fact that the dependence on the state of the system prior to the time  $t$ , which is contained in  $S(g'')$ , is eliminated by the corresponding part of the operator  $S^\dagger(g')$ .

Therefore, in the more general case, we shall also adopt the following *formulation of the condition of causality*: if there are two functions  $g''(y)$  and  $g'(y)$  which coincide for  $y^0$  smaller than a certain  $t$ , then the product  $S(g'')S^\dagger(g')$  should not depend on the state of the system for  $y^0 < t$ .

For future use, it will be very convenient to have the condition of causality formulated in differential form.

We shall often need to use the concept of the functional derivative which is a natural extension of the concept of a partial derivative. As is well known, the partial derivative  $\partial f / \partial z_i$  of a function of  $n$  variables  $z_1, \dots, z_n$  may be defined as the coefficient of  $dz_i$  in the sum

$$df = \sum_{(1 \leq i \leq n)} A_i dz_i, \quad (20.27)$$

which is the differential of this function.

Let us now consider a certain functional  $I(u)$  for which the variation  $\delta I(u)$ , defined as the principal part of the increment  $I(u + \delta u) - I(u)$ , may be represented by an integral of the form

$$\delta I(u) = \int_G A(x, u) \delta u dx,$$

where  $A$  is a functional of  $u$  which depends on the position of the point  $x$  in the region  $G$ :

$$A = A(x, u),$$

in a manner similar to the way in which  $A_i$  in the sum (20.27) is a function of  $z_1, \dots, z_n$

depending on the index  $i$ . By analogy with the preceding definition of a partial derivative, we now introduce the functional derivative  $\delta I(u)/\delta u(x)$  of the functional  $I(u)$  with respect to  $u$  at the point  $x$ , defining it by the relation

$$\frac{\delta I(u)}{\delta u(x)} = A(x, u).$$

In the same way, we can also introduce functional derivatives of higher orders, and it is not difficult to see that functional derivatives possess the main properties of ordinary derivatives.

If we set

$$g'(y) = g(y) \quad \text{and} \quad g''(y) = g(y) + \delta g(y),$$

where  $\delta g(y)$  is an infinitesimal variation of the function  $g(y)$  which differs from zero only for  $y^0 > t$ , then the matrix  $S(g'')$  may be written in the form

$$S(g'') = S(g) + \delta S(g), \quad (20.28)$$

where

$$\delta S(g) = \int_{y^0 > t} \frac{\delta S}{\delta g(y)} \delta g(y) dy.$$

It may now be seen that the expression

$$S(g'') \overset{\uparrow}{S}(g') = S(g) \overset{\uparrow}{S}(g) + \delta S(g) \overset{\uparrow}{S}(g) = 1 + \delta S(g) \overset{\uparrow}{S}(g)$$

does not depend on the behavior of the function  $g(x)$  for  $x^0 < t < y^0$ .

By going over to the variational derivative, we may therefore formulate the condition of causality as the condition that the expression

$$\frac{\delta S(g)}{\delta g(y)} \overset{\uparrow}{S}(g) \quad (20.29)$$

should be independent of the behavior of the function  $g(x)$  at the point  $x$  for  $x < y$ . From considerations of covariance, it also follows from this that the operator (20.29) cannot depend on the behavior of the function  $g(x)$  also at  $x \sim y$  ( $x \sim y$  denotes that the points  $x$  and  $y$  are separated by a space-like interval).

The condition of causality may evidently be written in the form

$$\frac{\delta}{\delta g(x)} \left( \frac{\delta S(g)}{\delta g(y)} \overset{\uparrow}{S}(g) \right) = 0 \quad \text{for } x \lesssim y. \quad (20.30)$$

Relation (20.30) represents the *formulation of the principle of causality in differential form* which is the *most convenient one* for subsequent considerations.

We thus have conditions of relativistic covariance (20.21), unitarity (20.22), and causality (20.30), which together provide a sufficient basis for the construction of the  $S$ -matrix. In §21 we shall give, with the aid of these conditions and the correspondence arguments, a method for the explicit determination of the coefficients in the expansion of  $S(g)$  into a functional series in  $g(x)$ . We also note that the preceding conditions should also be augmented with the requirement that the energy of stationary states should be positive. In the theory of free fields, the fulfillment of this requirement was considered separately. However, at the present stage of our presentation, we shall be unable to formulate this condition for the theory of interacting fields in view of the fact that at present, we are confined within the framework of perturbation theory and therefore cannot consider bound states.

20.6. “Classical Fields” as Arguments of Functionals. In conclusion, we note that, in the formulation of the theory adopted by us, the function  $g$ , which characterizes the extent of switching on the interaction, appears to play the role of a certain “classical field” which permits us to examine different regions in space-time.

The usual scattering matrix (20.14), which corresponds to the interaction of quantized wave fields over the entire space-time, does not depend on any ordinary functions, so that its matrix elements are numbers which contain no functional dependences. The introduction into the interaction Lagrangian of an unquantized function converts the  $S$ -matrix and its matrix elements into functionals. By computing functional derivatives of these functionals it turns out to be possible to examine the space-time properties of various quantities occurring in the theory (as has just been demonstrated in the formulation of the condition of causality). The “classical fields” (of the type  $g$ ) introduced in this procedure play the role of auxiliary variables in the intermediate stages of the argument and are usually eliminated from the final expressions by a suitable limiting process (for example,  $g \rightarrow 1$ ).

The classical field  $g(x)$  introduced above as a factor multiplying the interaction Lagrangian density (20.17) provides a convenient mathematical means for the formal expansion of the scattering matrix in powers of the interaction which we shall carry out in §21.

Naturally, this method of introducing the auxiliary field is not the only one. In some cases it turns out to be more convenient to introduce such a field in a more “physical” way, for example, in the form of a given external field  $u^{\text{ext}}(x)$ , or of a given external flux  $J^{\text{ext}}(x)$  of some kind of particles. Thus, to obtain Schwinger’s equations for Green’s functions of spinor electrodynamics (Chapter 6), one introduces into the interaction Lagrangian\*  $\mathcal{L}(x)$  an external unquantized electromagnetic potential  $A^{\text{ext}}(x)$  as a result of which  $\mathcal{L}(x)$  takes on the form

$$\mathcal{L}(x) = e : \bar{\psi}(x) \gamma^\mu \psi(x) [A_\mu(x) + A_\mu^{\text{ext}}(x)] :.$$

\*In future we shall always take  $L$  to stand for the interaction Lagrangian which was denoted earlier by  $L_{\text{int}}$ .

Here the classical field  $A^{\text{ext}}$  plays the role of an auxiliary functional argument which facilitates the analysis of the space-time properties of the theory, and in contrast to  $g(x)$ , the field  $A^{\text{ext}}$  (or more accurately its final limiting value) may have a direct physical meaning. The scattering matrix in this case turns out to be a functional of  $A^{\text{ext}}$  and, by repeating without any appreciable changes the argument leading to (20.30), we obtain in this case

$$\frac{\delta}{\delta A_n^{\text{ext}}(x)} \left[ \frac{\delta S(A^{\text{ext}})}{\delta A_m^{\text{ext}}(y)} \hat{S}(A^{\text{ext}}) \right] = 0 \quad \text{for } x \lesssim y. \quad (20.31)$$

By analogy with the foregoing, it is not difficult to reformulate the condition of causality to correspond to any other method of introducing classical fields.

## §21. The Interaction Lagrangian and the $S$ -Matrix

*21.1. Expansion of the  $S$ -Matrix in Powers of the Interaction.* We now begin the actual construction of the scattering matrix from the interaction Lagrangian  $\mathcal{L}(x)$ . We shall seek  $S(g)$  in the form of a formal functional expansion in powers of  $g(x)$ :

$$S(g) = 1 + \sum_{n \geq 1} \frac{1}{n!} \int S_n(x_1, \dots, x_n) g(x_1) \dots g(x_n) dx_1 \dots dx_n, \quad (21.1)$$

in which  $S_n(x_1, \dots, x_n)$  are operator expressions which depend on the complete field functions and on their partial derivatives at the points  $x_1, \dots, x_n$ . In order to guarantee the scalar nature of  $S_n$ , we shall also assume that the Fermi field operators occur in  $S_n$  only in even combinations. In other words, we shall require that the  $S_n(x_1, \dots, x_n)$  should be polylocal operators in the sense of the definition given in §17. It should be emphasized that the requirement that  $S_n$  should depend on the complete field functions, and not separately on their positive- and negative-frequency parts, represents a special physical condition. When this requirement is satisfied, then the relation

$$[S_n(x_1, \dots, x_n), S_m(y_1, \dots, y_m)] = 0, \quad (21.2)$$

holds when all the  $x_i$  are space-like with respect to all the  $y_j$ . Therefore, if the two functions  $g_1(x)$  and  $g_2(x)$  are localized in such space-time regions that any point of one region is space-like with respect to all the points of the other region, then  $S(g_1)$  commutes with  $S(g_2)$ . This essentially expresses the fact (which is also a manifestation of the principle of causality) that a signal cannot propagate with a speed greater than that of light and that the processes of switching on the interaction within two regions  $g_1$  and  $g_2$  which are space-like with respect to one another do not interfere with each other. It is solely in order to guarantee this important physical property that we shall assume that  $S_n$  is a polylocal operator.

To guarantee the convergence at least of the individual terms of (21.1), we shall assume that  $S_n$  are integrable operator functions. Indeed, for sufficiently smooth and sufficiently

rapidly decreasing functions  $g(x)$ , the individual integrals in (21.1) will converge. Naturally, the convergence of the individual terms has no relation to the convergence of the series as a whole. Recent investigations [Dyson (1953), Thirring (1953), Edwards (1953), and Ioffe (1954)] have shown that there are relatively good reasons to expect that this series may even turn out to be divergent. At best, one may hope, in the case of weak interactions, that by taking a certain small number of terms in the expansion one may obtain an approximation whose accuracy improves as the interaction is made weaker. In other words, in certain cases, one may regard the series (21.1) as a *source of asymptotic approximations*. One such case of practical importance is the case of electrodynamics.

However, even in the case of an interaction which is definitely known not to be weak (for example, the meson-nucleon interaction), the formal expansion (22.1) is of considerable interest, since it enables us to arrive fairly simply at a number of quantitative as well as qualitative properties of the matrix  $S(g)$  and hence attempt to establish them on a more rigorous basis at a later stage. Consequently, the study of this formal series must be of definite heuristic value which is enhanced by the fact that all attempts made to go beyond the framework of perturbation theory until now have so far led only to very limited results. We shall later present (see Chapters 8 and 10) a number of considerations with the aid of which it may turn out to be possible to construct a theory of interacting fields without making use of such formal expansions.

By returning to expression (21.1), we see that without restricting the generality of the argument,  $S_n(x_1, \dots, x_n)$  may be considered to be *symmetric* functions of their arguments  $x_1, \dots, x_n$ , since the weighting functions  $g(x_1) \dots g(x_n)$  enter the expressions in a symmetric manner.

**21.2. Conditions of Covariance, Unitarity, and Causality for  $S_n$ .** We shall now make use of the conditions that are satisfied by the matrix  $S(g)$  in order to determine the explicit form of the functions  $S_n$ .

From the condition (20.21) of relativistic covariance of  $S(g)$  we have

$$\begin{aligned} \int U_L S_n(x_1, \dots, x_n) \dot{U}_L^\dagger g(x_1) \dots g(x_n) dx_1 \dots dx_n = \\ = \int S_n(x_1, \dots, x_n) g(L^{-1}x_1) \dots g(L^{-1}x_n) dx_1 \dots dx_n. \end{aligned} \quad (21.3)$$

Introducing the change of variables  $x \rightarrow Lx$  in the right-hand side, we obtain

$$\begin{aligned} \int U_L S_n(x_1, \dots, x_n) \dot{U}_L^\dagger g(x_1) \dots g(x_n) dx_1 \dots dx_n = \\ = \int S_n(Lx_1, \dots, Lx_n) g(x_1) \dots g(x_n) dx_1 \dots dx_n, \end{aligned}$$

from which we arrive at the condition of Lorentz covariance for  $S_n$

$$\left. \begin{aligned} &U_L S_n(x_1, \dots, x_n) \dot{U}_L^\dagger = S_n(Lx_1, \dots, Lx_n) \\ &S_n(x_1, \dots, x_n) = \dot{U}_L^\dagger S_n(Lx_1, \dots, Lx_n) U_L. \end{aligned} \right\} \quad (21.4)$$

or

To take into account the unitary nature (20.22) of the  $S$ -matrix, we multiply the expansion (21.1) by its conjugate

$$\dot{S}(g) = 1 + \sum_{n \geq 1} \frac{1}{n!} \int \dot{S}_n(x_1, \dots, x_n) g(x_1) \dots g(x_n) dx_1 \dots dx_n \quad (21.5)$$

and after introducing for the sake of symmetry of the resulting expressions the notation

$$S_0 = 1,$$

we obtain

$$\begin{aligned} 1 &= \sum_{(k, m \geq 0)} \frac{1}{k!m!} \int S_k(x_1, \dots, x_k) g(x_1) \dots g(x_k) dx_1 \dots dx_k \times \\ &\quad \times \int \dot{S}_m(x_{k+1}, \dots, x_{k+m}) g(x_{k+1}) \dots g(x_{k+m}) dx_{k+1} \dots dx_{k+m} = \\ &= \sum_{(k, n)} \frac{1}{k! (n-k)!} \int S_k(x_1, \dots, x_k) \dot{S}_{n-k}(x_{k+1}, \dots, x_n) g(x_1) \dots g(x_n) dx_1 \dots dx_n. \end{aligned} \quad (21.6)$$

Collecting terms of the same "degree" in  $g(x)$  in (21.6) we obtain the identity ( $1 = 1$ ) for  $n = 0$ , while for  $n > 0$  we obtain

$$\sum_{0 \leq k \leq n} \frac{1}{k! (n-k)!} \int S_k(x_1, \dots, x_k) \dot{S}_{n-k}(x_{k+1}, \dots, x_n) g(x_1) \dots g(x_n) dx_1 \dots dx_n = 0. \quad (21.7)$$

For arbitrary  $g(x)$ , one cannot as yet conclude from this that the expression

$$\sum_k \frac{1}{k! (n-k)!} S_k(x_1, \dots, x_k) \dot{S}_{n-k}(x_{k+1}, \dots, x_n) \quad (21.8)$$

is equal to zero.

Such a conclusion could have been reached if (21.8) turned out to be symmetric in all its arguments  $x_1, \dots, x_n$ . Actually, symmetry exists in each term of (21.8) only within each of the two groups of arguments:  $x_1, \dots, x_k$  and  $x_{k+1}, \dots, x_n$ . To achieve complete symmetrization of (21.8), we introduce the symbol

$$P\left(\frac{x_1, \dots, x_k}{x_{k+1}, \dots, x_n}\right),$$

which denotes the sum over all the  $n!/k!(n-k)!$  ways of dividing the set of points  $x_1, \dots, x_n$  into two sets of  $k$  and  $n-k$  points. Permutations within each of these two sets are not taken into account, since the functions  $S_k$  are symmetric in their arguments. For example,

$$P\left(\frac{x_1}{x_2}\right) S_1(x_1) S_1(x_2) = S_1(x_1) S_1(x_2) + S_1(x_2) S_1(x_1),$$

$$P\left(\frac{x_1, x_3}{x_2}\right) S_2(x_1, x_2) S_1(x_3) = S_2(x_1, x_2) S_1(x_3) + S_2(x_1, x_3) S_1(x_2) + S_2(x_2, x_3) S_1(x_1).$$

To symmetrize (21.7), we rewrite it  $n!$  times, each time altering the notation for the arguments in such a way that the set of points  $x_1, \dots, x_n$  is distributed among the arguments of the functions  $S_k$  and  $S_{n-k}^\dagger$  in a different way. On adding the relations obtained in this way, we then obtain, because the weighting factor  $g(x_1) \dots g(x_n)$  remains unaltered,

$$\sum_k \int P\left(\frac{x_1, \dots, x_k}{x_{k+1}, \dots, x_n}\right) S_k(x_1, \dots, x_k) S_{n-k}^\dagger(x_{k+1}, \dots, x_n) g(x_1) \dots g(x_n) dx_1 \dots dx_n = 0,$$

from which it follows that the expression

$$\sum_{0 \leq k \leq n} P\left(\frac{x_1, \dots, x_k}{x_{k+1}, \dots, x_n}\right) S_k(x_1, \dots, x_k) S_{n-k}^\dagger(x_{k+1}, \dots, x_n)$$

vanishes due to its symmetry with respect to all  $n$  arguments. Remembering that  $S_0 = 1$  we obtain from the above:

$$S_n(x_1, \dots, x_n) + S_n(x_1, \dots, x_n) + \sum_{(1 \leq k \leq n-1)} P\left(\frac{x_1, \dots, x_k}{x_{k+1}, \dots, x_n}\right) S_k(x_1, \dots, x_k) S_{n-k}^\dagger(x_{k+1}, \dots, x_n) = 0. \quad (21.9)$$

We now turn to the condition of causality (20.30). We note from the outset that it is more convenient to deal not with the quantity

$$\frac{\delta S(g)}{\delta g(y)} S^\dagger(g),$$

which in view of the condition of unitarity

$$\frac{\delta}{\delta g(y)} (S S^\dagger) = \frac{\delta S}{\delta g(y)} S^\dagger + S \frac{\delta S^\dagger}{\delta g(y)} = 0$$

is anti-Hermitian, but with the expression

$$H(y; g) = i \frac{\delta S(g)}{\delta g(y)} S^\dagger(g). \quad (21.10)$$

which is evidently Hermitian. Utilizing (21.1) we find that



$$\begin{aligned}
H(y, g) &= i \sum_{n \geq 0} \frac{1}{n!} \int S_{n+1}(y, x_1, \dots, x_n) g(x_1) \dots g(x_n) dx_1 \dots dx_n \times \\
&\quad \times \sum_{m \geq 0} \frac{1}{m!} \int \dot{S}_m(x_{n+1}, \dots, x_{n+m}) g(x_{n+1}) \dots g(x_{n+m}) dx_{n+1} \dots dx_{n+m} = \\
&= \sum_{n \geq 0} \frac{1}{n!} \int H_n(y, x_1, \dots, x_n) g(x_1) \dots g(x_n) dx_1 \dots dx_n, \quad (21.11)
\end{aligned}$$

into which we have introduced the quantities

$$\begin{aligned}
H_n(y, x_1, \dots, x_n) &= \\
&= i \sum_{0 \leq k \leq n} P\left(\frac{x_1, \dots, x_k}{x_{k+1}, \dots, x_n}\right) S_{k+1}(y, x_1, \dots, x_k) \dot{S}_{n-k}^\dagger(x_{k+1}, \dots, x_n) = \\
&= i S_{n+1}(y, x_1, \dots, x_n) + \\
&+ i \sum_{0 \leq k \leq n-1} P\left(\frac{x_1, \dots, x_k}{x_{k+1}, \dots, x_n}\right) S_{k+1}(y, x_1, \dots, x_k) \dot{S}_{n-k}^\dagger(x_{k+1}, \dots, x_n), \quad (21.12)
\end{aligned}$$

which are symmetric in all their arguments with the exception of the first. On evaluating the functional derivative  $\partial/\partial g(x)$  of (21.11), we obtain

$$\begin{aligned}
i \frac{\delta}{\delta g(x)} \left( \frac{\delta S(g)}{\delta g(y)} \dot{S}^\dagger(g) \right) &= \\
&= \sum_{n \geq 1} \frac{1}{(n-1)!} \int H_n(y, x, x_1, \dots, x_{n-1}) g(x_1) \dots g(x_{n-1}) dx_1 \dots dx_{n-1},
\end{aligned}$$

from which, in view of the condition of causality (20.30) and on the basis of the symmetry of the functions  $H_n$  with respect to all their arguments with the exception of the first, it follows that

$$H_n(y, x_1, x_2, \dots, x_n) = 0, \quad (21.13)$$

if  $y \succ x_j$  for at least one  $x_j (j = 1, \dots, n)$ .

**21.3. Explicit Form of  $S_1(x)$  and  $S_2(x, y)$ .** From the conditions of covariance, unitarity, and causality for the complete matrix  $S(g)$  we have thus obtained the conditions of covariance, unitarity, and causality for the functions  $S_n(x_1, \dots, x_n)$ . We shall now show that (21.4), (21.9), and (21.13), together with certain additional considerations which express the correspondence principle, turn out to be sufficient to determine the explicit form of the functions  $S_n(x_1, \dots, x_n)$ .

It turns out that (21.9) and (21.13), regarded as recurrence relations, enable us to determine any arbitrary function  $S_n$  in terms of the "preceding"  $S_k (k = 1, 2, \dots, n-1)$  with the unitary condition (21.9) serving to determine the Hermitian part of the function  $S_n$ , while the condition of causality (21.13) determines the anti-Hermitian part of  $S_n$ . Because of this, the construction of the functions  $S_n(x_1, \dots, x_n)$  may be carried out

by induction, but for this it is necessary to have an explicit expression for  $S_1(x)$ . This function may be determined from considerations of correspondence.

Let us therefore consider the function  $S_1(x)$ . The condition of causality cannot as yet be formulated for a single function, but, as we shall see later, it follows from the condition of causality for  $n = 1$  that  $S_1(x)$  and  $S_1(y)$  must commute at points that are space-like with respect to one another:

$$[S_1(x), S_1(y)] = 0 \quad \text{for} \quad x \sim y, \quad (21.14)$$

for which it is evidently sufficient that  $S_1(x)$  should satisfy the condition of locality (in the sense of the definition given in §17).

By writing the condition of unitarity for  $n = 1$  we have

$$S_1(x) + \overset{\uparrow}{S}_1(x) = 0, \quad (21.15)$$

from which it follows that  $S_1(x)$  may be represented in the form

$$S_1(x) = i\Lambda_1(x), \quad (21.16)$$

where  $\Lambda_1(x)$  is an Hermitian operator. Finally, the condition of relativistic covariance (21.4) gives

$$S_1(x) = \overset{\uparrow}{U}_L S_1(Lx) U_L. \quad (21.17)$$

Thus,  $\Lambda_1(x)$  must be an Hermitian relativistically covariant operator satisfying the condition of locality.

We shall now relate  $\Lambda_1(x)$  to the interaction Lagrangian  $\mathcal{L}(x)$ . It is well known in classical theory that the interaction is taken into account by adding to the free field Lagrangian  $\mathcal{L}_0(x)$  the interaction Lagrangian  $\mathcal{L}(x)$ , and the "equations of motion" may be obtained with the aid of the principle of stationary action. Let us examine the action  $A$  of a system of classical fields in the case when the interaction is switched on with an intensity  $g(x)$ . We have

$$A = \int \mathcal{L}_0(x) dx + \int \mathcal{L}(x) g(x) dx, \quad (21.18)$$

where  $\mathcal{L}_0$  and  $\mathcal{L}$  contain field functions which satisfy appropriate equations of motion. In particular, by regarding  $g(x)$  as an infinitesimal of the first order, we find that these field functions differ from the free field functions also by infinitesimals of the first order. On the other hand, from the fact that the free field equations are obtained from the condition

$$\delta \int \mathcal{L}_0(x) dx = 0,$$

it follows that, if the field functions under the integral sign in the first term on the

right-hand side of (21.18) are given to within infinitesimal terms of the first order, then this will lead to an error of the second order of smallness in the value of the integral  $\int \mathcal{L}_0 dx$ .

Therefore, if the interaction is switched on with an infinitesimal intensity  $g(x)$ , the action of the system will be altered by the amount  $\int \mathcal{L}(x)g(x) dx$  in which  $\mathcal{L}(x)$  depends on the free-field functions.

It is well known that, in the quasi-classical case, the solution of the ordinary Schroedinger equation for the wave function  $\psi$

$$i \frac{\partial \psi}{\partial t} = H\psi$$

takes on the form  $\psi = \exp(iA)$  where  $A$  is the action of the system. The transition from the unperturbed expression

$$A_0 = \int \mathcal{L}_0(x) dx \quad (21.19)$$

to the action (21.18) will evidently correspond to the transformation of the wave function

$$\psi_0 = e^{iA_0} \rightarrow \psi = e^{iA} = e^{i \int \mathcal{L}(x)g(x) dx} \psi_0.$$

Thus, on taking into account the smallness of the quantity  $g$ , this transition will correspond to the following transformation of the wave function:

$$\psi \rightarrow \psi' = \psi + \delta\psi, \quad \delta\psi = i \int \mathcal{L}(x)g(x) dx \psi.$$

Starting with considerations of *correspondence*, we shall demand that the transformation law for the second quantized state amplitude  $\Phi$  should have the same form, i.e.,

$$\Phi \rightarrow \Phi' = \Phi + \delta\Phi, \quad \delta\Phi = i \int \mathcal{L}(x)g(x) dx \Phi.$$

In other words we shall assume that for an infinitesimal  $g$  the matrix  $S(g)$  shall have the form

$$B_{\text{int}}(t),$$

$$S(g) = 1 + i \int \mathcal{L}(x)g(x) dx.$$

By comparing this formula with (21.1), we see that  $S_1(x)$  actually does have the form (21.16) in which  $\Lambda_1(x)$  is equal to the interaction Lagrangian

$$S_1(x) = i\mathcal{L}(x). \quad (21.20)$$

It now follows from relations (21.14), (21.15), and (21.17) that the *interaction Lagrangian must be a local, Hermitian, and relativistically covariant combination of operator field functions*. We note here that it may readily be shown that scalar combinations of

operator functions satisfying (9.9) automatically ensure that the condition of relativistic covariance is satisfied, while the Hermitian conditions

$$\dot{\mathcal{L}}(x) = \mathcal{L}(x) \quad (21.21)$$

and the locality

$$[\mathcal{L}(x), \mathcal{L}(y)] = 0 \quad \text{for } x \sim y \quad (21.22)$$

of  $\mathcal{L}$  are subsidiary conditions that restrict the choice of the interaction Lagrangian.

We now proceed to the determination of  $S_2(x, y)$ . From the condition of causality (21.13) for  $n = 1$ , we obtain for  $x \gtrsim y$

$$H_1(x, y) = iS_2(x, y) + iS_1(x) \dot{S}_1(y) = 0,$$

from which on taking (21.15) into account, we have

$$S_2(x, y) = -\mathcal{L}(x) \mathcal{L}(y) \quad \text{for } x \gtrsim y. \quad (21.23)$$

In view of the symmetry of  $S_2$ , we also have

$$S_2(x, y) = S_2(y, x) = -\mathcal{L}(y) \mathcal{L}(x) \quad \text{for } y \gtrsim x. \quad (21.24)$$

The regions within which (21.23) and (21.24) are defined overlap for  $x \sim y$ , and this leads us to the condition of compatibility of the form (21.14) or (21.22), which represents the condition of locality for  $S_1$  or  $\mathcal{L}$ .

Thus we have obtained the definition of the function  $S_2(x, y)$  in the following form:

$$S_2(x, y) = \begin{cases} -\mathcal{L}(x) \mathcal{L}(y) & (x \gtrsim y), \\ -\mathcal{L}(y) \mathcal{L}(x) & (y \gtrsim x). \end{cases} \quad (21.25)$$

By taking the Hermitian conjugate, we obtain, in view of (21.21),

$$\dot{S}_2(x, y) = \begin{cases} -\mathcal{L}(y) \mathcal{L}(x) & (x \gtrsim y), \\ -\mathcal{L}(x) \mathcal{L}(y) & (y \gtrsim x). \end{cases} \quad (21.26)$$

The symmetry and the polylocal nature of  $S_2(x, y)$  follows from the above form in which it is expressed. It is evident that expression (21.25) satisfies the requirement of covariance. Also, it is not difficult to verify that the unitarity condition is satisfied. In accordance with (21.9), we must now check the relation

$$S_2(x, y) + \dot{S}_2(x, y) + S_1(x) \dot{S}_1(y) + S_1(y) \dot{S}_1(x) = 0.$$

Its validity follows directly from (21.20), (21.21), (21.25), and (21.26). Indeed, we have, for example, for  $x \gtrsim y$

$$\begin{aligned} S_2(x, y) + \overset{\dagger}{S}_2(x, y) + S_1(x) \overset{\dagger}{S}_1(y) + S_1(y) \overset{\dagger}{S}_1(x) = \\ = -\mathcal{L}(x) \mathcal{L}(y) - \mathcal{L}(y) \mathcal{L}(x) + \mathcal{L}(x) \mathcal{L}(y) + \mathcal{L}(y) \mathcal{L}(x) = 0. \end{aligned}$$

Similarly, we verify the validity of the above expression for  $y \gtrsim x$ . We have thus established that (21.25) satisfies all the requirements imposed on  $S_2$ .

**21.4. Chronological Product of Local Operators.** Before undertaking the construction of further functions  $S_k (k = 3, 4, \dots)$ , it is useful to introduce the concept of the *time-ordered*, or *chronological*, *product of operators*. For the time being, we introduce it only for local operators. The chronological product of a number of local operators  $\Lambda_1(x_1) \dots \Lambda_n(x_n)$  is denoted by the symbol  $T(\Lambda_1(x_1) \dots \Lambda_n(x_n))$  and, by definition,  $*$  is equal to the ordinary product of these operators taken in a definite order which corresponds to decreasing components of the arguments of the factors from left to right, i.e.,

$$T(\Lambda_1(x_1) \dots \Lambda_n(x_n)) = \Lambda_{j_1}(x_{j_1}) \Lambda_{j_2}(x_{j_2}) \dots \Lambda_{j_n}(x_{j_n}), \quad (21.27)$$

where the sequence  $x_{j_1}, \dots, x_{j_n}$  is determined by

$$x_{j_1}^0 \geq x_{j_2}^0 \geq \dots \geq x_{j_n}^0.$$

For the sake of brevity, we shall often refer to the chronological product as the  $T$ -product. We shall also need the concept of the *antichronological product* which, by definition, corresponds to increasing time components of the arguments from left to right. Taking, for example, the Hermitian conjugate of the right-hand side of (21.27), we obtain the expression

$$\overset{\dagger}{\Lambda}_{j_n}(x_{j_n}) \dots \overset{\dagger}{\Lambda}_{j_1}(x_{j_1}) \quad (x_{j_n}^0 \leq x_{j_{n-1}}^0 \leq \dots \leq x_{j_1}^0),$$

which may conveniently be called the antichronological product (or  $T^\dagger$ -product) of the operators  $\Lambda_1^\dagger(x_1)$   $T(\Lambda_1(x_1) \dots \Lambda_n(x_n)) = \Lambda_{j_1}(x_{j_1}) \Lambda_{j_2}(x_{j_2}) \dots \Lambda_{j_n}(x_{j_n})$ ,

$$\begin{aligned} \overset{\dagger}{T}(\Lambda_1(x_1) \dots \Lambda_n(x_n)) = \Lambda_{j_1}(x_{j_1}) \Lambda_{j_2}(x_{j_2}) \dots \Lambda_{j_n}(x_{j_n}) \\ (x_{j_1}^0 \leq x_{j_2}^0 \leq \dots \leq x_{j_n}^0). \end{aligned} \quad (21.28)$$

We shall show that the definition of the  $T$ -product is covariant in spite of the distinctive role played by the time. The chronological order of the points  $x_1, \dots, x_n$ , which may change in transforming from one Lorentz frame to another under a Lorentz rotation, is significant for the value of the  $T$ -product. At first glance it might seem that such a transformation might alter the value of the  $T$ -product. However this is not so. Indeed, suppose that the transformation consists of a large number of small transformations, which is always

\*Compare with the definition of the chronological product of two field operators in §15.

possible in view of the continuous nature of Lorentz rotations. The change in the chronological order of the points  $x_1, \dots, x_n$  will then be broken up into a certain number of stages, in each of which there will be a simultaneous change in the time order within a certain group  $x_j, \dots, x_k$  (of two or more) points. But the chronological order of a number of points  $x_j, \dots, x_k$  may be altered by a Lorentz rotation only if these points have a space-like relation to one another. The corresponding operators  $\Lambda_j(x_j) \dots \Lambda_k(x_k)$  will, in this case, commute among themselves by virtue of the property of locality, and their order will be immaterial. Therefore, at each stage of the transformation, the value of the  $T$ -product will not be altered, and, consequently, it will not be altered in general.

It is also not difficult to see that in accordance with the definition (21.27), *the local operators may be commuted within the  $T$ -product without changing its value*. From this it follows, in particular, that if all the operators  $\Lambda$  are the same, the  $T$ -product turns out to be a symmetric function of its arguments.

Returning to (21.25) we see that if we use the  $T$ -product,  $S_2$  may be rewritten in the form

$$S_2(x, y) = -T(\mathcal{L}(x) \mathcal{L}(y)). \quad (21.29)$$

Accordingly, (21.26) will take on the form

$$\dot{S}_2(x, y) = -\dot{T}(\mathcal{L}(x) \mathcal{L}(y)). \quad (21.30)$$

**21.5. Determination of the Functions  $S_n$  for Arbitrary  $n$ .** We shall now show that, in general, the expression

$$S_n(x_1, \dots, x_n) = i^n T(\mathcal{L}(x_1) \dots \mathcal{L}(x_n)), \quad (21.31)$$

which is a natural generalization of (21.20) and (21.29), satisfies all the formal conditions imposed on  $S_n$ . The conditions of symmetry, covariance, and polylocality are now evidently satisfied, and we need only check whether the conditions of unitarity and causality are likewise satisfied.

To do this, we find it more convenient to deal not with (21.31) but with the operator  $S(g)$  as a whole. Substituting (21.31) into (21.1), we obtain

$$S(g) = 1 + \sum_{n \geq 1} \frac{i^n}{n!} \int T(\mathcal{L}(x_1) \dots \mathcal{L}(x_n)) g(x_1) \dots g(x_n) dx_1 \dots dx_n. \quad (21.32)$$

We shall now examine some properties of the coefficients of the expansion (21.32). However, just as it is frequently more convenient in the theory of special functions to derive the interrelationship of the functions not from their explicit structure but from a certain general generating function, so in the present case it turns out to be simpler to verify the conditions of unitarity and causality for the series (21.32) as a whole. To do this we shall rewrite expression (21.32) in a slightly different form: we shall represent the  $n$ th term of the series in the form

$$\frac{i^n}{n!} T \left( \int \mathcal{L}(x_1) g(x_1) dx_1 \dots \int \mathcal{L}(x_n) g(x_n) dx_n \right) = \frac{i^n}{n!} T \left( \left\{ \int \mathcal{L}(x) g(x) dx \right\}^n \right). \quad (21.33)$$

The series (21.32) may now be formally summed by introducing the *T-exponential* [Feynman (1951)] :

$$S(g) = T \left\{ 1 + \sum_{n \geq 1} \frac{i^n}{n!} \left[ \int \mathcal{L}(x) g(x) dx \right]^n \right\} = T \left( \exp i \int \mathcal{L}(x) g(x) dx \right). \quad (21.34)$$

We have thus obtained a new expression for the scattering matrix  $S(g)$ .

We may also introduce the important concept of the *T-exponential* from a different point of view. Let us divide the region within which the interaction described by the function  $g(x)$  is switched on into an infinite number of infinitely thin layers  $\Delta_j$  by space-like surfaces  $t = \text{const}$ . We then have:

$$T \left( \exp i \int \mathcal{L}(x) g(x) dx \right) = T \left( \exp i \sum_j \int_{\Delta_j} \mathcal{L}(x) g(x) dx \right) = T \left( \prod_j \exp i \int_{\Delta_j} \mathcal{L}(x) g(x) dx \right).$$

It is then natural to define the *T-exponential* (21.34) as the limit of the *T-product*

$$T \left( \exp i \int \mathcal{L}(x) g(x) dx \right) = \lim_{\Delta_j \rightarrow 0} T \left\{ \prod_j \left( 1 + i \int_{\Delta_j} \mathcal{L}(x) g(x) dx \right) \right\}. \quad (21.35)$$

The proof of the unitary nature of the matrix  $S(g)$  is obvious from the representation (21.35). Indeed, the right-hand side of (21.35) is the ordinary product

$$\prod_j \left( 1 + i \int_{\Delta_j} \mathcal{L}(x) g(x) dx \right),$$

taken in the appropriate chronological sequence of the layers  $\Delta_j$ . But, for sufficiently small  $\Delta_j$ , each factor of this product is unitary to within quantities of a higher order of smallness, and therefore the product as a whole is unitary. This completes the proof of the unitary nature of  $S(g)$ .

Let us now check the condition of causality. By calculating the variational derivative of  $S(g)$  at the point  $y$ , we obtain

$$-i \frac{\delta S(g)}{\delta g(y)} = T \left( \mathcal{L}(y) \exp i \int \mathcal{L}(x) g(x) dx \right).$$

We divide the four-dimensional space into two parts  $G_+$  and  $G_-$  by a space-like surface  $x^0 = \text{const} = y^0$  with respect to which  $G_+$  lies in the "future," while  $G_-$  lies in the "past." We then have

$$\begin{aligned}
 -i \frac{\delta S(g)}{\delta g(y)} &= T \left( \mathcal{L}(y) \exp i \int_{G_+} \mathcal{L}(x) g(x) dx \exp i \int_{G_-} \mathcal{L}(z) g(z) dz \right) = \\
 &= T \left( \mathcal{L}(y) \exp i \int_{G_+} \mathcal{L}(x) g(x) dx \right) T \left( \exp i \int_{G_-} \mathcal{L}(z) g(z) dz \right). \quad (21.36)
 \end{aligned}$$

On the other hand, in a completely analogous manner we obtain

$$\begin{aligned}
 S(g) &= T \left( \exp \left[ i \int_{G_+} \mathcal{L}(x) g(x) dx + i \int_{G_-} \mathcal{L}(z) g(z) dz \right] \right) = \\
 &= T \left( \exp i \int_{G_+} \mathcal{L}(x) g(x) dx \right) T \left( \exp i \int_{G_-} \mathcal{L}(z) g(z) dz \right),
 \end{aligned}$$

and, likewise,

$$\dot{S}(g) = \left\{ T \left( \exp i \int_{G_-} \mathcal{L}(x) g(x) dx \right) \right\}^* \left\{ T \left( \exp i \int_{G_+} \mathcal{L}(z) g(z) dz \right) \right\}^*.$$

From this, taking into account the unitary nature of the expression  $T(\exp i \int_{G_-} \mathcal{L}(x) g(x) dx)$ , and using (21.36),

$$-i \frac{\delta S(g)}{\delta g(y)} \dot{S}(g) = T \left( \mathcal{L}(y) \exp i \int_{G_+} \mathcal{L}(z) g(z) dz \right) \left\{ T \left( \exp i \int_{G_+} \mathcal{L}(z) g(z) dz \right) \right\}^*.$$

Thus the product

$$-i \frac{\delta S(g)}{\delta g(y)} \dot{S}(g)$$

does not depend on the behavior of the function  $g(x)$  in the region  $G_-$ , i.e., for  $x^0 < y^0$ . From considerations of covariance, the same also holds for  $x \sim y$ . The condition of causality is therefore satisfied.

The above proofs of causality and unitarity of the operator  $S(g)$  are very simple and direct. However, it should be noted that from the purely mathematical point of view, they are not quite consistent. Indeed, in the course of the argument, we have associated the question of validity of the elementary relations (21.9) and (21.12) in the case of the product (21.31) with the entirely unclear problems of summing the series (21.1) as a whole, of making transitions to the limit, and so on. Strictly speaking, all these points are not at all required for the proof. Instead of dealing with the  $T$ -exponential, we may deal with the " $T$ -exponential accurate to a given order in  $g$ ," and in this way all the problems connected with the summation of the series are automatically avoided.

**21.6. Analysis of the Arbitrariness of the Functions  $S_n$  and the Most General Form of  $S(g)$ .** We have thus verified that the expression

$$S_n(x_1, \dots, x_n) = i^n T(\mathcal{L}(x_1) \dots \mathcal{L}(x_n)) \quad (21.31)$$



is acceptable in the sense that it satisfies all the requirements imposed on  $S_n$ . However, it turns out that *this expression is not the most general one* that satisfies all the above requirements. Let us therefore investigate the problem of constructing the most general expression for  $S_n(x_1, \dots, x_n)$ , which satisfies the conditions of symmetry, covariance, causality, and unitarity, and thus completely solve the given problem of constructing the operator  $S(g)$ .

To do this, let us first investigate the procedure of determining the function  $S_n(x_1, \dots, x_n)$ , given the preceding functions  $S_1, S_2, \dots, S_{n-1}$ . As a consequence of the condition of unitarity (21.9),  $S_n$  is determined in terms of these functions to within a certain anti-Hermitian operator which we shall denote by  $i\Lambda_n(x_1, \dots, x_n)$ . Moreover, the quantity  $\Lambda_n(x_1, \dots, x_n)$  must be a symmetric function of its arguments. The condition of causality, (21.13), completely determines the operator function  $S_n(x_1, \dots, x_n)$  in terms of the preceding functions within the domain of definition of its arguments in which  $x_1 \gtrsim x_j$  (for at least one of  $j = 2, 3, \dots, n$ ).

Therefore, the anti-Hermitian operator  $i\Lambda_n(x_1, \dots, x_n)$  must vanish in the aforementioned region. Because it is symmetric in all its arguments, it follows that it must also vanish whenever

$$x_i \neq x_j,$$

for at least one pair of arguments  $x_i$  and  $x_j$  and, consequently, it may differ from zero only when all its arguments are equal, i.e.,

$$x_1 = x_2 = \dots = x_n.$$

Thus, it follows from the conditions of causality, unitarity, and symmetry that the Hermitian operator  $\Lambda_n(x_1, \dots, x_n)$  is a quasi-local operator in the sense of the definition given in §17, and its coefficient functions have the form

$$Z\left(\dots \frac{\partial}{\partial x_i} \dots\right) \delta(x_1 - x_2) \dots \delta(x_1 - x_n),$$

where, because of invariance under translation,  $Z$  cannot depend on  $x_i$ .

We have thus established that the conditions of invariance, symmetry, unitarity, and causality determine  $S_n$  when  $S_1, S_2, \dots, S_{n-1}$  are given to within  $i\Lambda_n$ , where  $\Lambda_n(x_1, \dots, x_n)$  is a symmetric Hermitian quasi-local operator which transforms like a scalar. Therefore, to obtain expressions for  $S_1, S_2, \dots, S_n$ , one must specify not only the operator  $\mathcal{L}(x)$  but also the sequence of quasi-local operators

$$\Lambda_2(x_1, x_2), \dots, \Lambda_n(x_1, \dots, x_n).$$

We have thus arrived at results which, at first glance, seem somewhat strange. The matrix  $S(g)$  cannot be completely determined by specifying only the interaction Lagrangian: *it is also necessary to specify the infinite sequence of quasilocal operators*

$$\Lambda_2(x_1, x_2), \Lambda_3(x_1, x_2, x_3), \dots, \Lambda_n(x_1, \dots, x_n) \dots \quad (21.37)$$

To clarify this situation, let us approach this problem from a slightly different point of view. Consider the expression

$$T\left(\exp i \int \mathcal{L}(x; g) dx\right), \quad (21.38)$$

in which the "Lagrangian"  $\mathcal{L}(x; g)$  is defined by

$$\begin{aligned} \mathcal{L}(x; g) = & \mathcal{L}(x) g(x) + \\ & + \sum_{v \geq 2} \frac{1}{v!} \int \Lambda_v(x, x_1, \dots, x_{v-1}) g(x) g(x_1) \dots g(x_{v-1}) dx_1 \dots dx_{v-1}. \end{aligned} \quad (21.39)$$

In view of the quasilocal nature of the functions  $\Lambda_v$ , all the integrations in (21.39) disappear, and  $\mathcal{L}(x; g)$  in fact depends on the field functions  $u(x)$  at the point  $x$ , and is therefore a local operator. In addition to the field operators  $u(x)$ , the expression  $\mathcal{L}(x; g)$  also depends on the functions  $g(x)$  which may be regarded as a "classical" field. Consequently, (21.38) satisfies all the conditions imposed on  $S(g)$ , including the correspondence condition (21.20), and may be regarded as the expression for the scattering matrix  $S(g)$ . By expanding (21.20) into a series in powers of  $g$ , we obtain expressions for  $S_n(x_1, \dots, x_n)$  which satisfy all the requirements imposed on them. We have

$$T\left(\exp i \int \mathcal{L}(x; g) dx\right) = 1 + \sum_{m \geq 1} \frac{i^m}{m!} \int T(\mathcal{L}(x_1; g) \dots \mathcal{L}(x_m; g)) dx_1 \dots dx_m.$$

By substituting the (21.39) into the last expression, we obtain

$$\begin{aligned} T\left(\exp i \int \mathcal{L}(x; g) dx\right) = & 1 + \sum_{\left(\substack{v_1 \geq 0, \dots, v_m \geq 0 \\ m \geq 1}}\right) \frac{i^m}{m!} \frac{1}{v_1! \dots v_m!} \int T(\Lambda_{v_1}(x_1, \dots, x_{v_1}) \dots \\ & \dots \Lambda_{v_m}(x_{v_1+\dots+v_{m-1}+1}, \dots, x_{v_1+\dots+v_m})) g(x_1) \dots g(x_{v_1+\dots+v_m}) dx_1 \dots \\ & \dots dx_{v_1+\dots+v_m}, \end{aligned}$$

in which we have set  $\mathcal{L}(x) = \Lambda_1(x)$  in accordance with (21.16) and (21.20). The symbol

$$T(\Lambda_{v_1}(x_1, \dots, x_{v_1}) \dots \Lambda_{v_s}(x_{\lambda+1}, \dots, x_{\lambda+\lambda_s}))$$

denotes the product of the operators  $\Lambda_{v_1}, \dots, \Lambda_{v_s}$ , taken in the chronological order of their time arguments. The multiplicity of arguments in each of the  $\Lambda_v$  should give us no concern, since, by definition,  $\Lambda_v$  differs from zero only when all its arguments are equal.

Let us now rearrange this series in powers of  $g(x)$  by collecting together terms in which  $g(x)$  is raised to a definite  $n$ th power and which contain exactly  $n$  integrations:

$$T \left( \exp i \int \mathcal{L}(x; g) dx \right) = 1 + \sum_{n \geq 1} \sum_{\left( \begin{smallmatrix} 1 \leq m \leq n \\ \sum v_i = n \end{smallmatrix} \right)} \frac{i^m}{m!} \frac{1}{v_1! \dots v_m!} \int T(\Lambda_{v_1}(x_1, \dots, x_{v_1}) \dots \Lambda_{v_m}(x_{v_1+\dots+v_{m-1}+1}, \dots, x_n)) g(x_1) \dots g(x_n) dx_1 \dots dx_n.$$

This last expansion still differs from the expansion (21.1) by the unsymmetric nature of the coefficients of the various "powers" of the function  $g(x)$ . To symmetrize them, let us use the symmetry of the weighting factor  $g(x_1) \dots g(x_n)$  for any given  $n$ , and the symmetry of the functions  $\Lambda_v$ , to interchange  $n!/(v_1! \dots v_m!)$  times the designations of the variables  $x_1, \dots, x_n$ , so that after taking into account the symmetry of  $\Lambda_v$ , the sum of all the expressions thus obtained should be a symmetric function of all the arguments  $x_1, \dots, x_n$ . After dividing the result by the number  $n!/(v_1! \dots v_m!)$ , we arrive at the expression

$$T \left( \exp i \int \mathcal{L}(x; g) dx \right) = 1 + \sum_{n \geq 1} \sum_{\left( \begin{smallmatrix} 1 \leq m \leq n \\ \sum v_i = n \end{smallmatrix} \right)} \frac{i^m}{n!m!} \int P(x_1, \dots, x_{v_1} | x_{v_1+1} \dots | \dots, x_n) \times \\ \times T(\Lambda_{v_1}(x, \dots, x_{v_1}) \dots \Lambda_{v_m}(\dots, x_n)) g(x_1) \dots g(x_n) dx_1 \dots dx_n,$$

where  $P(x_1, \dots, x_{v_1} | x_{v_1+1} \dots | \dots, x_n)$  is the symmetrization operator over arbitrary decompositions of the set of  $n$  points into  $n!/(v_1! \dots v_m!)$  possible combinations of  $v_1, v_2, \dots, v_m$  points at a time ( $\sum v_i = n$ ). This operator is a natural generalization of the operator

$$P \left( \frac{x_1, \dots, x_v}{x_{v+1}, \dots, x_n} \right) = P(x_1, \dots, x_v | x_{v+1}, \dots, x_n)$$

introduced above. Thus, the matrix (21.38) is written in the form (21.1), where the coefficients  $S_n$  have the form

$$S_n(x_1, \dots, x_n) = \\ = \sum_{\left( \begin{smallmatrix} 1 \leq m \leq n \\ \sum v_i = n \end{smallmatrix} \right)} \frac{i^m}{m!} P(x_1, \dots, x_{v_1} | x_{v_1+1}, \dots | \dots, x_n) T(\Lambda_{v_1}(x_1, \dots, x_{v_1}) \dots \Lambda_{v_m}(\dots, x_n)). \quad (21.40)$$

This sum contains identical terms corresponding to equivalent subdivisions, i.e., subdivisions containing groups of equal values of indices  $v$ :

$$v_{\alpha_1} = \dots = v_{\alpha_k}; \quad v_{\beta_1} = \dots = v_{\beta_o}; \quad \dots; \quad v_{\gamma_1} = \dots = v_{\gamma_p}.$$

The number of such terms which differ, prior to symmetrization, only by values of the arguments, is  $m!/(k! q! \dots p!)$ . All these terms become identical as a result of symmetrization. Symmetrization within groups of the form  $v_{\alpha_1}, \dots, v_{\alpha_k}$  leads only to the appearance of the factor  $k! q! \dots p!$ . The corresponding terms are symmetric prior to the application

of the operator  $P$  because of the commutation property of the functions  $\Lambda_{\nu_{\alpha_1}}, \dots, \Lambda_{\nu_{\alpha_k}}$  under the chronological decomposition sign. The operation of symmetrization over all the decompositions, other than the equivalent decompositions, will be indicated by the symbol  $P'$ . Recalling that  $\Lambda_1(x) = \mathcal{L}(x)$ , we may rewrite (21.40) for  $S_n$  in the form

$$\begin{aligned} S_n(x_1, \dots, x_n) = & i^n T(\mathcal{L}(x_1) \dots \mathcal{L}(x_n)) + \\ & + \sum_{\substack{2 \leq m \leq n-1 \\ \sum_{i=1}^m \nu_i = n}} i^m P'(x_1, \dots, x_{\nu_1} | x_{\nu_1+1} \dots | \dots, x_n) \times \\ & \times T[\Lambda_{\nu_1}(x_1, \dots, x_{\nu_1}) \dots \Lambda_{\nu_m}(\dots x_n)] + i \Lambda_n(x_1, \dots, x_n). \end{aligned} \quad (21.41)$$

Here, for example, the first term corresponds to  $m! = n!$  equivalent terms, since it contains the group of identical indices  $\nu_1 = \dots = \nu_m = 1$ .

We now consider some simple special cases of (21.40). Thus, when  $n = 3$ , we have

$$\begin{aligned} S_3(1, 2, 3) = & i^3 T(\mathcal{L}(1) \mathcal{L}(2) \mathcal{L}(3)) + i^2 T\{\mathcal{L}(1) \Lambda_2(2, 3) + \mathcal{L}(2) \Lambda_2(1, 3) + \\ & + \mathcal{L}(3) \Lambda_2(1, 2)\} + i \Lambda_3(1, 2, 3). \end{aligned} \quad (21.42a)$$

When  $n = 4$ , we have

$$\begin{aligned} S_4(1, 2, 3, 4) = & i^4 T(\mathcal{L}(1) \mathcal{L}(2) \mathcal{L}(3) \mathcal{L}(4)) + \\ & + i^3 T\{\mathcal{L}(1) \mathcal{L}(2) \Lambda_2(3, 4) + \mathcal{L}(1) \mathcal{L}(3) \Lambda_2(2, 4) + \mathcal{L}(1) \mathcal{L}(4) \Lambda_2(2, 3) + \\ & + \mathcal{L}(3) \mathcal{L}(4) \Lambda_2(1, 2) + \mathcal{L}(2) \mathcal{L}(4) \Lambda_2(1, 3) + \mathcal{L}(2) \mathcal{L}(3) \Lambda_2(1, 4)\} + \\ & + i^2 T[\Lambda_2(1, 2) \Lambda_2(3, 4) + \Lambda_2(1, 3) \Lambda_2(2, 4) + \Lambda_2(1, 4) \Lambda_2(2, 3)] + \\ & + i^2 T\{\mathcal{L}(1) \Lambda_3(2, 3, 4) + \mathcal{L}(2) \Lambda_3(1, 3, 4) + \mathcal{L}(3) \Lambda_3(1, 2, 4) + \\ & + \mathcal{L}(4) \Lambda_3(1, 2, 3)\} + i \Lambda_4(1, 2, 3, 4). \end{aligned} \quad (21.42b)$$

Hence we see that each successive function  $S_n$  is expressed in terms of the preceding  $S_\nu$ ,  $\nu \leq n-1$  (or, more precisely, in terms of the chronological decompositions  $\Lambda_n$ ) to within a symmetric anti-Hermitian quasilocal operator  $i \Lambda_n(x_1, \dots, x_n)$ . The expression given by (21.40) is therefore the most general expression for  $S_n$ , while the expression

$$S(g) = T\left(\exp i \int \mathcal{L}(x; g) dx\right),$$

where  $\mathcal{L}(x; g)$  is defined by (21.39), is the most general expression for the scattering matrix. Thus, the *sequence of quasi-local operators* (21.37), which must be given in order that the scattering matrix  $S(g)$  should be completely determined, may in fact be included in the interaction "Lagrangian."

The question now arises as to the physical meaning of the linear combination (21.39) of  $\mathcal{L}(x) g(x)$  and of the integrals over quasilocal operators, which plays the role of the most general acceptable Lagrangian. The structure and the interrelationship of (21.39) and (21.40), which contain arbitrary quasilocal operators, is determined by a purely mathematical property, noted by Poincaré, of expansions occurring in perturbation theory in general.

To illustrate the aforementioned property, consider the differential equation

$$\frac{dy(x)}{dx} = \epsilon y(x), \quad (21.43)$$

whose solution  $y(x)$  contains one arbitrary constant  $C$ :

$$y(x) = Ce^{\epsilon x}.$$

Considering  $\epsilon$  to be a small parameter, we shall solve equation (21.43) by the method of successive approximations. At each stage of the approximation, we shall obtain on integration an arbitrary constant of the corresponding order of smallness in  $\epsilon$ . Thus, by solving equation (21.43) by perturbation theory, we obtain in place of a single arbitrary constant  $C$  an infinite set of arbitrary constants of various orders of smallness in  $\epsilon$ :

$$C_0, \epsilon C_1, \dots, \epsilon^n C_n, \dots$$

This is essentially equivalent to the constant  $C$  being expanded into a series in powers of  $\epsilon$ , i.e.,

$$C = C_0 + \epsilon C_1 + \dots + \epsilon^n C_n + \dots$$

In precisely the same way, we have used in our argument the nonphysical operation of “switching on” the interaction. It turns out that, in this procedure, part of the Lagrangian was “switched on” in the first approximation, part in the second, and part in the  $n$ th approximation, so that the interaction Lagrangian was broken up into a sequence of separate pieces. However, only the situation where the interaction is switched on completely has a physical meaning. The function  $g(x)$  is then equal to unity and (21.39) takes the form

$$\mathcal{L}(x; 1) = \mathcal{L}(x) + \sum_{v \geq 2} \frac{1}{v!} \int \Lambda_v(x, x_1, \dots, x_{v-1}) dx_1 \dots dx_{v-1}, \quad (21.44)$$

$$S(1) = S = T \left( \exp i \int \mathcal{L}(x; 1) dx \right). \quad (21.45)$$

Therefore, the actual scattering matrix  $S(1)$  is completely characterized by the actual interaction Lagrangian  $\mathcal{L}(x; 1)$  of the system, which in perturbation theory is sometimes represented in the form of a series.

In the usual presentation of field theory, based on Schroedinger's equation, the scattering matrix is obtained in the representation

$$T \left( \exp \left[ -i \int H(x) dx \right] \right), \quad (21.46)$$

where  $H(x)$  is the interaction Hamiltonian density. However,  $H(x)$  coincides with  $-\mathcal{L}$  only in particularly simple cases when  $\mathcal{L}$  does not depend on the derivatives of the field

functions. If, however, the derivatives are present, then we have the Wick  $T$ -product in (21.45) and the Dyson  $T$ -product in (21.46). The difference between the two products was noted in §15.3. In particular, the Dyson  $T$ -product, in which explicit  $\theta$ -functions are used for ordering, is noncovariant, and  $H$  contains, apart from  $-\mathcal{L}$ , some noncovariant terms. We emphasize that (21.45) and (21.46) refer to the same  $S$ -matrix, and transition from one type of  $T$ -product to the other can be used to transform one of these formulas to the other [see Sukhanov (1961) and Pavlov and Tavluiev (1971)].

## §22. Evaluation of Chronological Products

**22.1 Chronological Pairing.** Since we now have an explicit expression for the scattering matrix, we may proceed to evaluate its matrix elements between different states. In the course of this calculation, we shall have to reduce the matrix terms to the normal form, i.e., to a form in which all the annihilation operators are placed to the right and all the creation operators to the left in each term.

Let us therefore express the  $T$ -products of local operators  $\mathcal{L}(x)$  in terms of the normal products of the corresponding field operators. We shall find it convenient to extend the notion of the  $T$ -product to the case of the general system of linear operators, defined in §17.

We define the chronological, or the ordered, product

$$T(A_1(x_1) \dots A_n(x_n))$$

of linear operators  $A_1(x_1) \dots A_n(x_n)$  as their ordinary product in chronological order, multiplied by  $\eta = (-1)^p$ , where  $p$  is the parity of the Fermi permutations in the transition from the order  $1, \dots, n$  to the chronological order, i.e.,

$$T(A_1(x_1) \dots A_n(x_n)) = \eta A_{j_1}(x_{j_1}) \dots A_{j_n}(x_{j_n}) \quad (x_{j_1}^0 \geq x_{j_2}^0 \geq \dots \geq x_{j_n}^0), \quad (22.1)$$

where  $p$  is the parity of the permutation of the Fermi operators in the transition from the order  $(1, 2, \dots, n)$  to the order  $(j_1, j_2, \dots, j_n)$ .

The prescription for the evaluation of such products is given by *Wick's theorem for  $T$ -products* which is the analog of Wick's theorem for ordinary products. Before beginning the proof of this theorem we shall introduce the important concept of the *chronological pairing of the operators*.

To do this, we examine (22.1) in the case of two field operators:

$$T(u_1(x_1) u_2(x_2)) = \begin{cases} u_1(x_1) u_2(x_2) & (x_1^0 > x_2^0), \\ \eta u_2(x_2) u_1(x_1) & (x_2^0 > x_1^0). \end{cases}$$

In accordance with the definition of ordinary pairing, (see 15.6)

$$u_1(x_1) u_2(x_2) = : u_1(x_1) u_2(x_2) : + \underbrace{u_1(x_1) u_2(x_2)}$$

the above expression may be brought into the form

$$T(u_1(x_1) u_2(x_2)) = \begin{cases} :u_1(x_1) u_2(x_2): + \overline{u_1(x_1) u_2(x_2)} & (x_1^0 > x_2^0), \\ \eta :u_2(x_2) u_1(x_1): + \overline{\eta u_2(x_2) u_1(x_1)} = \\ = :u_1(x_1) u_2(x_2): + \overline{\eta u_2(x_2) u_1(x_1)} & (x_2^0 > x_1^0). \end{cases} \quad (22.2)$$

From this it may be seen that, in any arbitrary case,  $T(u_1(x_1) u_2(x_2))$  differs from  $:u_1(x_1) u_2(x_2):$  by a  $c$ -number which we shall call the *chronological pairing*  $\overline{u_1(x_1) u_2(x_2)}$ , i.e., by definition

$$T(u_1(x_1) u_2(x_2)) = :u_1(x_1) u_2(x_2): + \overline{u_1(x_1) u_2(x_2)} \quad (22.3)$$

and

$$\overline{u_1(x_1) u_2(x_2)} = \begin{cases} \overline{u_1(x_1) u_2(x_2)} & \text{for } x_1^0 > x_2^0, \\ \overline{\eta u_2(x_2) u_1(x_1)} & \text{for } x_2^0 > x_1^0. \end{cases} \quad (22.4)$$

First of all, let us note an important property of chronological pairing: *the order of factors can be interchanged within a chronological pairing*, just as it can within a normal product:

$$\overline{u_1(x_1) u_2(x_2)} = \overline{\eta u_2(x_2) u_1(x_1)},$$

which follows directly from (22.4).

We shall now define chronological pairings for the operators of the principal wave fields. To do this, we note that by evaluating the vacuum expectation value  $\Phi_0^* \dots \Phi_0 = \langle \dots \rangle_0$  of (22.3), and taking into account the fundamental property of the normal product and the normalization of the amplitude of the vacuum state, we obtain

$$\langle T(u_1(x_1) u_2(x_2)) \rangle_0 = \overline{u_1(x_1) u_2(x_2)}, \quad (22.5)$$

i.e., that the chronological pairing of two field operators is equal to the vacuum expectation value of the chronological product of these operators. On the other hand, it was established in §15 that such expectation values are equal to causal Green's functions of the corresponding fields to within a factor of  $i$ . From the formulas obtained in that section, we find that

for the scalar field

$$\overline{\varphi(x) \varphi(y)} = \langle T(\varphi(x) \varphi(y)) \rangle_0 = \frac{1}{i} D^c(x-y) = \frac{1}{(2\pi)^4 i} \int \frac{e^{ik(x-y)}}{m^2 - k^2 - i\epsilon} dk, \quad (22.6)$$

for the electromagnetic field

$$\overline{A_I(x) A_n(y)} = \langle T(A_I(x) A_n(y)) \rangle_0 = i g^{In} D_0^c(x-y) = \frac{g^{In}}{(2\pi)^4 i} \int \frac{e^{ik(x-y)}}{k^2 + i\varepsilon} dk, \quad (22.7)$$

for the vector field

$$\overline{U_n(x) U_I(y)} = \langle T(U_n(x) U_I(y)) \rangle_0 = i D_{nI}^c(x-y) = \frac{i}{(2\pi)^4} \int \frac{\left(g^{nI} - \frac{k_n k_I}{m^2}\right)}{m^2 - k^2 - i\varepsilon} e^{ik(x-y)} dk, \quad (22.8)$$

and for the spinor field

$$\overline{\psi_\alpha(x) \bar{\psi}_\beta(y)} = \langle T(\psi_\alpha(x) \bar{\psi}_\beta(y)) \rangle_0 = \frac{1}{i} S_{\alpha\beta}^c(x-y) = \frac{1}{(2\pi)^4 i} \int \frac{(m - \hat{p})_{\alpha\beta}}{m^2 - p^2 - i\varepsilon} e^{ip(x-y)} dp. \quad (22.9)$$

However, we note that the formulas which lead to (22.6)–(22.9) were derived in §15 only for  $x^0 > y^0$  and for  $x^0 < y^0$ . From the covariance of  $T$ -products established in §21 it follows that these formulas are also valid for  $x \sim y$ . Thus (22.6)–(22.9) may be considered to hold everywhere for  $x \neq y$ .

The rules for integrating these expressions over an infinitesimal neighborhood of the point  $x = y$  *may be fixed arbitrarily*. For example, it is always possible to add to the right-hand side of each of these expressions an arbitrary coefficient function of a quasi-local operator

$$P\left(\frac{\partial}{\partial x}\right) \delta(x-y),$$

where  $P(\partial/\partial x)$  is a polynomial in  $\partial/\partial x^\alpha$ . This necessity of making an additional definition of pairing in an infinitesimal neighborhood of the point  $x = y$  is a particular expression of the arbitrariness contained in the  $T$ -product. Indeed, the  $T$ -products are completely specified by our formal “definition” (21.27) and (22.2) only when their arguments are all different. Therefore, it is necessary to complete their definition within infinitesimal neighborhoods of those points at which the arguments coincide by specifying the rules for integrating their coefficient functions, i.e., in other words, the coefficient functions of  $T$ -products must be defined as integrable improper functions.

We thus arrive at the conclusion that it is necessary not only to choose an interaction Lagrangian, but also to supplement the definitions of the  $T$ -products.

However, it should be noted that the influence on  $S(g)$  of a change in the  $T$ -product may be taken into account by changing the Lagrangian  $\mathcal{L}(x)$ . Indeed, by changing the  $T$ -products of field functions, we introduce into the  $T$ -products of the interaction Lagrangians various quasilocal operators which, as was shown in the preceding section, is equivalent to the addition of certain expressions to the interaction Lagrangian.



Consequently, to obtain the matrix elements of the scattering matrix  $S(g)$  that determine the structure of physical processes, it is necessary to specify simultaneously the interaction Lagrangian and the rules for integrating  $T$ -products. If these rules are already specified, the interaction Lagrangian  $\mathcal{L}(x)$  must be chosen in a way consistent with these rules.

The dependence of the form of the Lagrangian on certain subsidiary considerations is by no means a specific peculiarity of quantum field theory. For example, in order to determine the form of the Lagrangian in classical physics, it is necessary first of all to choose the independent dynamic variables [compare the treatments of the scalar field in the usual way (§3) and in Kemmer's formalism (§4.4)].

Thus, we must first complete the definition of all the pairings, and also of their products, in such a way that the products are integrable functions. This requirement ensures that the  $T$ -products turn out to be completely determined, and it is then possible to determine the Lagrangian.

We shall give a detailed treatment of these problems, which are connected with the problem of regularization of the  $S$ -matrix in the next chapter. For the present, we confine our attention to completing the definition of the pairings of field functions. We shall adopt the convention that if for  $x \neq y$  the pairing coincides with a certain Green's function

$$\Delta_{\alpha\beta}^c(x-y) = P_{\alpha\beta} \left( \frac{\partial}{\partial x} \right) D^c(x-y),$$

then it also coincides with this function in an infinitesimal neighborhood of the point  $x = y$ . In particular, we shall consider that formulas (22.6)-(22.9) are also valid in the infinitesimal neighborhood of the point  $x = y$ . In other words, within the framework of the terminology introduced in §15, chronological pairings are defined as the expectation values of the Wick chronological products  $T_W$ .

Let us now consider the general case where the field is described by a set of equations of the first order. We saw in §15 that the field functions then satisfy the commutation relations

$$\begin{aligned} [u_\alpha(x), u_\beta^*(y)] &= \frac{1}{i} \Delta_{\alpha\beta}(x-y) = \frac{1}{i} P_{\alpha\beta} \left( \frac{\partial}{\partial x} \right) D(x-y), \\ [u_\alpha^-(x), u_\beta^{*-}(y)] &= \frac{1}{i} \Delta_{\alpha\beta}^+(x-y) = \frac{1}{i} P_{\alpha\beta} \left( \frac{\partial}{\partial x} \right) D^\pm(x-y). \end{aligned}$$

Therefore, by evaluating the ordinary pairing, we obtain

$$\overline{u_\alpha(x) u_\beta^*(y)} = \frac{1}{i} \Delta_{\alpha\beta}^-(x-y),$$

from which we find by standard methods the following expressions for chronological pairing:

$$\overline{u_\alpha(x) u_\beta^*(y)} = \begin{cases} -i \Delta_{\alpha\beta}^-(x-y) & (x^0 > y^0), \\ i \Delta_{\alpha\beta}^+(x-y) & (y^0 > x^0), \end{cases}$$

which for  $x \neq y$  coincide with

$$\frac{1}{i} \Delta_{\alpha\beta}^c(x-y) = \frac{1}{i} P_{\alpha\beta} \left( \frac{\partial}{\partial x} \right) D^c(x-y).$$

We therefore assume that for arbitrary  $x$  and  $y$ ,

$$\overline{u_\alpha(x) u_\beta^*(y)} = \frac{1}{i} \Delta_{\alpha\beta}^c(x-y). \quad (22.10)$$

In a number of cases, derivatives of field functions may appear in the interaction Lagrangian. It is therefore useful to give here a complete definition of their pairings. Noting that for  $x^0 \geq y^0$

$$\frac{\overline{\partial^k u_\alpha(x) \partial^q u_\beta^*(y)}}{(\partial x_0)^{k_0} \dots (\partial x_3)^{k_3} (\partial y_0)^{q_0} \dots (\partial y_3)^{q_3}} = \frac{\partial^k}{(\partial x_0)^{k_0} \dots (\partial x_3)^{k_3}} \frac{\partial^q}{(\partial y_0)^{q_0} \dots (\partial y_3)^{q_3}} \overline{u_\alpha(x) u_\beta^*(y)}, \quad (22.11)$$

we assume that by definition, (22.11) is valid for any  $x$  and  $y$ .

We note that (22.10) and (22.11) define pairings for the Wick  $T$ -product (§15.3). The corresponding formulas for the Dyson variant are

$$\begin{aligned} \overline{i u_\alpha(x) u_\beta^*(y)} &= \theta(x-y) \Delta_{\alpha\beta}^-(x-y) + \theta(y-x) \Delta_{\beta\alpha}^-(y-x), \\ i \frac{\overline{\partial^k u_\alpha(x) \partial^q u_\beta^*(y)}}{(\partial x_0)^{k_0} \dots (\partial x_3)^{k_3} (\partial y_0)^{q_0} \dots (\partial y_3)^{q_3}} &= \theta(x-y) \frac{\partial^k}{(\partial x_0)^{k_0} \dots (\partial x_3)^{k_3}} \times \\ &\quad \times \frac{\partial^q}{(\partial y_0)^{q_0} \dots (\partial y_3)^{q_3}} \Delta_{\alpha\beta}^-(x-y) + \theta(y-x) \frac{\partial^{k+q}}{(\partial x_0)^{k_0} \dots (\partial y_3)^{q_3}} \Delta_{\alpha\beta}^-(y-x). \end{aligned}$$

We also note some properties of chronological pairings and  $T$ -products.

As may be easily seen, the free-field operators which occur in the normal products and in the ordinary pairings remain "free" in the sense that the above expressions reduce to zero when acted upon by the differential operators of the free field equations. Thus, for the scalar field

$$(\square_x - m^2) : \varphi(x) \varphi(y) : = 0, \quad (22.12)$$

$$(\square_x - m^2) \overline{\varphi(x) \varphi(y)} = 0. \quad (22.13)$$

It might seem that this property should also be retained by the chronological pairing defined in (22.4) in terms of ordinary pairings, and consequently also by the  $T$ -product. However, in fact (because the definition (22.4) is not satisfied at the point  $x = y$ ) this does not occur. Indeed, as we have just shown, chronological pairings may be expressed in terms of causal Green's functions satisfying inhomogeneous field equations. Thus, for a scalar field, we have in accordance with (22.5)

$$(\square_x - m^2) \overline{\varphi(x)} \varphi(y) = i\delta(x-y). \quad (22.14)$$

Consequently, the same holds also in the case of  $T$ -products. In particular, for a scalar field, we find with the aid of (22.3), (22.12), and (22.14) that

$$(\square_x - m^2) T(\varphi(x) \varphi(y)) = i\delta(x-y). \quad (22.15)$$

Thus, in the interaction representation, the field operators within a  $T$ -product must be regarded as not satisfying the homogeneous field equations.

**22.2. Wick's Theorem for Chronological Products.** Wick's theorem for chronological products asserts that *the  $T$ -product of a system of  $n$  linear operators is equal to the sum of their normal products with all possible chronological pairings (including the term with no pairings).*

The proof in fact reduces to the proof of Wick's theorem for ordinary products. Indeed, according to the definition given by (22.1), the  $T$ -product is equal to a certain ordinary product

$$T(A_1(x_1) \dots A_n(x_n)) = \eta A_{j_1}(x_{j_1}) \dots A_{j_n}(x_{j_n}).$$

By applying Wick's theorem to this ordinary product, we see that it is equal to the sum of the normal products of the operators  $A_{j_1}(x_{j_1}) \dots A_{j_n}(x_{j_n})$  with all possible ordinary pairings. But since the order in the sequence  $x_{j_1}, \dots, x_{j_n}$  is chronologically correct, the ordinary pairings coincide with the chronological ones, i.e.,  $T(A_1(x_1) \dots A_n(x_n))$  is equal to the product of  $\eta$  and the sum of the normal products of the operators  $A_{j_1}(x_{j_1}) \dots A_{j_n}(x_{j_n})$  with all possible chronological pairings.

As has been noted before, linear operators within a chronological pairing may be commuted (taking possible changes of sign into account). As a result, we may reestablish the normal order of the factors  $1, 2, \dots, n$  within normal products with all possible chronological pairings, at the same time leaving out the factor  $\eta$ . This completes the proof of the theorem.

Let us now consider the  $T$ -product of several normal products of linear field operators  $A_i(x) \dots D_j(z)$

$$T(:A_1(x) A_2(x) \dots A_n(x) : \dots :D_1(z) \dots D_m(z):). \quad (22.16)$$

$T$ -products of just this form will be needed for the evaluation of the  $T$ -products of local operators, since, by definition, the local operator  $\mathcal{L}(x)$  is expressed as a linear combination of terms of the type

$$:A_1(x) A_2(x) \dots A_n(x):.$$

The formulation of Wick's theorem for  $T$ -products such as (22.16) has only the single special feature that mutual chronological pairings of operators occurring within the same normal product should not be taken into account.

As an illustration, let us reduce to the normal form the  $T$ -product of two current operators for the spinor field

$$J^m(x) = :\bar{\psi}(x) \gamma^m \psi(x): \quad \text{and} \quad J^n(y) = :\bar{\psi}(y) \gamma^n \psi(y):.$$

Utilizing Wick's theorem, taking into account the vanishing of the pairings  $\overline{\psi(x) \psi(y)}$  and  $\overline{\bar{\psi}(x) \bar{\psi}(y)}$ , and not taking into account the pairings  $\overline{\psi(x) \bar{\psi}(x)}$  and  $\overline{\bar{\psi}(y) \psi(y)}$ , we obtain:

$$\begin{aligned} T(J^m(x), J^n(y)) &= T(:\bar{\psi}(x) \gamma^m \psi(x):, :\bar{\psi}(y) \gamma^n \psi(y):) = \\ &= :\bar{\psi}(x) \gamma^m \psi(x) \bar{\psi}(y) \gamma^n \psi(y): + \overline{\bar{\psi}(y) \gamma^n \psi(x)} \bar{\psi}(y) \gamma^n \psi(y): + \\ &\quad + :\bar{\psi}(x) \gamma^m \psi(x) \overline{\bar{\psi}(y) \gamma^n \psi(y)}: + \overline{\bar{\psi}(x) \gamma^m \psi(x)} \bar{\psi}(y) \gamma^n \psi(y):. \end{aligned}$$

Taking into account the fact that, according to (22.9),

$$\overline{\psi(x) \bar{\psi}(y)} = \frac{1}{i} S^c(x-y)$$

and

$$\overline{\bar{\psi}(x) \psi(y)} = -\overline{\bar{\psi}(y) \psi(x)} = i S^c(y-x),$$

and carrying out the summation over the spinor indices, we finally obtain

$$\begin{aligned} T(J^m(x), J^n(y)) &= :J^m(x) J^n(y): - i:\bar{\psi}(y) \gamma^n S^c(y-x) \gamma^m \psi(x): - \\ &\quad - i:\bar{\psi}(x) \gamma^m S^c(x-y) \gamma^n \psi(y): + \text{Sp}(S^c(x-y) \gamma^n S^c(y-x) \gamma^m). \end{aligned} \quad (22.17)$$

In conclusion, we note that, starting with (22.16), one may also define a  $T$ -product of normal products of the more general type

$$T(:A_1(x_1) \dots A_n(x_n): \dots :D_1(z_1) \dots D_m(z_m):) \quad (22.18)$$

as a sum of normal products of the operators

$$A_1(x_1) \dots D_m(z_m)$$

with all possible pairings, excluding mutual pairings of operators occurring in the same normal product.

On the basis of this definition of the  $T$ -product, it is possible to introduce the  $T$ -product of polylocal operators

$$T(A(x_1, \dots, x_n) \dots D(z_1, \dots, z_m)), \quad (22.19)$$

by expressing it as a linear combination of expressions such as (22.18). The direct definition of the  $T$ -product (22.19) using the chronological criterion is not convenient in the present case because of the multiplicity of arguments of the operators  $A, \dots, D$ .

It may be seen from the above that in effect, the  $T$ -product represents a new algebraic operation which may be introduced independently of ordinary products.

From the mathematical point of view, the  $T$ -product is particularly attractive due to the fact that in contrast to ordinary products, the operators within a  $T$ -product may be permuted as if they commute or anticommute exactly.

### §23. Reduction of the $S$ -Matrix to the Normal Form

**23.1. Structure of the Coefficients of the Scattering Matrix.** We now proceed to formulate a convenient set of prescriptions for reducing the normal form of the  $T$ -products of Lagrangians, which define the operator expressions  $S_n(x_1, \dots, x_n)$  in the scattering matrix. For the sake of clarity, it is convenient to begin by considering a specific case. Let us therefore consider the interaction between the electromagnetic and the spinor electron-positron fields. In accordance with well-established tradition, we take electrons to be the fundamental particles of the spinor field and the positrons to be the antiparticles. Therefore, the operators of the spinor field  $a^{*\nu,+}$  and  $a^{\nu,-}$  (see §13) describe creation and annihilation of electrons, while the operators  $a^{\nu,+}$  and  $a^{*\nu,-}$  refer to positrons. Naturally, in view of the complete symmetry of our presentation, one could just as well regard positrons as the fundamental particles, and electrons as the antiparticles. The roles played by the operators  $a^{*\nu,+}$ ,  $a^{\nu,-}$  and  $a^{\nu,+}$ ,  $a^{*\nu,-}$  would then be interchanged, and one would only need to change the sign of the electric current, which, in the usual form adopted by us, appears as follows:

$$j^\mu(x) = -eJ^\mu(x) = -e:\bar{\psi}(x)\gamma^\mu\psi(x):, \quad (23.1)$$

where  $e$  is the magnitude of the charge of the electron multiplied by  $\sqrt{4\pi}$  and is given by

$$e = \sqrt{\frac{4\pi}{137}} \quad (23.2)$$

in the natural system of units which we employ (§1).\*

In accordance with (8.15), the interaction Lagrangian for the electromagnetic and the electron-positron fields has the form

$$\mathcal{L}(x) = -:j^\mu(x)A_\mu(x): = e:\bar{\psi}(x)\gamma^\mu\psi(x)A_\mu(x):. \quad (23.3)$$

Taking into account the elementary chronological pairings (22.7) and (22.9),

\*A more accurate value is  $\alpha = e^2/4\pi = \frac{1}{137.036}$  (see also (41.43)).

$$\overline{A_m(x)} A_n(y) = i g^{mn} D_0^c(x-y), \quad \overline{\psi_\sigma(x)} \bar{\psi}_\rho(y) = -\overline{\bar{\psi}_\rho(y)} \psi_\sigma(x) = \frac{1}{i} S_{\sigma\rho}^c(x-y),$$

we consider the process of reduction to the normal form of the individual terms of the  $S$ -matrix

$$S(1) = 1 + \sum_{n \geq 1} \frac{1}{n!} \int S_n(x_1, \dots, x_n) dx_1 \dots dx_n, \\ S_n(x_1, \dots, x_n) = i^n T(\mathcal{L}(x_1) \dots \mathcal{L}(x_n)), \quad (21.31)$$

which depend on the Lagrangian (20.3). The first-order term is already written in the normal form:

$$S_1(x) = i\mathcal{L}(x) = ie: \bar{\psi}(x) \hat{A}(x) \psi(x) :. \quad (23.4)$$

In the second order we obtain

$$S_2(x, y) = i^2 T(\mathcal{L}(x) \mathcal{L}(y)) = -e^2 T(:J^m(x) A_m(x) : :J^n(y) A_n(y):).$$

Since the operators of the electromagnetic field commute with those of the electron-positron field, the aforementioned  $T$ -product may be expressed as a product of two  $T$ -products:

$$T(J^m(x) J^n(y)) T(A_m(x) A_n(y)).$$

The first factor  $T(J^m(x) J^n(y))$  was evaluated in (22.17), while the second factor may be evaluated in an elementary way by means of (22.3) and of the pairing of the electromagnetic field (22.7). We therefore obtain

$$S_2(x, y) = -e^2 (:A_m(x) A_n(y): + i g^{mn} D_0^c(x-y)) \times \\ \times \{ :J^m(x) J^n(y): - i: \bar{\psi}(y) \gamma^n S^c(y-x) \gamma^m \psi(x): - \\ - i: \bar{\psi}(x) \gamma^m S^c(x-y) \gamma^n \psi(y): + \text{Tr}(S^c(x-y) \gamma^n S^c(y-x) \gamma^m) \} = \\ = -e^2 \{ :J^m(x) A_m(x) J^n(y) A_n(y): + i: J_n(x) J^n(y): D_0^c(x-y) - \\ - i: \bar{\psi}(y) \gamma^n A_n(y) S^c(y-x) \gamma^m A_m(x) \psi(x): - \\ - i: \bar{\psi}(x) \gamma^m A_m(x) S^c(x-y) \gamma^n A_n(y) \psi(y): + \\ + : \bar{\psi}(y) \gamma_m S^c(y-x) D_0^c(x-y) \gamma^m \psi(x): + \\ + : \bar{\psi}(x) \gamma^m S^c(x-y) D_0^c(x-y) \gamma_m \psi(y): + \\ + \text{Tr} : (S^c(x-y) \gamma^n A_n(y) S^c(y-x) \gamma^m A_m(x): + \\ + i D_0^c(x-y) \text{Tr}(S^c(x-y) \gamma^n S^c(y-x) \gamma_n) \}. \quad (23.5)$$

In (23.4) and (23.5) we have adopted the order of writing down the matrix operators such that within a normal product the spinor operators  $\bar{\psi}$  and  $\psi$  are always ordered in

accordance with a definite principle. In accordance with (23.3), each  $\mathcal{L}$  contains one spinor  $\bar{\psi}$  and one spinor  $\psi$ . Therefore, the  $n$ th-order term of a  $T$ -product contains  $n$  operators  $\bar{\psi}$  and  $n$  operators  $\psi$ . In the process of reducing the  $T$ -product to the normal form, certain pairs of operators  $\bar{\psi}(x_i)\psi(x_j)$  will be replaced by the corresponding pairings. Thus, after the individual terms of the  $T$ -product have been reduced to the normal form, each of the terms of the resulting expression [for example, the terms (23.5)] will contain the same number of operators  $\bar{\psi}$  and  $\psi$ . Moreover, each  $\bar{\psi}(x_k)$  may be made to correspond to a definite  $\psi(x_{j_k})$  by following through the sequence

$$\overbrace{\psi(x_k)\bar{\psi}(x_{i_k})}, \quad \overbrace{\psi(x_{i_k})\bar{\psi}(x_{i_k})}, \quad \dots, \quad \overbrace{\psi(x_{m_k})\bar{\psi}(x_{j_k})}. \quad (23.6)$$

The free operators  $\psi$  and  $\bar{\psi}$  in the normal products may therefore always be divided into pairs with the same arguments (if both operators  $\psi$  and  $\bar{\psi}$  from some  $\mathcal{L}(x)$  do not undergo pairing) or with arguments related by a sequence of arguments of corresponding  $S_c$ -functions.

We shall agree to write down the free operators in such a way that the operators forming pairs should be next to each other (only  $c$ -functions may stand between them), and that, within each pair,  $\bar{\psi}(x_k)$  should always be to the left of  $\psi(x_{j_k})$ , i.e., in the form

$$:\bar{\psi}(x_1)\psi(x_{j_1})\bar{\psi}(x_2)\psi(x_{j_2})\dots\bar{\psi}(x_k)\psi(x_{j_k}):. \quad (23.7)$$

Permutations of pairs among themselves do not alter the value of the normal product, and therefore the order of the pairs is immaterial. All the terms in (23.4) and (23.5) are written in the form of (23.7). In future, we shall always make use of this method of writing our formulas; this will enable us to obtain in a simple way the rule for determining the sign of any arbitrary term obtained by evaluating a  $T$ -product of the form of (21.31).

Expressions having a structure similar to (23.5) may also be obtained without difficulty for higher-order terms. Using Wick's theorem for  $T$ -products, it may be shown immediately that the coefficient  $S_n$  (to within  $i^n$ ) is equal to the sum of normal products of Lagrangians  $\mathcal{L}(x_1), \dots, \mathcal{L}(x_n)$  with all possible pairings. This theorem can also be used to formulate rules for automatically writing down the elements of the scattering matrix in normal form, and, consequently, the matrix elements as well.

**23.2. Feynman Diagrams and Rules of Correspondence.** To obtain such rules, the elements of the  $S$ -matrix are made to correspond to certain graphical diagrams, first introduced by Feynman (1949a, b), which we shall call Feynman diagrams.

In order to introduce Feynman diagrams, it is necessary to establish certain definite rules of correspondence. As we have established above, the operator functions  $S_n(x_1, \dots, x_n)$  may be represented, in accordance with Wick's theorem, by the sum of terms each of which is a product of a certain number of pairings of the electromagnetic field  $\overline{A}(x)A(y)$  and the spinor field  $\overline{\psi}(x)\psi(y)$ , the normal product of the free operators  $\bar{\psi}, \psi, A$ , and a certain number of matrices  $\gamma^n$ .

It is obvious that in order to give a graphical description of the terms of the scattering matrix in the case of (23.3), it is sufficient to specify the graphical representation of the two pairings

$$\overline{A_m(x_i)} A_n(x_j), \quad \overline{\psi(x_k)} \bar{\psi}(x_l),$$

of the three free operators

$$\psi(x_i), \quad \bar{\psi}(x_j), \quad A_n(x_k)$$

and of the Dirac matrices  $\gamma^n$ .

We therefore agree to represent the pairing

$$\overline{A_m(x_i)} A_n(x_j) = \overline{A_n(x_j)} A_m(x_i) = ig^{mn} D_0^c(x_i - x_j),$$

which is symmetrical in its arguments  $x_i$  and  $x_j$ , by a wavy line connecting the points  $x_i$  and  $x_j$ :



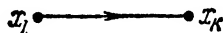
which may be considered to be a representation of the motion of a photon between the points  $x_i$  and  $x_j$  in the diagram.

The nonsymmetric pairing

$$\overline{\psi(x_k)} \bar{\psi}(x_l) = \frac{1}{i} S^c(x_k - x_l)$$

will correspond in the diagram to a directed line connecting the points  $x_i$  and  $x_j$ . It is convenient to choose the direction of this line by a direct inspection of the situation. We have already agreed to adopt the convention that the operator  $\psi(x)$  describes the creation of a positron and the annihilation of an electron at the point  $x$ , while the operator  $\bar{\psi}(y)$  describes the creation of an electron and the annihilation of a positron at the point  $y$ . In the other words, the operator  $\psi(x)$  appears to correspond to an electron arriving at the point  $x$ , while  $\bar{\psi}(x)$  appears to correspond to an electron leaving the point  $x$  (and *vice versa* for a positron).

Since electrons are taken to be the fundamental particles of the spinor field  $\psi$ , it is natural to make the pairing  $\overline{\psi(x_k)} \bar{\psi}(x_l)$  correspond in the diagram to a line directed from the point  $x_l$  to the point  $x_k$ , i.e.,



The directed lines introduced in this manner may be regarded as representing the motion



of an electron between the points  $x_l$  and  $x_k$  of the diagram (or as representing the motion of a positron from the point  $x_k$  to the point  $x_l$ ).\*

We shall make the free operators  $A(x)$ ,  $\psi(x)$ , and  $\bar{\psi}(x)$  correspond to lines connecting the point  $x$  with the edge of the diagram. In so doing we make the operator  $A_m(x)$  correspond to an undirected wavy (photon) line:



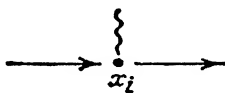
the operator  $\psi(x)$  to a directed (electron) line entering the point  $x$ :



and the operator  $\bar{\psi}(x)$  to a directed (electron) line leaving the point  $x$ :



The Dirac matrix  $\gamma^k$  from the Lagrangian  $\mathcal{L}(x_i)$  multiplied by the interaction constant  $e$ , will correspond in the diagram to the point  $x_i$  at which one photon line, one entering electron line, and one emerging electron line all meet:

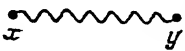



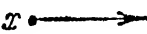
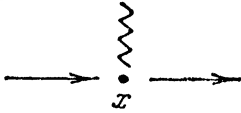


We shall call such points the *vertices* (or *junctions*) of the diagram. Table I gives a summary of the rules of correspondence between the various factors of the operator functions  $S_n$  and the elements of Feynman diagrams.

The rules of correspondence have been chosen in such a way that the diagram corresponding to one of the terms expressed in the normal form in the operator function  $S_n(x_1, \dots, x_n)$  consists of  $n$  vertices and a certain number of internal and external photon and electron lines. By virtue of the local nature of the interaction Lagrangian (23.3), one electron line enters and one electron line leaves each vertex. Thus, the electron lines of the complete diagram are continuous at the vertices and form either closed loops or continuous zig-zag lines which begin and end at the edges of the diagram. The sequence of arguments of the pairings (23.6) determines the sequence of the vertices of the individual electron

\*The correspondence between the diagrams introduced in this way and real processes is discussed in greater detail in § 24.

Table I. Rules of Correspondence

Factor in scattering matrix	Element of Feynman diagram
1. Pairing of operators of the electromagnetic field $\overline{A}_n(x) A_m(y) = i g^{mn} D_0^c(x-y)$	1. Internal photon line 
2. Pairing of operators of the spinor field $\overline{\psi}_a(x) \psi_b(y) = \frac{1}{i} S_{ab}^c(x-y)$	2. Internal electron line 
3. Free operator $A_m(x)$ inside a normal product	3. External photon line 
4. Free operator $\psi(x)$ inside a normal product	4. Entering external electron line 
5. Free operator $\bar{\psi}(x)$ inside a normal product	5. Emerging external electron line 
6. Dirac matrix $\gamma^k$ from the Lagrangian $\mathcal{L}(x)$ multiplied by $e$ : $e\gamma^k$	6. Vertex 

lines in the diagram, while the pairs of free operators  $\bar{\psi}(x_i)\psi(x_{j_i})$  within a normal product correspond to the beginning ( $x_{j_i}$ ) and to the end ( $x_i$ ) of individual electron lines which do not form loops. A characteristic feature of the rules of correspondence is the fact that the coefficient functions  $K_i(x_1, \dots, x_n)$  of the operator expressions

$$S_n(x_1, \dots, x_n) = \sum_i K_i(x_1, \dots, x_n) : \dots A(x_{j_i}) \dots \bar{\psi}(x_k) \dots \psi(x_l) \dots :,$$

which consist entirely of chronological pairings of the operators  $\psi$ ,  $\bar{\psi}$ , and  $A$ , correspond to the internal lines of the diagram, while the free operators within a normal product correspond to external lines.

As an example, consider one of the second-order terms in (23.5):

$$i^2 e^2 : \overbrace{\bar{\psi}(x) \hat{A}(x) \psi(x) \bar{\psi}(y) \hat{A}(y) \psi(y)} := e^2 \text{Sp} \{ i : S^c(y-x) \hat{A}(x) i S^c(x-y) \hat{A}(y) : \} = \\ = -e^2 \text{Tr} \{ : S^c(y-x) \hat{A}(x) S^c(x-y) \hat{A}(y) : \}. \quad (23.8)$$

Utilizing the rules of correspondence, we obtain the appropriate Feynman diagram (Fig. 3)



Fig. 3. Photon self-energy diagram.

(the so-called “photon self-energy diagram”; for more details see §27).

As a second example, consider one of the third-order terms in  $S_3(x, y, z)$ :

$$\begin{aligned}
 (ie)^3 : \bar{\psi}(x) \gamma^m A_m(x) \psi(x) \bar{\psi}(y) \hat{A}(y) \psi(y) \bar{\psi}(z) \gamma^n A_n(z) \psi(z) : = \\
 = -e^3 D_0^c(x-z) : \bar{\psi}(x) \gamma^n S^c(x-y) \hat{A}(y) S^c(y-z) \gamma_n \psi(z) :. \quad (23.9)
 \end{aligned}$$

In this case, the rules of correspondence lead us to the diagram shown in Fig. 4.

An examination of these simplest diagrams (Figs. 3 and 4) shows that motion along an electron line corresponds exactly to the order of the matrix elements from right to left in the corresponding term of the  $S$ -matrix. For example, in the second case, by representing the diagram of Fig. 4 in the form of Fig. 5, we obtain exactly the same order of noncommuting matrix factors as in (23.9).

We note that the construction of normal products with pairings corresponding to various diagrams in accordance with the rules formulated above may lead to an error in sign. The rules of correspondence established above must therefore be supplemented by a *sign rule*.

To formulate this rule, we note that the error in sign occurs only in expressions which correspond to closed electron loops. Indeed, in accordance with the rules of correspondence, the diagram of Fig. 3 corresponds to the expression

$$(ie)^2 \bar{\psi}(x) \bar{\psi}(y) \psi(y) \psi(x),$$

which differs in sign from (23.8). It is also evident that in the more general case, such a difference in sign will occur for any arbitrary group of pairings corresponding to each of the closed loops of the diagram, irrespective of the order of these loops. From this it follows that the expression which is obtained by means of the rules of correspondence must, in addition, be multiplied by

$$\eta = (-1)^l, \quad (23.10)$$

where  $l$  is the number of closed loops in the diagram under consideration.

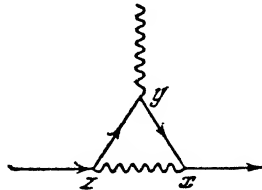


Fig. 4. Vertex diagram of the third order.

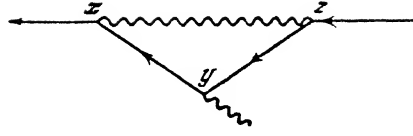


Fig. 5.

23.3. *Examples.* We shall illustrate the foregoing by examples of the application of the rules of correspondence and the sign rule (23.10). We construct the term of the fourth-order operator function  $S_4(x, y, z, t)$  which corresponds to the diagram of Fig. 6. By following the electron loop from the point  $x$  we obtain the matrix factors

$$\begin{aligned}
 & \begin{array}{ccccccc}
 x & & t & & z & & y \\
 \bullet & & \bullet & & \bullet & & \bullet \\
 ie\gamma^l; & \longleftarrow & ie\gamma^k; & \longleftarrow & ie\gamma^n; & \longleftarrow & ie\gamma^m;
 \end{array} \\
 & \frac{1}{i} S^c(x-t) = \frac{1}{i} S^c(t-z) = \frac{1}{i} S^c(z-y) = \frac{1}{i} S^c(y-x) = \\
 & = \overline{\psi(x)} \psi(t); \quad = \overline{\psi(t)} \psi(z); \quad = \overline{\psi(z)} \psi(y); \quad = \overline{\psi(y)} \psi(x).
 \end{aligned} \quad (23.11)$$

By multiplying the product (23.11) by the photon factors

$$A_n(z), \quad ig^{mk} D_0^c(y-t), \quad A_l(x)$$

and by

$$\eta = (-1)^1 = -1,$$

we obtain after carrying out the summation over the spinor indices and over the polarization indices of the electromagnetic field ( $k, l, m, n$ ):

$$\begin{aligned}
 & -ie^4 \text{Sp} [ : (S^c(x-t) \gamma_n D_0^c(t-y) S^c(t-z) \hat{A}(z) S^c(z-y) \gamma^n S^c(y-x) \hat{A}(x)) : ] = \\
 & = -e^4 \text{Sp} [ : \overline{\psi(x)} \psi(t) \gamma^k A_k(t) \overline{\psi(t)} \psi(z) \hat{A}(z) \overline{\psi(z)} \psi(y) \gamma^n A_n(y) \overline{\psi(y)} \psi(x) \hat{A}(x) : ] = \\
 & = i^4 e^4 : (\overline{\psi(t)} \gamma^k A_k(t) \overline{\psi(t)} \psi(z) \hat{A}(z) \overline{\psi(z)} \psi(y) \gamma^n A_n(y) \overline{\psi(y)} \psi(x) \hat{A}(x) \psi(x)) :
 \end{aligned}$$

i.e., one of the terms in the expansion of the function

$$S_4(x, y, z, t) = i^4 T(\mathcal{L}(x) \mathcal{L}(y) \mathcal{L}(z) \mathcal{L}(t))$$

into normal products with chronological pairings.

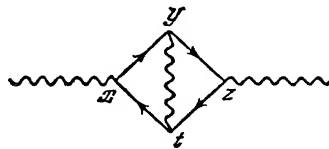


Fig. 6.

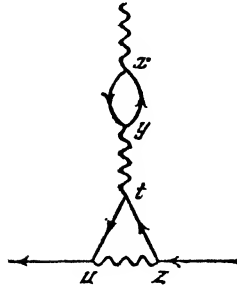


Fig. 7.

As a second example, we construct the term of the fifth-order operator function  $S_5(x, y, z, t, u)$  which corresponds to the diagram with two electron loops (Fig. 7). We write out in turn the matrix factors which correspond to the closed cycle

$$\begin{array}{c} y \quad x \\ \bullet \quad \bullet \\ \leftarrow ie\gamma^l \frac{1}{i} S^c(y-x) \quad \leftarrow ie\gamma^k \frac{1}{i} S^c(x-y) \end{array}$$

and to the open cycle

$$\begin{array}{c} u \quad t \quad z \\ \bullet \quad \bullet \quad \bullet \\ \leftarrow \bar{\psi}(u); ie\gamma^p; \frac{1}{i} S^c(u-t); ie\gamma^m; \frac{1}{i} S^c(t-z); ie\gamma^n; \psi(z) \end{array}$$

On multiplying them by the photon factors

$$\begin{array}{ccc} x & y & t \\ \bullet & \bullet & \bullet \\ \text{wavy line} & \text{wavy line} & \text{wavy line} \\ A_k(x); & ig^{lm} D_0^c(y-t); & ig^{np} D_0^c(z-u) \end{array}$$

and by

$$\eta = (-1)^l = -1,$$

we obtain, after carrying out the summation over the polarization indices ( $k, l, n, m, p$ ) and the spinor indices, the expression

$$\begin{aligned} & - (ie)^5 : \text{Tr} \left( \gamma_m \frac{1}{i} S^c(y-x) \hat{A}(x) \frac{1}{i} S^c(x-y) iD_0^c(y-t) \times \right. \\ & \quad \times \bar{\psi}(u) \gamma_n \frac{1}{i} S^c(u-t) \gamma^m \frac{1}{i} S^c(t-z) iD_0^c(z-u) \gamma^n \psi(z) \Big) : = \\ & = - (ie)^5 : \text{Tr} \left( \overbrace{\gamma^l \psi(y) \bar{\psi}(x)} \hat{A}(x) \overbrace{\psi(x) \bar{\psi}(y)} \overbrace{\bar{A}_l(y) A_m(t)} \times \right. \\ & \quad \times \overbrace{\bar{\psi}(u) \gamma^p \psi(u) \bar{\psi}(t) \gamma^m \psi(t) \bar{\psi}(z) \bar{A}_n(z) A_p(u) \gamma^n \psi(z)} : = (ie)^5 \times \\ & \quad \times : \bar{\psi}(u) \gamma^p A_p(u) \psi(u) \bar{\psi}(t) \gamma^m A_m(t) \psi(t) \bar{\psi}(z) \gamma^n A_n(z) \psi(z) \bar{\psi}(y) \gamma^l A_l(y) \psi(y) \bar{\psi}(x) \hat{A}(x) \psi(x) : , \end{aligned}$$

which corresponds exactly to one of the terms in the expansion into normal products with chronological pairings of the function

$$S_5(x, y, z, t, u) = i^5 T(\mathcal{L}(x) \mathcal{L}(y) \mathcal{L}(z) \mathcal{L}(t) \mathcal{L}(u)) .$$

**23.4. Concluding Remarks.** We have thus shown that the rules of correspondence together with the sign rule do in fact enable us to write down automatically the terms of the functions  $S_n(x_1, \dots, x_n)$  in normal form. It is evident that in order to obtain the complete expression for the operator function of the  $n$ th order  $S_n(x_1, \dots, x_n)$ , it is necessary to write out in accordance with Wick's theorem all the normal products which correspond to all the possible diagrams of the  $n$ th order, i.e., to all the possible diagrams containing  $n$  vertices. For example, for  $n = 2$ , it is necessary to construct terms which correspond to all those diagrams (Fig. 8) which exhaust all possible second-order diagrams. The sum of the corresponding normal products gives exactly (23.5) for  $S_2(x, y)$ .

We have carried out a detailed analysis of the structure of the terms of the  $S$ -matrix which depends on the interaction Lagrangian for the electromagnetic and the electron-positron fields (23.3). It is quite clear that the method described above for the construction of the terms of the  $S$ -matrix with the aid of Feynman diagrams may be applied without difficulty to any other arbitrary local interaction. Of course, the rules of correspondence must be suitably modified. For example, the Lagrangian for the pseudoscalar interaction of a nucleon spinor field and a pseudoscalar meson field has the form

$$\mathcal{L}(x) = ig : \bar{\Psi}(x) \gamma^5 \tau \Psi(x) \varphi(x) : . \quad (23.12)$$

The rules of correspondence for this Lagrangian are completely analogous to the rules given in Table I. The role of the spinors  $\psi$  is now played by the nucleon spinors  $\Psi$ , the potential  $A$  has been replaced by  $\varphi$ , and the matrices  $e\gamma^n$  by  $g\gamma^5 \tau_i$ . One meson and two nucleon lines will meet at each vertex of the corresponding diagrams.

Lagrangians of the form of (8.31) are used in the study of  $\beta$ -processes:

$$\mathcal{L}(x) = G : \bar{\Psi}(x) O \Psi(x) \bar{\psi}(x) O v(x) : , \quad (23.13)$$

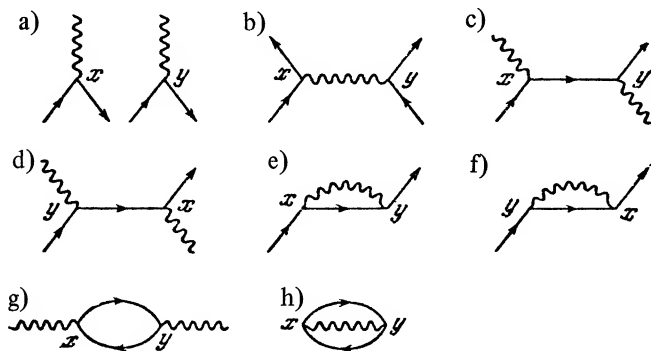


Fig. 8.

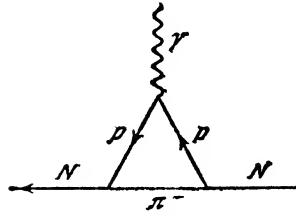


Fig. 9.

They contain three kinds of spinor fields—the nucleon field  $\Psi$ , the electron-positron field  $\psi$ , and the neutrino field  $\nu$ . The rules of correspondence in this case must be formulated for the operators  $\Psi, \bar{\Psi}, \psi, \bar{\psi}, \nu, \bar{\nu}$  and for the pairings between them. Four lines will meet at the vertices of the corresponding diagrams—two nucleon lines, one electron line, and one neutrino line. Only the nucleon lines of such a diagram will be continuous.

Complex processes in which interactions of different nature are combined are also of interest. The magnetic moment of an electrically neutral particle such as the neutron can, evidently, be represented as the result of a multistage process of interaction between the neutron and the electromagnetic field of the following type. At first, the neutron is virtually transformed into a  $\pi$ -meson and a proton, then one of the charged particles ( $\pi^-$  or  $p$ ) interacts with the electromagnetic field, and finally, the  $\pi^-$  and the proton are transformed back into a neutron. The process may be pictorially represented in the form of the diagram shown in Fig. 9. The corresponding term of the  $S$ -matrix obviously corresponds to the  $T$ -product of two Lagrangians  $\mathcal{L}_{\text{mes}}$  of the form (23.12) and of one Lagrangian  $\mathcal{L}_{\text{e.d.}}$  of the form (23.3):

$$T(\mathcal{L}_{\text{mes}}(x) \mathcal{L}_{\text{e.d.}}(y) \mathcal{L}_{\text{mes}}(z)).$$

The rules of correspondence for such diagrams and for the terms of the  $S$ -matrix are given by the sum of the rules of correspondence for the Lagrangians (23.3) and (23.12).

We thus see that the representation of the terms of the  $S$ -matrix by Feynman diagrams is a universal method which is suitable for any local interaction Lagrangians, and enables us to obtain automatically the terms of the  $S$ -matrix corresponding to any arbitrary process of interest to us.

## §24. Feynman's Rules for the Evaluation of Matrix Elements of the Scattering Matrix

**24.1. Transition to the Momentum Representation.** We shall now investigate the process of evaluating matrix elements of the scattering matrix, which occupies a prominent place in the calculation of effective cross sections for various processes of scattering and of mutual transformation of particles.

The evaluation of matrix elements is most conveniently carried out in the momentum representation, since, in this representation, the field operators  $u(x)$  and the causal functions  $\Delta^c(x - y)$  have a simple structure:

$$\begin{aligned}
 u(x) &= \frac{1}{(2\pi)^{3/2}} \int e^{ikx} \delta(k^2 - m^2) u(k) dk, & \Delta^c(x) &= \frac{1}{(2\pi)^4} \int e^{-ikx} \Delta(k) dk, \\
 u(k) &= \sum_{\sigma} v^{\sigma}(k) a_{\sigma}(k), & \Delta(k) &= \frac{P(k)}{m^2 - k^2 - i\varepsilon}.
 \end{aligned}$$

We write the  $S$ -matrix in the momentum representation. After reduction to the normal form, the  $n$ th-order term of the scattering matrix may be expressed in accordance with Wick's theorem as a sum of terms of the form

$$\int dx_1 \dots \int dx_n K_{\dots \alpha \dots \beta \dots}(x_1, \dots, x_n) : \dots u_{\alpha}(x_i) \dots \overset{*}{u}_{\beta}(x) \dots : \quad (24.1)$$

The coefficient functions  $K_{\dots \alpha \dots \beta \dots}(x_1, \dots, x_n)$  in the preceding expression correspond to the internal lines of Feynman diagrams and consist of multiple products of pairings occurring in the analysis of field functions. The normal product

$$: \dots u_{\alpha}(x_i) \dots \overset{*}{u}_{\beta}(x_i) \dots : \quad (24.2)$$

contains free operators corresponding to the external lines of the diagram.

To be specific, let us again restrict ourselves to the case of interacting electromagnetic and electron-positron spinor fields described by the Lagrangian

$$\mathcal{L}(x) = e : \bar{\psi}(x) \hat{A}(x) \psi(x) :. \quad (23.3)$$

In this case, the coefficient functions are products of pairings of the electromagnetic field

$$\left. \begin{aligned}
 \overline{A_m(x) A_n(y)} &= i g^{mn} D_0^c(x-y), \\
 D_0^c(x-y) &= -\frac{1}{(2\pi)^4} \int \frac{e^{-ik(x-y)}}{k^2 + i\varepsilon} dk
 \end{aligned} \right\} \quad (24.3)$$

and of pairings of the electron-positron field

$$\left. \begin{aligned}
 \overline{\psi_{\alpha}(x) \bar{\psi}_{\beta}(y)} &= \frac{1}{i} S_{\alpha\beta}^c(x-y), \\
 S_{\alpha\beta}^c(x-y) &= \frac{1}{(2\pi)^4} \int e^{-ip(x-y)} \frac{(m + \hat{p})_{\alpha\beta}}{m^2 - p^2 - i\varepsilon} dp,
 \end{aligned} \right\} \quad (24.4)$$

while the normal products (24.2) contain operator functions of the electromagnetic and the electron-positron fields which have the following momentum expansions:



$$\left. \begin{aligned} A_m(x) &= A_m^+(x) + A_m^-(x), \\ A_m^\pm(x) &= \frac{1}{(2\pi)^{3/2}} \int e^{\pm i k x} \delta(k^2) A_m^\pm(k) dk, \\ A_m^\pm(k) &= \sum_{\nu=0,1,2,3} e_\nu^\pm a_\nu^\pm(k); \end{aligned} \right\} \quad (24.5)$$

$$\left. \begin{aligned} \psi(x) &= \psi^+(x) + \psi^-(x), \\ \psi_\sigma^\pm(x) &= \frac{1}{(2\pi)^{3/2}} \int e^{\pm i p x} \delta(p^2 - m^2) \psi_\sigma^\pm(p) dp, \\ \psi_\sigma^\pm(p) &= \sum_{\nu=1,2} v_\sigma^{\nu,\pm}(p) a_\nu^\pm(p); \end{aligned} \right\} \quad (24.6)$$

$$\left. \begin{aligned} \bar{\psi}(x) &= \bar{\psi}^+(x) + \bar{\psi}^-(x), \\ \bar{\psi}_\sigma^\pm(x) &= \frac{1}{(2\pi)^{3/2}} \int e^{\pm i p x} \delta(p^2 - m^2) \bar{\psi}_\sigma^\pm(p) dp, \\ \bar{\psi}_\sigma^\pm(p) &= \sum_{\nu=1,2} \bar{v}_\sigma^{\nu,\pm}(p) a_\nu^\pm(p). \end{aligned} \right\} \quad (24.7)$$

Substituting (24.3)-(24.7) into (21.1), we see that the integration over the variables  $x_1, \dots, x_n$  now reduces to the evaluation of independent integrals of the form

$$\int dx_j e^{i x_j (p_1 - p_2 + k)} = (2\pi)^4 \delta(p_1 - p_2 + k). \quad (24.8)$$

We could at this point establish the rules of correspondence governing the construction of the elements of the  $S$ -matrix in the momentum representation. However, it is more convenient to investigate the evaluation of the matrix elements of the terms of the  $S$ -matrix and to formulate directly the rules of correspondence for the matrix elements. These rules were first proposed by Feynman (1944a, b) and are known as *Feynman's rules*.

**24.2. Evaluation of Matrix Elements.** We now examine the matrix elements of the normal products (24.2), taken between various states containing particles with given momenta. As was shown in §9, the amplitude of the state containing  $n$  particles of different kinds with definite momenta is given by an expression of the form

$$\begin{aligned} \Phi_{\dots k \dots} &= a_1^+(k_1) a_2^+(k_2) \dots a_n^+(k_n) \Phi_0 = \\ &= \int dk_1^0 \sqrt{2k_1^0} \delta(k_1^2 - m_1^2) a_1^+(k_1) \dots \int dk_n^0 \sqrt{2k_n^0} \delta(k_n^2 - m_n^2) a_n^+(k_n) \Phi_0. \end{aligned} \quad (24.9)$$

In the evaluation of the matrix element

$$\Phi_{\dots k' \dots}^* \dots u_\alpha(x_j) \dots \bar{u}_\beta(x_i) \dots : \Phi_{\dots k \dots} \quad (24.10)$$

the creation operators  $u^{(+)}$  and  $u^{\dagger(+)}$  must be commuted with the annihilation operators  $a_i^{(-)}$  from the amplitude  $\Phi_{\dots k' \dots}^*$  while the operators  $u^{(-)}$  and  $u^{\dagger(-)}$  must be commuted with the operators  $a_i^{(+)}$  from the amplitude  $\Phi_{\dots k \dots}$  until one of them acts on  $\Phi_0$  or on  $\Phi_0^*$ , which yields zero.

The process of calculating such matrix elements was examined in detail at the beginning

of §17. It was established there that the matrix element (24.10) differs from zero if for each operator

$$u(x) = u^+(x) + u^-(x)$$

in the normal product one can find an operator  $a^{*(+)}$  in  $\Phi$ , or  $a^{*(-)}$  in  $\Phi^*$ , corresponding to the same field, which will "cancel" the operator  $u$  as a result of being commuted with it. Thus (24.10) will differ from zero when the sum of the number of particles of each field in the initial state  $\Phi_{\dots k' \dots}$  and in the final state  $\Phi_{\dots k \dots}$  will be exactly equal to the number of operator functions of the given field in the normal product (24.2). The matrix element (24.10) will also differ from zero when in addition to the operators which "cancel" the normal product,  $\Phi_{\dots k' \dots}$  and  $\Phi_{\dots k \dots}$  also contain operators which cancel one another. The total number of particles in the states  $\Phi_{\dots k' \dots}^*$  and  $\Phi_{\dots k \dots}$  exceeds the number of operators in (24.1) by an even number. However, such matrix elements differ from zero only when the momenta of the above "extra" particles are the same in states  $\Phi_{\dots k' \dots}$  and  $\Phi_{\dots k \dots}$ .

By restricting ourselves to the case where no particles have the same momenta in the initial and final states, we arrive at the conclusion that the matrix element (24.10) may be represented by a product of the result of commuting

$$u^-(x_j) \text{ with } \int \theta(k^0) \hat{a}_Q^+(k) \delta(k^2 - m^2) \sqrt{2k^0} dk^0 = \hat{a}_Q^+(k)$$

and

$$u^+(x_i) \text{ with } \int \theta(k^0) \hat{a}_Q^-(k) \delta(k^2 - m^2) \sqrt{2k^0} dk^0 = \hat{a}_Q^-(k),$$

which on being evaluated with the aid of the commutation relations written in the form

$$\delta(p^2 - m^2) \delta(k^2 - m^2) \{\hat{a}_Q^-(p), \hat{a}_Q^+(k)\} = \delta_{\sigma Q} \delta(p - k) \delta(k^2 - m^2),$$

yield

$$\begin{aligned} \{u^-(x_j), \hat{a}^+(k)\} &= \frac{1}{(2\pi)^{3/2}} \int dk^0 \theta(k^0) \sqrt{2k^0} \int dpe^{-ipx_j} \sum_{\sigma} v^{\sigma, -}(p) \times \\ &\quad \times \{\hat{a}_Q^-(p), \hat{a}_Q^+(k)\} \delta(k^2 - m^2) \delta(p^2 - m^2) = \\ &= \frac{1}{(2\pi)^{3/2}} \int dk^0 \theta(k^0) \sqrt{2k^0} \int dpe^{-ipx_j} \sum_{\sigma} v^{\sigma, -}(p) \delta(p - k) \delta(k^2 - m^2) \delta_{\sigma Q} = \\ &= \frac{1}{(2\pi)^{3/2}} \int dk^0 \theta(k^0) \sqrt{2k^0} \delta(k^2 - m^2) e^{-ikx_j} v^{Q, -}(k) = \frac{1}{(2\pi)^{3/2}} v^{Q, -}(k) e^{-ikx_j}, \quad (24.11) \end{aligned}$$

where

$$v^{Q, -}(k) = \left. \frac{v^{Q, -}(k)}{\sqrt{2k^0}} \right|_{k^0 = \sqrt{k^2 + m^2}},$$

and also by a similar calculation

$$\{a_{\mathbf{Q}}^*(\mathbf{k}), u^+(x_i)\} = \frac{1}{(2\pi)^{3/2}} v_{\mathbf{Q}+}(\mathbf{k}) e^{ikx_i}. \quad (24.12)$$

Thus, after the commutations have been carried out the matrix element (24.10) is in fact expressed in the form of the product

$$\prod_{(\mathbf{k})} \frac{v^-(\mathbf{k})}{(2\pi)^{3/2}} e^{-ikx_j} \prod_{(\mathbf{k}')} \frac{v^+(\mathbf{k}')}{(2\pi)^{3/2}} e^{ik'x_i}, \quad (24.13)$$

where the factors  $(2\pi)^{-3/2} e^{-ikx} v^-(\mathbf{k})$  correspond to particles in the initial state, while the factors  $(2\pi)^{-3/2} e^{ik'x} v^+(\mathbf{k}')$  correspond to particles in the final state.

Let us now consider the above results from the point of view of their correspondence with Feynman diagrams. Of greatest significance is the fact that from the point of view of the structure of the matrix elements, to each external line of the diagram there corresponds a real particle either in the initial or in the final state. This allows us to regard Feynman diagrams as a schematic representation of real processes involving the interaction of elementary particles. The nature of these processes is determined by the structure of the vertices of the diagram which in turn depends on the interaction Lagrangian.

The vertices of the Lagrangian of spinor electrodynamics,  $e\bar{\psi}\hat{A}\psi$ , may evidently be regarded as schematic representation of the process of absorption or emission of a photon by a positron or an electron, and of the process of creation or annihilation of an electron-positron pair with the absorption or emission of a photon, i.e., processes of the type shown in Fig. 1.

The vertices of the Lagrangian for  $\beta$ -interactions describe interaction events in which four particles take part simultaneously: two nucleons, an electron (positron), and a neutrino.

The  $\delta(p - p' + k \dots)$  which arise in the transition to the momentum representation (24.8) evidently express the law of conservation of energy-momentum of the interacting particles in each individual event. By adding together the arguments of all such functions we obtain the condition

$$\sum p - \sum p' = 0, \quad (24.14)$$

which expresses the law of conservation of the total energy-momentum four-vector for all the real particles participating in the process under consideration.

For the formulation of specific rules of correspondence between matrix elements and Feynman diagrams, we return to spinor electrodynamics (23.3). It may readily be seen that an electron in the initial state with momentum  $p$  corresponds to the result of commuting  $\psi(x)$  with  $a_{\nu}^{*+}(p)$ , which is

$$\frac{1}{(2\pi)^{3/2}} \bar{v}^{\nu} \cdot -(\mathbf{p}) e^{-i\mathbf{p}\cdot\mathbf{x}}.$$

A positron with momentum  $\mathbf{p}$  corresponds to the factor

$$\frac{1}{(2\pi)^{3/2}} \bar{v}^{\nu} \cdot -(\mathbf{p}) e^{-i\mathbf{p}\cdot\mathbf{x}},$$

an electron in the final state corresponds to the factor

$$\frac{1}{(2\pi)^{3/2}} \bar{v}^{\nu} \cdot +(\mathbf{p}) e^{i\mathbf{p}\cdot\mathbf{x}},$$

a positron in the final state corresponds to the factor

$$\frac{1}{(2\pi)^{3/2}} v^{\nu} \cdot +(\mathbf{p}) e^{i\mathbf{p}\cdot\mathbf{x}},$$

a photon in the initial state with momentum  $\mathbf{k}$  and polarization  $e_n$  corresponds to the result of commuting  $A_m(x)$  with  $a_n^{\dagger}(\mathbf{k})$  which is

$$-\frac{1}{(2\pi)^{3/2}} \frac{g^{nl} e_m^l}{\sqrt{2k^0}} e^{-i\mathbf{k}\cdot\mathbf{x}} = \frac{e_m^n}{(2\pi)^{3/2} \sqrt{2k^0}} e^{-i\mathbf{k}\cdot\mathbf{x}}$$

and, finally, a photon in the final state corresponds to the factor

$$-\frac{1}{(2\pi)^{3/2}} \frac{g^{nl} e_m^l}{\sqrt{2k^0}} e^{i\mathbf{k}\cdot\mathbf{x}} = \frac{e_m^n}{(2\pi)^{3/2} \sqrt{2k^0}} e^{i\mathbf{k}\cdot\mathbf{x}}.$$

We see from the above that particles which arrive at the point  $x$  with momentum  $\mathbf{k}$  always correspond to the negative-frequency exponential  $\exp(-i\mathbf{k}\cdot\mathbf{x})$ , while particles emerging from the point  $x$  correspond to the factor  $\exp(i\mathbf{k}\cdot\mathbf{x})$ . It is therefore convenient to adopt a similar convention for the internal lines of the diagram as well.

According to (24.4), we shall therefore assume that the pairing  $\overline{\psi(x)} \psi(y)$  describes an electron with four-momentum  $p$  leaving the point  $y$  and entering the point  $x$ :

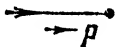
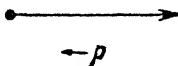
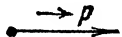
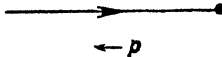

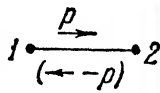

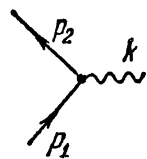
$$\overline{\psi(x)} \psi(y) \quad \sim \quad \bullet \xleftarrow{\quad} \bullet \quad (24.15)$$

$x \qquad y$

We note that the pairing and the diagram of (24.15) simultaneously describe a positron with four-momentum  $-p$ , i.e., moving in the opposite direction.

Table II. Correspondence Rules for Matrix Elements in the Momentum Representation

(The sign rule and the symmetry properties have not been taken into account.)

No.	Particle and its state	Factor in the matrix element	Element of the diagram
1.	Electron in the initial state with momentum $\mathbf{p}$	$\frac{v^{v,-}(\mathbf{p})}{(2\pi)^{3/2}}$	
2.	Positron in the initial state with momentum $\mathbf{p}$	$\frac{\bar{v}^{v,-}(\mathbf{p})}{(2\pi)^{3/2}}$	
3.	Electron in the final state with momentum $\mathbf{p}$	$\frac{\bar{v}^{v,+}(\mathbf{p})}{(2\pi)^{3/2}}$	
4.	Positron in the final state with momentum $\mathbf{p}$	$\frac{v^{v,+}(\mathbf{p})}{(2\pi)^{3/2}}$	
5.	Photon in the initial or final state with polarization $e_n$ and momentum $\mathbf{k}$	$\frac{e_n^n}{(2\pi)^{3/2} \sqrt{2k^0}} \quad (n \neq 0)$	
6.	Motion of an electron from 1 to 2 (or of a positron from 2 to 1)	$\frac{1}{(2\pi)^4 i} \int \frac{m + \hat{p}}{m^2 - p^2 - i\epsilon} dp$	
7.	Motion of a photon between vertices with summation indices $m$ and $n$	$\frac{g^{mn}}{(2\pi)^4 i} \int \frac{1}{k^2 + i\epsilon} dk$	
8.	Vertex with summation index $m$ with electron line $p_1$ and photon line $k$ entering, and electron line $p_2$ emerging.	$ie\gamma^m (2\pi)^4 \delta(p_2 - p_1 - k)$	

The correspondence rules listed in Table II are obtained by integrating with respect to the variables  $x_1, \dots, x_n$  and using (24.8).

24.3. *Taking Account of Symmetry Properties.* When matrix elements are constructed in the momentum representation, it is also necessary to take into account the symmetry properties of the coefficients  $S_n(x_1, \dots, x_n)$  of the scattering matrix. For example, the symmetry of the coefficient  $S_2(x_1, x_2)$  with respect to its arguments  $x_1$  and  $x_2$  leads to the fact that to diagrams of the Compton scattering type (Fig. 8c, d) there correspond two terms of  $S_2(x_1, x_2)$  in the  $x$ -representation:

$$:\bar{\psi}(x_1) \hat{A}(x_1) \overbrace{\psi(x_1) \psi(x_2)} \hat{A}(x_2) \psi(x_2):$$

and

$$\overbrace{:\bar{\psi}(x_1) \hat{A}(x_1) \psi(x_1) \bar{\psi}(x_2) \hat{A}(x_2) \psi(x_2):} = : \bar{\psi}(x_2) \hat{A}(x_2) \psi(x_2) \overbrace{\bar{\psi}(x_1) \hat{A}(x_1) \psi(x_1)}:,$$

which lead to identical expressions after the integration over  $x_1$  and  $x_2$  has been carried out.

In general, the symmetry of  $S_n(x_1, \dots, x_n)$  under any interchange of the arguments leads to the cancellation of the factor  $(n!)^{-1}$  after transition to the momentum representation. It is also important to remember that the symmetry of the Lagrangian and of the above diagrams may lead to the appearance of additional numerical factors. As an example, consider the simple scattering diagram (Fig. 10) in the model theory with the Lagrangian

$$\mathcal{L}(x) = \frac{\lambda}{3!} \varphi^3(x), \quad (24.16)$$

where  $\varphi$  is the scalar field (for further details about this models see §36).

The second-order perturbation-theory matrix element

$$\begin{aligned} M_2 &= \langle a^-(p') a^-(k') | \frac{1}{2!} \int S_2(x_1, x_2) dx_1 dx_2 | a^+(p) a^+(k) \rangle = \\ &= \frac{\lambda^2}{2! (3!)^2} \int dx_1 dx_2 \langle a^-(p') a^-(k') | T(\varphi^3(x_1) \varphi^3(x_2)) | a^+(p) a^+(k) \rangle \end{aligned} \quad (24.17)$$

can be evaluated with the aid of Wick's theorem. Applying Wick's theorem to the extreme left-hand operator, and using the fact that  $x_1$  and  $x_2$  are dummy arguments, we have

$$6 \frac{\lambda^2}{2! (3!)^2} \int dx_1 dx_2 \underbrace{a^-(p') \varphi(x_1)} \langle a^-(k') T(\varphi^2(x_1) \varphi^3(x_2)) a^+(p) a^+(k) \rangle.$$

The factor 6 corresponds to the six possible pairings of the operator  $a^-(p')$  and the operators  $\varphi(x_1)$ ,  $\varphi(x_2)$  from  $S_2(x_1, x_2)$ . Applying Wick's theorem again to  $a^-(k')$ , etc., and using the topology of the diagram in Fig. 10, we obtain

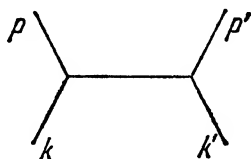


Fig. 10.

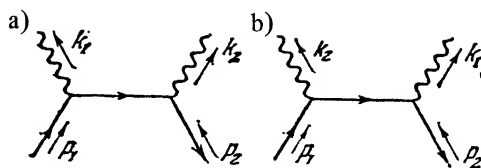


Fig. 11.

$$\begin{aligned}
 M_2 &= 6 \cdot 2 \frac{\lambda^2}{2! (3!)^2} \int dx_1 dx_2 \underbrace{a^-(p')} \varphi(x_1) \underbrace{a^-(k')} \varphi(x_1) \langle T(\varphi(x_1) \varphi^2(x_2) a^+(p) a^+(k)) \rangle = \\
 &= 6 \cdot 2 \cdot 3 \frac{\lambda^2}{2! (3!)^2} \int dx_1 dx_2 \underbrace{a^-(p')} \varphi(x_1) \underbrace{a^-(k')} \varphi(x_1) \overbrace{\varphi(x_1) \varphi(x_2)} \langle \varphi^2(x_2) a^+(p) a^+(k) \rangle = \\
 &= \lambda^2 \int dx_1 dx_2 \underbrace{a^-(p')} \varphi(x_1) \underbrace{a^-(k')} \varphi(x_1) \overbrace{\varphi(x_1) \varphi(x_2)} \underbrace{\varphi(x_2) a^+(p)} \underbrace{\varphi(x_2) a^+(k)}.
 \end{aligned}
 \tag{24.18}$$

Thus, the symmetry of the interaction Lagrangian (24.16) under all three operators  $\varphi$  which it contains leads to an effective cancellation of the factor  $3! = 6$  in the denominator of the coupling constant. Finally, we note that when the initial and (or) final states contain  $N$  quanta of the same field (a few photons, electrons, mesons, and so on), the corresponding normalized amplitude will contain the additional normalizing factors of the form  $(N!)^{-1/2}$ , which must also be taken into account.

Of course, the above symmetry properties can be formalized and the Feynman rules given in Table II can be suitably augmented.

However, experience shows that it is useful to have the correspondence rules for constructing the matrix elements to within the numerical factors and the sign. These factors and the signs are best determined with the aid of Wick's theorem.

It should be noted that the Feynman diagrams introduced above in fact describe *several processes simultaneously*.

Thus, the diagram of Fig. 11a describes first of all a process in which the electron  $p_1$  initially emits a photon  $k_1$ , and then annihilates with the positron  $p_2$ , emitting the photon  $k_2$ . In addition, the same diagram describes the process in which the positron  $p_2$  initially emits the photon  $k_2$  and then annihilates with the electron  $p_1$  emitting a photon  $k_1$ . In the first case, the internal electron line describes the motion of an electron in the direction of the electron line, in the second case it describes the motion of a positron in the opposite direction.

In the general case, one may consider that each internal electron line describes either an electron with four-momentum  $p$  (with  $p^0 > 0$ ) or a positron with four-momentum  $-p$  (with  $p^0 < 0$ ), while each internal photon line describes photons moving in one of the two possible directions as long as the whole overall picture of the motion of the particles does not contradict the energy-momentum conservation laws

$$p - p' \pm k = 0,$$

which follow from the structure of the vertex parts of the matrix element. In such a case

the positive-frequency part of the causal function  $\Delta^c$  describes motion in one direction, while the negative-frequency part describes motion in the opposite direction.

Those particles which, in this method of description, appear only in the intermediate states (for example, the photon in Fig. 8b) are called virtual particles. The intermediate states of real particles such as the electron state  $p_1 - k_1$  (or the positron state  $p_2 - k_2 = -(p_1 - k_1)$ ) in the diagram of Fig. 11a are called virtual states.

Virtual states of real particles and states of virtual particles differ from states of real particles by the fact that, in such states, the energy and the momentum are independent of each other, i.e., the following condition does not hold in virtual states:

$$p^2 - m^2 = 0.$$

**24.4. Scattering by External Fields.** We now turn to the structure of matrix elements for processes of scattering of particles by external classical fields. Such fields are described by an unquantized potential  $u^{\text{ext}}(x)$  which is a given function of the space coordinates  $x$  and the time  $x^0$ . For example, an external classical electromagnetic field is described by the potential  $A_n^{\text{ext}}(x)$ . The complete interaction Lagrangian for spinor electrodynamics, which we are considering, can now be represented in the form

$$\mathcal{L}(x) = -j^n(x) (A_n(x) + A_n^{\text{ext}}(x)). \quad (24.19)$$

Corresponding to this terms will appear in the scattering matrix which contain the external unquantized field  $A^{\text{ext}}$ . The matrix elements will now contain momentum representations of the external potential

$$A_n^{\text{ext}}(x) = \frac{1}{(2\pi)^{3/2}} \int e^{iqx} A_n^{\text{ext}}(q) dq, \quad (24.20)$$

which in view of their nonoperator nature need not commute with the photon operators of the state amplitudes and are not connected with real particles represented by external lines in Feynman diagrams.

We shall therefore agree to make the factor

$$\frac{1}{(2\pi)^{3/2}} \int A^{\text{ext}}(k) dk \quad (24.21)$$

correspond to a wavy line leaving a vertex of the diagram which corresponds to the term  $j^n(x)A_n^{\text{ext}}(x)$  in the Lagrangian and terminating in a shaded circle which we will use to denote symbolically a scattering center (Fig. 12).

The representation of the interaction with a classical field by a virtual photon which the interacting particle (in this case an electron) exchanges with the source of the classical field corresponds to the physical nature of the phenomenon. The classical field differs from a quantized field only by the fact that it is specified in advance and that the interaction



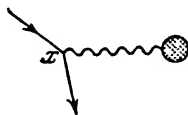


Fig. 12.

processes do not affect it. Such a field may be regarded as the limit of an ordinary quantized field as its intensity is made infinitely large. The interaction with such a field is actually brought about, even in the limiting case, by virtual photons whose propagation is described in the limiting case not by a causal photon function  $D_0^C$  but by a given potential  $A^{\text{ext}}$ .

If in going over to the momentum representation we carry out the integrations over the variables  $x_i$ , and integrate the expression

$$j^n(x_k) A_n^{\text{ext}}(x_k) \quad (24.22)$$

over the variable  $x_k$ , we obtain the factor  $\delta(p_1 - p_2 + q)$  which expresses the transfer of a part of the energy and of the momentum to the virtual photon. Thus, the law of conservation of energy-momentum for the real particles (24.14) does not hold for processes of scattering by a classical field.

Let us examine in greater detail the important special use of scattering by the *static* classical field

$$\begin{aligned} A_n^{\text{ext}}(x) &= a_n(x), \\ a_n(x) &= \frac{1}{(2\pi)^{3/2}} \int e^{-iqx} \varphi_n(q) dq. \end{aligned} \quad (24.23)$$

The variable  $q^0$  is absent from the expansion of the static potential, and therefore the virtual photons which describe the interaction with the static field have the energy  $q^0 = 0$  and transfer only momentum. Integrating (24.22) over  $x$ , we obtain the  $\delta$ -function

$$\delta(p - p' + q) = \delta(p^0 - p'^0) \delta(\mathbf{p} - \mathbf{p}' + \mathbf{q}),$$

which expresses the law of conservation of energy ( $p^0 = p'^0$ ) and the law of transfer of momentum ( $\mathbf{p}' - \mathbf{p} = \mathbf{q}$ ). The second factor,  $\delta(\mathbf{p} - \mathbf{p}' + \mathbf{q})$ , is then removed by integration over  $\mathbf{q}$ . Accordingly, the complete conservation law (24.14) will not longer hold and will be replaced by the law of conservation of the energy alone

$$\Sigma p^0 - \Sigma p'^0 = 0. \quad (24.24)$$

**24.5. General Structure of the Matrix Elements.** We shall examine, in general terms, the structure of the matrix elements of the scattering matrix

$$S = S(1),$$

defined in §20 as the limit of the matrix  $S(g)$  when the region in which  $g(x) = 1$  is expanded indefinitely to include the whole space-time.

It is therefore necessary also to investigate the structure of the matrix elements of the matrix  $S(g)$

$$\Phi_{\dots p' \dots}^* S(g) \Phi_{\dots p \dots} \quad (24.25)$$

and the singularities of the limiting transition  $g(x) \rightarrow 1$ .

From the structure of the matrix  $S(g)$  represented in the form

$$S(g) = T \left( \exp i \int \mathcal{L}(x; g) dx \right), \quad (21.38)$$

it follows that the matrix elements (24.24) differ from the matrix elements

$$\Phi_{\dots p' \dots}^* S(1) \Phi_{\dots p \dots}$$

only by the presence of an additional factor  $g(x)$  in the interaction Lagrangian  $\mathcal{L}(x)$ . Therefore  $g(x)$  may be tentatively regarded as a certain external classical field, an interaction with which takes place at each vertex of a Feynman diagram. The Fourier transform of the  $g$ -field

$$\tilde{g}(x) = \frac{1}{(2\pi)^4} \int e^{ixx} g(x) dx \quad (24.26)$$

may be made to correspond in the diagrams to lines similar to the one shown in Fig. 13, which emerge from each vertex, but which do not reach the edge of the diagram. Such a line may be shown, for example, as a dotted line (Fig. 13). Moreover, the introduction into the Lagrangian of the function  $g(x)$  evidently violates the law of conservation of energy-momentum at each vertex and, consequently, in the diagram as a whole.

Wick's theorem enables us to write the matrix elements (24.25) in the form

$$\begin{aligned} & \Phi_{\dots p' \dots}^* S(g) \Phi_{\dots p \dots} = \\ & = \sum_{(n, i_n)} \int K_{in}(x_1, \dots, x_n) g(x_1) \dots g(x_n) \Phi_{\dots p' \dots}^* \dots u_a(x_i) \dots \Phi_{\dots p \dots} dx_1 \dots dx_n. \end{aligned} \quad (24.27)$$

The matrix elements appearing on the right-hand side yield, after the commutations have been performed,

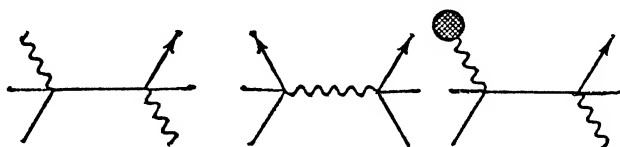


Fig. 13.

$$\begin{aligned} & \Phi_{p' \dots}^* \dots u_\alpha(x_j) \dots \Phi_{\dots p \dots} = \\ & = \sum A(\dots p' \dots p \dots) \exp \{i(p'_1 x_{j_1} + \dots + p'_s x_{j_s} - p_1 x_{i_1} - \dots - p_r x_{i_r})\} \quad (i, j = 1, \dots, n). \end{aligned} \quad (24.28)$$

Because the coefficient functions  $K(x_1, \dots, x_n)$  are invariant under translations, their momentum representation contains a  $\delta$ -function:

$$K(x_1, \dots, x_n) = \int \delta\left(\sum k_j\right) Q(k_1, \dots, k_n) e^{i \sum k_j x_j} dk_1 \dots dk_n.$$

Hence, using (24.26) we obtain

$$\begin{aligned} K(x_1, \dots, x_n) g(x_1) \dots g(x_n) &= \\ &= \int \delta\left(\sum k_j\right) Q(k_1, \dots, k_n) \tilde{g}(k'_1) \dots \tilde{g}(k'_n) e^{i \sum (k_j + k'_j) x_j} dk \dots dk' \dots = \\ &= \int K_g(k_1, \dots, k_n) e^{i \sum k_j x_j} dk \dots, \end{aligned} \quad (24.29)$$

where

$$\begin{aligned} K_g(k_1, \dots, k_n) &= \\ &= \int \delta\left(\sum k_j - \sum k'_j\right) Q(k_1 - k'_1, \dots, k_n - k'_n) \tilde{g}(k'_1) \dots \tilde{g}(k'_n) \dots dk' \dots \end{aligned} \quad (24.30)$$

Substituting (24.28) and (24.29) into (24.27) we obtain

$$\Phi_{\dots p' \dots} S(g) \Phi_{\dots p \dots} = \sum A(\dots p' \dots p \dots) K_g(\dots k \dots) (2\pi)^{4n}, \quad (24.31)$$

where each  $k$  is equal either to one of the  $p'$ , or to one of the  $p$ , or to the difference between  $p$  and  $p'$ , or to zero. In any case,

$$\sum k_j = \sum p' - \sum p.$$

We have already noted that  $g(x)$  is a function of the region in four-space in which the interaction occurs. If we assume that the process occurs within a volume  $V$  and within a time interval  $T$ , we find that

$$g(x) = 1 \quad \text{for} \quad x \in V, \quad x^0 \in T$$

and that it approaches zero sufficiently rapidly near the boundaries of this region. By increasing the dimensions of the region  $V$  and the value of  $T$  without limit, we eventually arrive at the situation in which the interaction is switched on in the infinitely remote past, is switched off in the infinitely remote future, and is effective over all space. The scattering matrix  $S(1)$  is usually defined by a limit of this kind.

The above Fourier transform  $\tilde{g}(k)$  tends to a quantity which is proportional to  $\delta(k)$ . To obtain the coefficient of proportionality we note that

$$\int \tilde{g}(k) dk = g(0) = 1,$$

and, therefore, we have approximately

$$\tilde{g}(k) \approx \delta(k). \quad (24.32)$$

If we now carry out the integration in (24.30), and use (24.32), we obtain

$$\begin{aligned} K_g(k_1, \dots, k_n) &= \int Q(k'_2 + \dots + k'_n - k_2 - \dots - k_n, k_2 - k'_2, \dots, k_n - k'_n) \times \\ &\quad \times \tilde{g}\left(\sum k_j - \sum k'_j + k'_1\right) \tilde{g}(k'_2) \dots \tilde{g}(k'_n) dk'_2 \dots dk'_n \approx \\ &\approx \tilde{g}\left(\sum k_j\right) Q(-k_2 - \dots - k_n, \dots, k_2, \dots, k_n) \approx \tilde{g}\left(\sum k_j\right) Q(k_1, \dots, k_n). \end{aligned} \quad (24.33)$$

In order to be able to carry out this integration, it is important that the function  $Q$  be continuous in the neighborhood of the point  $(k_1, \dots, k_n)$ .

Substituting (24.33) into (24.31), we now obtain

$$\Phi_{\dots p' \dots}^* S(g) \Phi_{\dots p \dots} \approx \tilde{g}\left(\sum p - \sum p'\right) F(p', p), \quad (24.34)$$

and therefore in view of (21.32)

$$\Phi_{\dots p' \dots}^* S(1) \Phi_{\dots p \dots} = \delta\left(\sum p - \sum p'\right) F(p', p). \quad (24.35)$$

When the probabilities of the various scattering processes are determined, one has to evaluate squares of matrix elements of the form

$$\left| \Phi_{\dots p' \dots}^* S \Phi_{\dots p \dots} \right|^2, \quad (24.36)$$

which may be naturally defined as the asymptotic forms of the expressions

$$\left| \Phi_{\dots p' \dots}^* S(g) \Phi_{\dots p \dots} \right|^2 \approx \left| \tilde{g}\left(\sum p - \sum p'\right) \right|^2 |F(p', p)|^2,$$

as  $V \rightarrow \infty, T \rightarrow \infty$ .

In the limit of (24.32), the expression  $|\tilde{g}(k)|^2$  tends formally to  $\delta^2(k)$ . Here, however, we have to use the fact (known from quantum mechanics) that we shall be interested in the probability of the process *per unit time per unit volume*.

Therefore, according to (24.26), we have the approximate expression

$$|\tilde{g}(k)|^2 = \tilde{g}(k) \tilde{g}(k) \simeq \delta(k) \tilde{g}(0) = \delta(k) \frac{1}{(2\pi)^4} \int g(x) dx \Big|_{g(x) \rightarrow 1} \rightarrow \delta(k) \frac{VT}{(2\pi)^4}.$$

For the square of the matrix element (24.36) in which we are interested, we now obtain a limiting expression of the form

$$|\Phi_{\dots p' \dots}^* S(1) \Phi_{\dots p \dots}|^2 = \frac{VT}{(2\pi)^4} \delta\left(\sum p - \sum p'\right) |F(p', p)|^2. \quad (24.37)$$

A similar calculation may be carried out for the scattering of particles by a classical stationary field such as (24.23). In this case, as we have already established, momentum is not conserved, but energy is, and in place of (24.35) and (24.37) we obtain:

$$\Phi_{\dots p' \dots}^* S(1) \Phi_{\dots p \dots} = \delta\left(\sum p^0 - \sum p'^0\right) F(p', p), \quad (24.38)$$

$$|\Phi_{\dots p' \dots}^* S(1) \Phi_{\dots p \dots}|^2 = \frac{T}{2\pi} \delta\left(\sum p^0 - \sum p'^0\right) |F(p', p)|^2. \quad (24.39)$$

## §25. Probabilities of Scattering Processes and Effective Cross Sections

*25.1. Normalization of the State Amplitude.* We shall now establish the connection between the matrix elements of the  $S$ -matrix and the probabilities of scattering processes. We first consider the case where the interaction is switched on with intensity  $g(x)$  localized within a finite space-time region, and discuss the usual problem of scattering theory in which prior to switching on the interaction there are  $s$  fluxes of particles with precisely determined momenta  $p_1, \dots, p_s$  and inner quantum numbers  $\sigma_1, \dots, \sigma_s$  characterizing the mass, the charge, and the spin, and in which it is required to find the average number of scattered particles with momenta lying within the infinitesimal intervals  $dp'_1, \dots, dp'_r$  and with inner quantum numbers  $\sigma'_1, \dots, \sigma'_r$ .

First of all it is evident that just as in the case of analogous problems of ordinary quantum mechanics, we are here dealing with nonnormalizable state amplitudes and must therefore use normalization per unit volume. In order to obtain the state amplitude with such normalization, it is natural to make use of a limiting transition and to consider a sequence of state amplitudes with an indefinitely increasing ordinary norm.

We first obtain this amplitude for the case of a single particle. We take the amplitude of the single-particle state

$$\Phi_1 = \int \chi_\sigma(p) a_\sigma^+(p) dp \Phi_0 \quad (25.1)$$

and note that its norm is

$$\Phi_1^\dagger \Phi_1 = \int |\chi_\sigma(\mathbf{p})|^2 d\mathbf{p} = N. \quad (25.2)$$

Therefore, by setting  $N = 1$ , we may consider that the expression

$$|\chi_\sigma(\mathbf{p})|^2 d\mathbf{p}$$

gives the probability that the particle characterized by the inner quantum number  $\sigma$  has its momentum within the interval  $d\mathbf{p}$  about the average value  $\mathbf{p}$ . The function  $\chi(\mathbf{p})$  itself is thus the wave function of the particle in the momentum representation. Therefore, its Fourier transform

$$\varphi_\sigma(\mathbf{x}) = \frac{1}{(2\pi)^{3/2}} \int e^{i\mathbf{p}\cdot\mathbf{x}} \chi_\sigma(\mathbf{p}) d\mathbf{p} \quad (25.3)$$

is the wave function in the configuration representation with the norm

$$\int |\varphi_\sigma(\mathbf{x})|^2 d\mathbf{x} = \int |\chi_\sigma(\mathbf{p})|^2 d\mathbf{p} = N.$$

When  $N = 1$ , the quantity

$$|\varphi(\mathbf{x})|^2 d\mathbf{x} \quad (25.4)$$

may be interpreted as the probability in configuration space. By setting  $N$  equal to the number of particles much larger than unity, we find that (25.4) is the average number of particles within the infinitesimal volume element  $d\mathbf{x}$ .

Now, by letting the norm  $N$  increase indefinitely in such a way that  $\chi_\sigma(\mathbf{p})$  tends to the expression

$$(2\pi)^{3/2} \delta(\mathbf{p} - \mathbf{p}_0), \quad (25.5)$$

we obtain a wave packet of increasingly better defined momentum. From (25.3) we find that

$$|\varphi_\sigma(\mathbf{x})| \rightarrow 1,$$

and that therefore also

$$|\varphi_\sigma(\mathbf{x})|^2 \rightarrow 1, \quad (25.6)$$

i.e., in the limit, the situation in which there is one particle per unit volume.

By going to the limit in (25.1) we obtain the expression for the amplitude of the single-particle state normalized per unit volume:

$$\Phi_1 = (2\pi)^{3/2} a_\sigma^\dagger(\mathbf{p}) \Phi_0. \quad (25.7)$$

In case of several different particles it is necessary to consider instead of (25.1) the expression

$$\Phi_s = \int \chi_{\sigma_1}(\mathbf{p}_1) a_{\sigma_1}^\dagger(\mathbf{p}_1) d\mathbf{p}_1 \dots \int \chi_{\sigma_s}(\mathbf{p}_s) a_{\sigma_s}^\dagger(\mathbf{p}_s) d\mathbf{p}_s \Phi_0, \quad (25.8)$$

where all the  $\sigma_i (i = 1, 2, \dots, s)$  are different. The norm of this amplitude will obviously be given by the product of the norms of  $s$  single-particle states:

$$\Phi_s^* \Phi_s = \prod_{1 \leq i \leq s} \int |\chi_{\sigma_i}(\mathbf{p}_i)|^2 d\mathbf{p}_i = \prod_i N_i. \quad (25.9)$$

By repeating the argument given above separately for each of the factors in (25.8) and (25.9), we conclude that the amplitude of the many-particle state normalized per unit volume for each of the particles present in this state has the form

$$\Phi_s = (2\pi)^{\frac{3s}{2}} a_{\sigma_1}^\dagger(\mathbf{p}_1) a_{\sigma_2}^\dagger(\mathbf{p}_2) \dots a_{\sigma_s}^\dagger(\mathbf{p}_s) \Phi_0. \quad (25.10)$$

*25.2. Calculation of Transition Probabilities.* We now return to the problem stated at the beginning of this section. We see that as a result of the interaction, the system will go over from the state (25.10) into the state described by the amplitude

$$S(g) (2\pi)^{\frac{3s}{2}} \Phi_{\rho_1 \sigma_1 \dots \rho_s \sigma_s},$$

where

$$\Phi_{\rho_1 \sigma_1 \dots \rho_s \sigma_s} \equiv a_{\sigma_1}^\dagger(\mathbf{p}_1) \dots a_{\sigma_s}^\dagger(\mathbf{p}_s) \Phi_0, \quad (25.11)$$

and therefore the average number of particles that will be found in the final state described by the normalized amplitude  $\Phi_\alpha$  is, in accordance with the general rule of quantum mechanics,

$$(2\pi)^{3s} \frac{|\Phi_\alpha^* S(g) \Phi_{\rho_1 \sigma_1 \dots \rho_s \sigma_s}|^2}{\Phi_\alpha^* \Phi_\alpha}. \quad (25.12)$$

It is assumed in connection with the above choice of normalization that in the initial state, the average numbers of the various particles per unit volume are equal to unity. When these numbers are respectively equal to  $n_1, \dots, n_s$ , (25.12) must be replaced by

$$n_1 n_2 \dots n_s (2\pi)^{3s} \frac{|\Phi_\alpha^* S(g) \Phi_{\rho_1 \sigma_1 \dots \rho_s \sigma_s}|^2}{\Phi_\alpha^* \Phi_\alpha}. \quad (25.13)$$

We are now interested in the number of particles scattered into the momentum intervals  $\Delta p'_1, \dots, \Delta p'_r$  centered on  $p'_1, \dots, p'_r$ . In carrying out this calculation we shall assume, as mentioned in §24.5, that the momentum of each particle has different values in the initial and the final states. Experimentally, this requirement corresponds to the fact that particles with unchanged momentum are always included in the primary beam, while only those particles whose momentum has changed are considered as scattered.

We shall therefore take the amplitude of the final state in the form

$$\Phi_\alpha = (2\pi)^{\frac{3r}{2}} \int_G \Phi_{k_1 \sigma_1 \dots k_r \sigma_r} dk_1 \dots dk_r,$$

where  $G$  is equal to the product of the volumes  $\Delta p'_1, \dots, \Delta p'_r$ . On calculating the norm of  $\Phi_\alpha$ , we now find that

$$\Phi_\alpha^* \Phi_\alpha = (2\pi)^{3r} \Delta p'_1 \dots \Delta p'_r,$$

and, in place of (25.13) we obtain

$$\begin{aligned} \frac{n_1 \dots n_s (2\pi)^{3s}}{\Delta p'_1 \dots \Delta p'_r} \int_G dk_1 \dots dk_r (\Phi_{k_1 \sigma_1 \dots k_r \sigma_r}^* S(g) \Phi_{p_1 \sigma_1 \dots p_s \sigma_s})^* \times \\ \times \int_G dk'_1 \dots dk'_r (\Phi_{k'_1 \sigma'_1 \dots k'_r \sigma'_r}^* S(g) \Phi_{p_1 \sigma_1 \dots p_s \sigma_s}), \end{aligned}$$

which, by virtue of the definition of  $G$ , is equal to

$$n_1 \dots n_s (2\pi)^{3s} \left| \Phi_{\dots p' \dots}^* S(g) \Phi_{\dots p \dots} \right|^2 \Delta p'_1 \dots \Delta p'_r, \quad (25.14)$$

where, for the sake of brevity, the symbol  $\Phi_{p_1 \sigma_1 \dots p_r \sigma_r}$  has been replaced by  $\Phi_{\dots p \dots}$ .

The structure of the squares of matrix elements of the form

$$\left| \Phi_{\dots p' \dots}^* S(g) \Phi_{\dots p \dots} \right|^2$$

and the limiting transition  $g(x) \rightarrow 1$  were investigated in §24. By substituting (25.37) into (25.14), and dividing it by  $VT$ , we obtain the number of particles scattered into the interval  $dp'_1 \dots dp'_r$  per unit time and per unit volume:

$$(2\pi)^{3s-4} n_1 \dots n_s |F(p', p)|^2 \delta\left(\sum p - \sum p'\right) dp'_1 \dots dp'_r. \quad (25.15)$$

Making use of formula (25.39) we also find the corresponding expression for scattering per unit time by a stationary classical field in the form

$$(2\pi)^{3s-1} n_1 \dots n_s |F(p', p)|^2 \delta\left(\sum p^0 - \sum p'^0\right) dp'_1 \dots dp'_r. \quad (25.16)$$



We emphasize once again that (25.15) and (25.16) are meaningful only if the function  $F(p', p)$  is *continuous* in the neighborhood of the given values of the momenta  $p', p$ . In specific calculations, we obtain for  $F(p', p)$  an expression in the form of a sum of several terms corresponding to a perturbation-theory expansion. The appearance of singularities in  $F(p', p)$  may therefore lead to a failure of perturbation theory.

Thus, in order to calculate the number of scattered particles or the corresponding probabilities it is first necessary to calculate the matrix element (25.37) (or (25.39)) and to substitute the resulting function  $F(p', p)$ , which multiplies the  $\delta$ -function, into (25.15) (or (25.16)). The final result will be meaningful only if  $F(p', p)$  is continuous in the neighborhood of the given values of its arguments.

We note that, because of the normalization of state vectors introduced in §25.1, it is convenient to change the normalization of the matrix element  $F$ . Instead of (24.35), let

$$\Phi_{p_1 \sigma_1 \dots p_s \sigma_s}^* S(1) \Phi_{q_1 \nu_1 \dots q_r \nu_r} = \frac{i (2\pi)^{4 - \frac{3}{2}(s+r)} \delta^4(\sum p - \sum q)}{\left\{ \prod_i (2p_i^0) \prod_k (2q_k^0) \right\}^{1/2}} M(p, q). \quad (25.17)$$

The change in the power of  $2\pi$  corresponds to the difference between (25.10) and (25.11), and the denominator takes into account the structure of the momentum representation of (spinless) field functions. The amplitude introduced in this way has a momentum dimensionality equal to  $4 - r - s$  and, in the simplest cases, is a Lorentz-invariant quantity.

The formula given by (25.15) assumes the form

$$(2\pi)^4 n_1 \dots n_s \frac{|M(p, q)|^2 \delta\left(\sum_k p_k - \sum_i q_i\right)}{\prod_k (2p_k^0)} \prod_i \frac{dp_i}{2p_i^0 (2\pi)^3}. \quad (25.18)$$

**25.3. Two-Particles Scattering.** We now consider the special case of the scattering of two particles ( $i = s = 2$ ). The left-hand side of (25.17) is then nonzero in the absence of interaction for  $S = 1$ :

$$\Phi_0^* a_1^-(p') a_2^-(q') a_1^+(q) a_2^+(p) \Phi_0 = \delta(p' - p) \delta(q - q') \pm \delta_{12} \delta(p' - q) \delta(q' - p).$$

The second term differs from 0 only if the quantum numbers of the two particles are identical, i.e., if the particles are identical. The upper sign corresponds to bosons and the lower to fermions.

To make sure that we are dealing with a pure interaction effect, the “diagonal” matrix element is removed from the total, i.e., we consider the expression

$$\Phi_{\dots p' \dots}^* (S - 1) \Phi_{\dots p \dots}, \quad (25.19)$$

which is the same as the original expression for  $p'_i \neq p_i$ . In accordance with (24.35), we substitute (and at the same time modify the normalization)

$$\Phi_0^* a_{\bar{1}}(p') a_{\bar{2}}(q') [S - 1] a_{\bar{1}}^{\dagger}(q) a_{\bar{1}}^{\dagger}(p) \Phi_0 = \frac{i \delta^4(p' + q' - p - q)}{2\pi \sqrt{q^0 q'^0 p^0 p'^0}} f(p', q'; p, q). \quad (25.20)$$

The function  $f$  in this expression is the *scattering amplitude* in the relativistic case. The normalization factor on the right-hand side of (25.20) corresponds to the scattering of spinless particles. It is readily verified that the relativistic amplitude  $f$  is a dimensionless function. It can be shown that it is also a Lorentz-invariant function (this is responsible for the appearance of the factor  $(q^0 q'^0 p^0 p'^0)^{-1/2}$ ). Finally, the numerical normalization is chosen from the point of view of convenience in relation to the effective cross section (see below).

In case of a boson (creation operator  $a_n^{\dagger}(k)$ ) scattered by a fermion (creation operator  $b_{\nu}^{\dagger}(p)$ ), the scattering amplitude is usually introduced in a somewhat different way:

$$\begin{aligned} \Phi_0^* a_m^-(k') b_{\mu}^-(p') (S - 1) a_n^+(k) b_{\nu}^+(p) \Phi_0 = \\ = \frac{i \delta(k' + p' - k - p)}{2\pi \sqrt{k_0 k'_0}} f_{n\nu, m\mu}(k, p; k', p'). \end{aligned} \quad (25.21)$$

where  $n(m)$  is the set of boson polarization and isospin indices and  $\nu(\mu)$  are the fermion spin and isospin indices. The representation given by (25.21) is valid, for example, in the case of scattering of light by electrons (Compton scattering), or for pion-nucleon scattering. The matrix amplitude for scattering,  $f_{n\nu, m\mu}$ , has the dimensions  $[m^{-1}] = cm$  and can be made dimensionless by introducing the corresponding mass. This is not, however, usually done because of traditional considerations involving correspondence between its non-relativistic limit and the dimensional quantum-mechanical scattering amplitude.

**25.4. Effective Scattering Cross Sections.** For  $r = s = 2$ , the replacement of  $F$  with the amplitude  $f_{n\nu, m\mu}$ , introduced in accordance with (25.21), means that, instead of (25.15), we now have the expression

$$\frac{n_1 n_2}{k_0 k'_0} |f_{n\nu, m\mu}(k, p; k', p')|^2 \delta(p + k - p' - k') dk' dp',$$

the dimensions of which in our system of units are  $cm^{-4}$ . Using the three-dimensional  $\delta$ -function to remove integration with respect to  $p'$ , we obtain

$$\frac{n_1 n_2}{k_0 k'_0} |f|^2 \delta(p_0 + k_0 - p'_0 - k'_0) dk', \quad (25.22)$$

where we have put  $p' = p + k - k'$  in the argument of  $f$ .

The expression given by (25.22) is the number of particles (in the present case, bosons) scattered into the momentum interval  $dk'$  per unit volume per unit time, and is usually represented by the product

$$n_1 n_2 \nu(k) d\sigma,$$

where  $\nu(k)$  is the modulus of the incident-boson velocity in the laboratory frame (equal to unity in the case of a photon and  $|k|/k^0$  for particles of finite rest mass). The factor  $d\sigma$  has the dimensions of area ( $\text{cm}^2$ ) and is proportional to the solid angle element containing the boson after the scatter. It is called the *differential cross section*. According to (25.22), this cross section can be written in the form

$$d\sigma = \frac{|f|^2 \delta(p_0 + k_0 - p'_0 - k'_0) d\mathbf{k}'}{k'_0 |\mathbf{k}|} = |f|^2 \delta(p_0 + k_0 - p'_0 - k'_0) \frac{|\mathbf{k}'|^2 d|\mathbf{k}'| d\Omega'}{k'_0 |\mathbf{k}|}.$$

To dispose of the  $\delta$ -function, we use the fact that, since  $(k'_0)^2 - m^2 = (k')^2$ ,

$$|\mathbf{k}'| d|\mathbf{k}'| = k'_0 dk'_0,$$

and the integration with respect to  $dk'_0$  can be removed with the aid of the remaining  $\delta$ -function. However, it must be remembered that the energies  $p_0$ ,  $k_0$ ,  $p'_0$ , and  $k'_0$ , in its argument are independent functions because of the conservation of three-momentum:  $\mathbf{p}' = \mathbf{p} + \mathbf{k} - \mathbf{k}'$ . Therefore, substituting

$$p'_0 + k'_0 - p_0 - k_0 = \varphi(k'_0),$$

we obtain

$$\frac{d\sigma}{d\Omega'} = \frac{|\mathbf{k}'|}{|\mathbf{k}| \varphi'(k'_0)} |f_{n\nu, m\mu}(\mathbf{k}, \mathbf{p}; \mathbf{k}', \mathbf{p}')|^2. \quad (25.23)$$

Integrating the differential cross section (25.23) with respect to the total solid angle  $\Omega$ , we obtain the total cross section

$$\sigma = \frac{1}{|\mathbf{k}|} \int_{\Omega} \frac{|\mathbf{k}'| d\Omega}{\varphi'(k'_0)} |f_{n\nu, m\mu}(\mathbf{k}, \mathbf{p}; \mathbf{k}', \mathbf{p}')|^2. \quad (25.24)$$

In the laboratory frame (for  $\mathbf{p} = 0$ )

$$\varphi'(k'_0) = \frac{k_0 k'_0 M - \mu^2 p'_0}{|\mathbf{k}'|^2 p'_0}, \quad (25.25)$$

where  $\mu$  is the boson mass and  $M$  is the fermion mass. Thus, for example, in the static limit of pion-nucleon scattering,

$$\frac{d\sigma}{d\Omega'} = |f|^2. \quad (25.26)$$

This expression is similar in form to the corresponding formula of nonrelativistic quantum mechanics. This in fact explains the particular choice of normalization in (25.21).

We also note that, as a rule, when total cross sections are evaluated, one is not interested in the final-state spin indices, and the results are summed over these indices. If, on the other hand, the initial indices are not separated out in an experiment (for example, in the case of scattering of unpolarized particles by an unpolarized target), then an average must be taken over them. Denoting these summation and averaging operations by the symbol  $\Sigma_{\sigma}$ , we obtain the following expression for the differential cross section:

$$\begin{aligned} d\sigma &= \frac{(2\pi)^2 |k'| |k'_0|}{v(k) \varphi'(k'_0)} \sum_{\sigma} |F(k, p; k', p')|^2 d\Omega'_1 = \\ &= \frac{|k'|}{|k| |\varphi'(k'_0)|} \sum_{\sigma(m, \mu)} |f_{n\nu, m\mu}(k, p; k', p')|^2 d\Omega'_1. \end{aligned} \quad (25.27)$$

Similarly, the general formulas (25.15) and (25.16) can be used to obtain expressions for the probabilities and cross sections corresponding to other possible processes. We conclude by deriving the formula for the scattering of a particle with momentum  $p_1$  by a stationary external field. To be specific, consider the case where the particle scattered by the potential creates a further particle  $k$ , i.e., we have the bremsstrahlung type process. Substituting  $s = 1, r = 2$  in (25.16), we obtain

$$(2\pi)^2 n_1 |F(p', p)|^2 \delta(E_1 - E'_1 - E_k) dp'_1 dk, \quad (25.28)$$

which, in our system of units has the dimensions  $[T^{-1}] = cm^{-1}$ . If we write (25.28) in the form  $n_1 v_1 d\sigma_1$ , and integrate with respect to  $E'_1$  and sum over the spin indices, we find that the differential cross section is given by

$$d\sigma_1 = \frac{(2\pi)^2}{v_1} |F(p', p)|^2 p'_1 E'_1 d\Omega'_1 dk |_{E_1 - E'_1 - E_k = 0}. \quad (25.29)$$

**25.5. Two-Particle Decay.** We now suppose that in the initial state, we have one particle ( $s = 1$ ) of mass  $M$  and, in the final state, two particles ( $r = 2$ ) with masses  $m_1$  and  $m_2$ . This case corresponds to the two-particle decay of the particle  $M$ , which is kinematically possible for

$$M > m_1 + m_2. \quad (25.30)$$

According to (25.18), the mean number of particles in the final state, which enter the momentum interval  $dp_1, dp_2$  per unit time per unit volume, is

$$\frac{n}{32\pi^2 q_0} |M(p_1, p_2; q)|^2 \delta^{(4)}(q - p_1 - p_2) \frac{dp_1 dp_2}{p_1^0 p_2^0}.$$

In the rest system of the initial particle:

$$q = (M, 0), \quad p_1 = (E_1, \mathbf{p}), \quad p_2 = (E_2, -\mathbf{p}); \quad E_i = \sqrt{m_i^2 + \mathbf{p}^2}.$$

Integration with respect to  $d\mathbf{p}_2$  yields

$$\frac{|M(\dots)|^2}{32\pi^2 M E_1 E_2} \delta(M - E_1 - E_2) d\mathbf{p}. \quad (25.31)$$

The integration with respect to  $|\mathbf{p}|$  can be removed with the aid of the remaining  $\delta$ -function:

$$\delta(M - E_1 - E_2) = \frac{2E_1 E_2}{M} \delta(\mathbf{p}^2 - p_*^2), \quad 4p_*^2 = M^2 - 2(m_1^2 + m_2^2) + \frac{(m_1^2 - m_2^2)^2}{M^2}. \quad (25.32)$$

Integration with respect to  $|\mathbf{p}|$  yields

$$\frac{p_*}{32\pi^2 M^2} |M(p_*, \Omega)|^2 d\Omega.$$

Integrating this with respect to the solid angle, we obtain the total probability of the two-particle decay process per unit time:

$$w = \frac{p_*}{32\pi^2 M^2} \int |M(p_*, \Omega)|^2 d\Omega. \quad (25.33)$$

The quantity  $w$  has the dimensions of mass (or, in the ordinary set of units  $\text{sec}^{-1}$ ) and is equal to the energy width:

$$w = \Gamma. \quad (25.34)$$

When there are a number of decay channels, the total width is obtained by summing expressions such as (25.33):

$$\Gamma_{\text{tot}} = \sum_i \Gamma_i, \quad (25.35)$$

and the lifetime of the original particle is then defined by

$$\tau = \frac{1}{\Gamma_{\text{tot}}} \left( = \frac{\hbar}{\Gamma_{\text{tot}}} \right). \quad (25.36)$$

## §26. Examples of Calculation of Second-Order Processes

As examples of the application of the methods presented in §25, we shall calculate, in the first nonvanishing approximation, the cross sections for the Compton scattering, for the two-photon annihilation of an electron-positron pair, and for the bremsstrahlung of an electron in the field of a nucleus. Since the formulas obtained below are well known, we shall not go into a discussion of them.

**26.1. Compton Scattering.** We shall calculate the cross section for the scattering of a photon by a free electron. The initial state contains a photon of momentum  $\mathbf{k}_1$ , energy  $k_1 = |\mathbf{k}_1|$ , and an electron of four-momentum  $p_1$ . Without loss of generality we shall assume that the electron is at rest, i.e.,  $\mathbf{p}_1 = 0$ ,  $p_1^0 = m$ . In the final state we have the scattered photon of momentum  $\mathbf{k}_2$  and energy  $k_2 = |\mathbf{k}_2|$ , and an electron which has received the recoil momentum  $\mathbf{p}_2 = \mathbf{k}_1 - \mathbf{k}_2$  and which has the energy

$$p_2^0 = E_2 = \sqrt{(\mathbf{k}_1 - \mathbf{k}_2)^2 + m^2} = \sqrt{k_1^2 + k_2^2 - 2|\mathbf{k}_1||\mathbf{k}_2|\cos\theta + m^2},$$

where  $\theta$  is the angle between the vectors  $\mathbf{k}_1$  and  $\mathbf{k}_2$ .

In accordance with the general formula (25.27), the differential cross section for this process is

$$d\sigma = (2\pi)^2 \frac{k_2^3}{|\Phi'(k_2)|} \sum_{\sigma} |F(p', p)|^2 d\Omega = (2\pi)^2 \frac{k_2^3 E_2}{k_1 m} \sum_{\sigma} |F(p', p)|^2 d\Omega. \quad (26.1)$$

The matrix element  $F(p', p)$  is given by (24.35). In order to calculate it, we shall restrict ourselves to the possible diagrams of the second order, shown in Fig. 14. The matrix elements corresponding to these diagrams can be obtained from the rules of correspondence formulated in §24.

We now shall write down in detail the matrix element for the diagram of Fig. 14a. The factor

$$(2\pi)^{-3/2} \bar{v} v + (p_2),$$

will be taken to correspond to the external line of the electron emerging from the vertex 2 with the four-momentum  $p_2$ , while the factor

$$\frac{1}{(2\pi)^{3/2}} \frac{e_m^2}{\sqrt{2k_2}} \quad (m \neq 0) \quad (21.31)$$

will be taken to correspond to the external line of the photon emerging from the vertex 2 with momentum  $\mathbf{k}_2$ . Since the factor

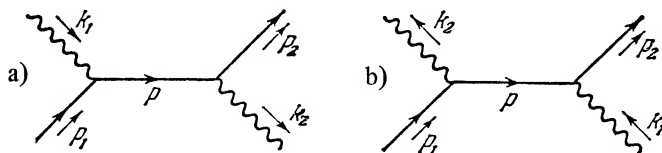


Fig. 14.

$$ie\gamma^n (2\pi)^4 \delta(p - k_2 - p_2),$$

corresponds to this vertex the product of the last two factors gives

$$\delta(p - k_2 - p_2) \frac{ie}{\sqrt{2k_2}} (2\pi)^{3/2} \sum_{\alpha=1,2,3} e_2^\alpha \gamma^\alpha = -\delta(p - k_2 - p_2) \frac{ie (2\pi)^{3/2}}{\sqrt{2k_2}} \hat{e}_2.$$

The factor

$$\frac{1}{(2\pi)^4 i} \frac{m + \hat{p}}{m^2 - p^2 - i\varepsilon},$$

corresponds to the internal electron line, the expression

$$\delta(p_1 + k_1 - p) \frac{ie (2\pi)^{3/2}}{\sqrt{2k_1}} \hat{e}_1,$$

corresponds to vertex 1 together with the external photon line  $k_1$ , and finally the factor

$$(2\pi)^{-3/2} u^{v',-}(p_1)$$

corresponds to the external electron line  $p_1$ .

Multiplying together all the preceding factors in the sequence corresponding to the motion along the electron line, and integrating over  $p$ , we obtain the expression

$$\delta(p_1 + k_1 - p_2 - k_2) F_a(p, k),$$

where

$$F_a(p, k) = \frac{e^2}{8\pi^2 i} \frac{1}{\sqrt{k_1 k_2}} \bar{u}^{v,+}(p_2) \hat{e}_2 \frac{\hat{p}_1 + \hat{k}_1 + m}{(p_1 + k_1)^2 - m^2} \hat{e}_1 u^{v',-}(p_1).$$

The diagram of Fig. 14b yields an analogous term

$$F_b(p, k) = \frac{e^2}{8\pi^2 i} \frac{1}{\sqrt{k_1 k_2}} \bar{u}^{v,+}(p_2) \hat{e}_1 \frac{\hat{p}_1 - \hat{k}_2 + m}{(p_1 - k_2)^2 - m^2} \hat{e}_2 u^{v',-}(p_1).$$

Noting that

$$(p_1 + k_1)^2 - m^2 = 2(p_1 k_1) = 2m k_1, \quad (p_1 - k_2)^2 - m^2 = -2(p_1 k_2) = -2m k_2,$$

we obtain, on adding  $F_a$  and  $F_b$ ,

$$F(p, k) = \frac{e^2}{16\pi^2 i} \frac{1}{m (k_1 k_2)^{1/2}} \bar{u}^{v,+}(p_2) M u^{v',-}(p_1), \quad (26.2)$$

where

$$M = k_2 \hat{e}_2 (\hat{p}_1 + \hat{k}_1 + m) \hat{e}_1 - k_1 \hat{e}_1 (\hat{p}_1 - \hat{k}_2 + m) \hat{e}_2.$$

In calculating  $\sum_{\sigma} |F|^2$  we note that we must average over the initial spin index  $\nu'$ , and sum over the final spin index  $\nu$ . Therefore, we obtain with the aid of (7.20) and (7.21):

$$\begin{aligned} \sum_{\sigma} |F|^2 &= \frac{1}{2} \sum_{\nu, \nu'} FF^* = \frac{e^4}{2 (4\pi)^4 m^3 (k_1 k_2)^3} \sum_{\nu \nu'} \bar{v}^{\nu, +}(p_2) M v^{\nu', -}(p_1) \bar{v}^{\nu', +}(p_1) \overline{M} v^{\nu, -}(p_2) = \\ &= \frac{e^4}{8 (4\pi)^4 E_2 (m k_1 k_2)^3} \text{Sp } A, \end{aligned} \quad (26.3)$$

where

$$A = (\hat{p}_2 + m) M (\hat{p}_1 + m) \overline{M}. \quad (26.4)$$

Since the four-vectors  $k$ ,  $p$ , and  $e$  are real,  $\gamma^0$  is Hermitian, and  $\gamma$  is anti-Hermitian, we find that, if  $M = \sum_i c_i \hat{a}_1 \hat{a}_2 \dots \hat{a}_i$ , then

$$\overline{M} = \sum_i^* c_i^* \hat{a}_i \hat{a}_{i-1} \dots \hat{a}_1,$$

where  $a_1, \dots, a_i$  are vectors such as  $k$ ,  $p$ , and  $e$ .

Before evaluating (26.4) let us simplify  $M$  somewhat. Since, in accordance with the definition of the Dirac matrices,

$$\hat{a}\hat{b} + \hat{b}\hat{a} = 2(ab),$$

we may represent  $M$  in the form

$$M = k_2 \hat{e}_2 \hat{k}_1 \hat{e}_1 - k_2 \hat{e}_2 \hat{e}_1 (\hat{p}_1 - m) + 2k_2 \hat{e}_2 (p_1 e_1) + k_1 \hat{e}_1 k_2 \hat{e}_2 + k_1 \hat{e}_1 \hat{e}_2 (\hat{p}_1 - m) - 2k_1 \hat{e}_1 (p_1 e_2). \quad (26.5)$$

But since  $M$  occurs in the combination

$$\bar{v}^{\nu, +}(p_2) M v^{\nu', -}(p_1),$$

and, in accordance with the field equation,

$$(\hat{p}_1 - m) v^{\nu', -}(p_1) = 0$$

and since

$$(p_1 e_2) = (p_1 e_1) = 0,$$

the matrix  $M$  may be represented in the form

$$M = k_2 \hat{e}_2 \hat{k}_1 \hat{e}_1 + k_1 \hat{e}_1 \hat{k}_2 \hat{e}_2.$$

The matrix  $A$  may now be written as the sum of three terms



$$A = A_1 + A_2 + A_3,$$

where

$$A_1 = (\hat{p}_1 + m) M (\hat{p}_1 + m) \bar{M}, \quad A_2 = (\hat{k}_1 - \hat{k}_2) M \hat{p}_1 \bar{M}, \quad A_3 = m (\hat{k}_1 - \hat{k}_2) M \bar{M}.$$

For the calculation of traces, we shall use the conditions of transversality of the electromagnetic field

$$(e_1 k_1) = (e_2 k_2) = 0$$

and the relations

$$(\rho_1 e_1) = (\rho_1 e_2) = 0, \quad (\rho_1 k_1) = m k_1, \quad (\rho_1 k_2) = m k_2, \quad (k_1 k_1) = (k_2 k_2) = 0, \quad e_1^2 = e_2^2 = -1.$$

The term  $A_3$  contains the product of an odd number of Dirac matrices and, therefore,

$$\text{Tr } A_3 = 0.$$

In order to calculate  $\text{Tr } A_1$  it is sufficient to commute the first two factors in  $A_1$  and to use the third of formulas (6.16). This gives

$$\text{Tr } A_1 = 32 m^2 k_1^2 k_2^2 (e_1 e_2)^2 = 32 m^2 k_1^2 k_2^2 \cos^2 \theta. \quad (26.6)$$

For the calculation of  $A_2$  it is convenient to use the fact that the factors of the product whose trace is being sought may be cyclically interchanged, and to write  $A_2$  in the form

$$A^2 = \bar{M} (\hat{k}_1 - \hat{k}_2) M \hat{p}_1 + \tilde{A}_2 \quad (\text{Tr } \tilde{A}_2 = 0).$$

By carrying out the commutations in such a way that identical factors turn out to be next to each other, we obtain

$$(\hat{k}_1 - \hat{k}_2) M = a + b,$$

where

$$a = k_2 \hat{e}_2 \hat{k}_2 \hat{k}_1 \hat{e}_1 - k_1 \hat{e}_1 \hat{k}_1 \hat{k}_2 \hat{e}_2, \quad b = 2 (k_1 e_2) k_2 \hat{k}_1 \hat{e}_1 - 2 (k_2 e_1) k_1 \hat{k}_2 \hat{e}_2.$$

Further, we find that

$$\text{Tr } \bar{M} a \hat{p}_1 = 8 k_1 k_2 m (k_1 k_2) (k_1 - k_2) + 8 k_1 k_2 m \{k_1 (k_2 e_1)^2 - k_2 (k_1 e_2)^2\}, \quad (26.7)$$

$$\text{Tr } \bar{M} b \hat{p}_1 = 8 k_1 k_2 m \{k_2 (k_1 e_2)^2 - k_1 (k_2 e_1)^2\}. \quad (26.8)$$

By adding expressions (26.6)–(26.8), and by taking into account the following relation, whose validity may be easily checked,

$$(k_1 k_2) = m (k_1 - k_2),$$

we obtain

$$\text{Tr } A = 8m^2 k_1^2 k_2^2 \left\{ 4 \cos^2 \theta + \frac{k_1}{k_2} + \frac{k_2}{k_1} - 2 \right\}.$$

By substituting this expression together with (26.3) into (26.1), we obtain the differential cross section

$$d\sigma = \frac{e^4}{(8\pi)^2 m^2} \frac{k_2^2}{k_1^2} \left\{ 4 \cos^2 \theta + \frac{k_1}{k_2} + \frac{k_2}{k_1} - 2 \right\} d\Omega.$$

Introducing the “electron radius”

$$r_0 = \frac{e^2}{4\pi m},$$

we obtain the well-known *Klein-Nishina-Tamm formula*:

$$d\sigma = \frac{r_0^2}{4} \frac{k_2^2}{k_1^2} \left\{ 4 \cos^2 \theta + \frac{k_1}{k_2} + \frac{k_2}{k_1} - 2 \right\} d\Omega. \quad (26.9)$$

**26.2. Annihilation of an Electron-Positron Pair.** We shall now investigate the process of mutual annihilation of an electron and a positron. The simplest diagram which corresponds to this process (Fig. 15) is the only first-order diagram.

However, it may readily be seen that the one-photon annihilation described by this diagram is forbidden by the energy-momentum conservation laws. Indeed, the conservation laws give

$$\begin{aligned} \mathbf{k} &= \mathbf{p}_1 + \mathbf{p}_2, \\ |\mathbf{k}| &= \sqrt{|\mathbf{p}_1|^2 + m^2} + \sqrt{|\mathbf{p}_2|^2 + m^2}, \end{aligned}$$

and by transforming, for example, to the system in which the center of mass of the electron and the positron is at rest ( $\mathbf{p}_1 + \mathbf{p}_2 = 0$ ), we obtain an obvious contradiction.

The two-photon annihilation is described by the two second-order diagrams shown in Fig. 11.

We shall carry out the calculation in the system in which the center of mass of the electron and the positron is at rest. Then by setting

$$\mathbf{p}_1 = \mathbf{p}, \quad \mathbf{p}_2 = -\mathbf{p}, \quad \mathbf{k}_1 = \mathbf{k}, \quad \mathbf{k}_2 = -\mathbf{k},$$

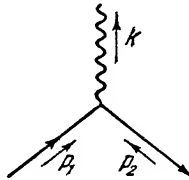


Fig. 15.

we obtain:

$$\begin{aligned} \text{electron: } & (E_p = \sqrt{p^2 + m^2}, p), \\ \text{positron: } & (E_p, -p), \\ \text{first photon: } & (k^0 = |\mathbf{k}|, \mathbf{k}), \\ \text{second photon: } & (k^0, -\mathbf{k}). \end{aligned}$$

In accordance with the basic formula (22.21) and taking into account the normalization of the amplitude of the two-photon state and the relation  $|f'(E')| = 2$ , the cross section for the foregoing process is equal to

$$d\sigma = \frac{(2\pi)^2 (k^0)^2}{4v(p_1)} \sum_{\sigma} |F|^2 d\Omega = (2\pi)^2 \frac{(k^0)^2}{4p} \sum_{\sigma} |F|^2 d\Omega, \quad (26.10)$$

where we have used the notation  $p = |\mathbf{p}|$ .

Making use of the rules of correspondence, we construct the matrix elements

$$\begin{aligned} F_a(p, k) &= \frac{e^2}{8\pi^2 i k^0} \bar{v}^{v, -}(p_2) \hat{e}_2 \frac{\hat{p}_1 - \hat{k}_1 + m}{(p_1 - k_1)^2 - m^2} \hat{e}_1 v^{v, -}(p_1), \\ F_c(p, k) &= \frac{e^2}{8\pi^2 i k^0} \bar{v}^{v, -}(p_2) \hat{e}_1 \frac{\hat{p}_1 - \hat{k}_2 + m}{(p_1 - k_2)^2 - m^2} \hat{e}_2 v^{v, -}(p_1); \end{aligned}$$

and by adding them we find that

$$F(p, k) = \frac{ie^2}{8\pi^2 k^0} \frac{\bar{v}^{v, -}(p_2) M v^{v, -}(p_1)}{2(p_1 k_1)(p_1 k_2)},$$

where

$$M = (p_1 k_2) \hat{e}_2 (\hat{p}_1 - \hat{k}_1 + m) \hat{e}_1 + \hat{e}_1 (\hat{p}_1 - \hat{k}_2 + m) \hat{e}_2 (p_1 k_1).$$

By commuting  $\hat{p}_1 \hat{e}_1$  and  $\hat{p}_2 \hat{e}_2$  and by making use of the field equations

$$(\hat{p}_1 - m) v(p_1) = 0,$$

we bring  $M$  into the form

$$M = (p_1 k_2) [2(p_1 e_1) \hat{e}_2 - \hat{e}_2 \hat{k}_1 \hat{e}_1] + (p_1 k_1) [2(p_1 e_2) \hat{e}_1 - \hat{e}_1 \hat{k}_2 \hat{e}_2],$$

from which it also follows that

$$\bar{M} = (p_1 k_2) [2(p_1 e_1) \hat{e}_2 - \hat{e}_1 \hat{k}_1 \hat{e}_2] + (p_1 k_1) [2(p_1 e_2) \hat{e}_1 - \hat{e}_2 \hat{k}_2 \hat{e}_1].$$

In calculating  $\sum_{\sigma} |F|^2$  it is necessary to average over the spins of both the electron and the positron. Therefore, by taking into account formulas (7.20) and (7.21) we obtain

$$\sum_{\sigma} |F|^2 = \frac{e^4}{(8\pi^2 k^0)^2} \frac{1}{16 (p_1 k_1)^2 (p_1 k_2)^2} \sum_{\nu, \nu'} \bar{v}^{\nu, -}(p_2) M v^{\nu', -}(p_1) \bar{v}^{\nu', +}(p_1) \bar{M} v^{\nu, +}(p_2) =$$

$$= \frac{e^4}{(4\pi k^0)^4} \frac{1}{16 (p k_1)^2 (p_1 k_2)^2} \text{Tr } A, \quad (26.11)$$

where

$$A = (\hat{p}_2 - m) M (\hat{p}_1 + m) \bar{M}.$$

Making use of the law of conservation of four-momentum

$$p_2 = k_1 + k_2 - p_1,$$

we shall write  $A$  in the form of the sum

$$A = A_1 + A_2 + A_3,$$

where

$$A_1 = (\hat{k}_1 + \hat{k}_2) M \hat{p}_1 \bar{M},$$

$$A_2 = -(\hat{p}_1 + m) M (\hat{p}_1 + m) \bar{M},$$

$$\text{Tr } A_3 = 0.$$

For the calculation of  $\text{Tr } A_1$  and  $\text{Tr } A_2$  we shall utilize (6.16) and the following relations which follow directly from the corresponding definitions:

$$(k_1 e_1) = (k_2 e_2) = (k_1 e_2) = (k_2 e_1) = 0,$$

$$(k_1 k_2) = 2 (k^0)^2, \quad k_1^2 = k_2^2 = 0,$$

$$(k_1 p_1) = (k^0)^2 - k_1 p_1 = k^0 (k^0 - p \cos \theta),$$

$$(k_2 p_1) = (k^0)^2 - k_2 p_1 = k^0 (k^0 + p \cos \theta),$$

where  $\theta$  is the angle between the vectors  $k_1$  and  $p_1$ .

The general method of calculating traces consists, as in the preceding case, of successive displacements of identical matrix four-vectors towards each other. Omitting the elementary and tedious calculations, we merely state the results:

$$\text{Tr } A_1 = 32 (k^0)^2 (p k_1) (p k_2) \{2 (e_1 e_2) (p e_1) (p e_2) + k_0^2 + (p e_1)^2 + (p e_2)^2\},$$

$$\text{Tr } A_2 = -32 (k^0)^2 (p k_1) (p k_2) \{2 (e_1 e_2) (p e_1) (p e_2) + (p e_1)^2 + (p e_2)^2\} - 32 \{2 (k^0)^2 (p e_1) (p e_2) + (e_1 e_2) (p k_1) (p k_2)\}^2,$$

$$\text{Tr } A = 32 (p k_1)^2 (p k_2)^2 \left\{ \frac{(k^0)^4}{(p k_1) (p k_2)} - \frac{4 (k^0)^4 (p e_1)^2 (p e_2)^2}{(p k_1)^2 (p k_2)^2} - \frac{4 (k^0)^2 (e_1 e_2) (p e_1) (p e_2)}{(p k_1) (p k_2)} - (e_1 e_2)^2 \right\}.$$

By summing over the polarization directions of the photons and by taking into account the relations

$$\begin{aligned} \sum_{e_1} (pe_1)^2 &= \sum_{e_2} (pe_2)^2 = p^2 \sin^2 \theta, & \sum_{e_1, e_2} (pe_1)^2 (pe_2)^2 &= p^4 \sin^4 \theta, \\ \sum_{e_1, e_2} (e_1 e_2)^2 &= 2, & \sum_{e_1, e_2} (e_1 e_2) (pe_1) (pe_2) &= -p^2 \sin^2 \theta, \end{aligned}$$

we obtain:

$$\text{Tr } A = 64 (pk_1)^2 (pk_2)^2 \left[ \frac{(k^0)^2 + p^2 + p^2 \sin^2 \theta}{(k^0)^2 - p^2 \cos^2 \theta} - \frac{2p^4 \sin^4 \theta}{((k^0)^2 - p^2 \cos^2 \theta)^2} \right].$$

By substituting this expression into (26.11) and (26.10) we obtain the well-known formula for the differential effective cross section for the annihilation of an electron-positron pair [Heitler (1954)] :

$$d\sigma = \frac{e^4}{4(4\pi)^2 k^0 p} \left\{ \frac{(k^0)^2 + p^2 + p^2 \sin^2 \theta}{(k^0)^2 - p^2 \cos^2 \theta} - \frac{2p^4 \sin^4 \theta}{((k^0)^2 - p^2 \cos^2 \theta)^2} \right\} d\Omega. \quad (26.12)$$

**26.3. Bremsstrahlung.** We now discuss the bremsstrahlung of an electron in the field of a nucleus. The process consists of the fact that the electron in passing close to a nucleus emits a photon under the influence of the electric field of the nucleus. In this process, the nucleus takes up a part of the momentum required for the conservation of energy. By Coulomb's law, the potential of a nucleus of charge  $Ze$  is

$$a_0(x) = \frac{Ze}{4\pi |x|},$$

and therefore

$$\varphi_0(q) = \frac{Ze}{(2\pi)^{3/2} |q|^2}. \quad (26.13)$$

In the lowest-order nonvanishing approximation, the bremsstrahlung is described by the two second-order diagrams shown in Fig. 16, where  $p_0$  is the initial four-momentum of the electron,  $q$  is the momentum given up to the nucleus,  $p$  is the final four-momentum of the electron, and  $k$  is the momentum of the emitted photon.

In accordance with the basic formula (25.29), the cross section for this process is

$$d\sigma = \frac{(2\pi)^2 EE_0 p}{p_0} \sum_{\sigma} |F(p', p)|^2 d\Omega_p d\Omega_k (k^0)^2 dk^0 \quad (26.14)$$

where

$$p_0 = |p_0| \quad \text{and} \quad p = |p|.$$

To evaluate the matrix elements, we introduce a fictitious "polarization vector" of the time-like recoil pseudophoton  $e_q$ , which has only a time component, i.e.,

$$e_q^\mu = \delta_{\mu 0}.$$

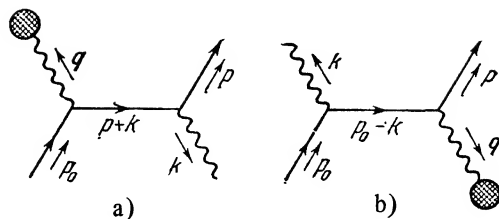


Fig. 16.

By applying the usual methods we then obtain

$$F(p', p) = \frac{e^2 \varphi(q)}{i(2\pi)^2} \bar{v}^{v'} + (p) \left\{ \frac{\hat{e}}{\sqrt{2k^0}} \frac{\hat{p} + \hat{k} + m}{2(pk)} \hat{e}_q - \hat{e}_q \frac{\hat{p}_0 - \hat{k} + m}{2(p_0 k)} \frac{\hat{e}}{\sqrt{2k^0}} \right\} v^{v'} - (p_0)$$

or

$$F(p', p) = \frac{e^2 \varphi(q)}{2i(2\pi)^2 \sqrt{2k^0}} \bar{v}^{v'} + (p) M v^{v'} - (p_0), \quad (26.15)$$

where after the usual simplifications  $M$  may be brought into the form

$$M = \frac{a}{(pk)} + \frac{b}{(p_0 k)}$$

$$(a = \hat{e} \hat{k} \hat{e}_q + 2(e\rho) \hat{e}_q, \quad b = \hat{e}_q \hat{k} \hat{e} - 2(e\rho_0) \hat{e}_q).$$

On squaring (26.15), summing over the spin  $v$  and the polarization  $e$ , and averaging over the spin  $v'$ , we obtain

$$\sum_{\sigma} |F|^2 = \frac{Z^2 e^6}{64(2\pi)^7 k^0 E E_0} \frac{1}{(q^2)^2} \text{Tr } \tilde{A}, \quad (26.16)$$

where

$$A = (\hat{p} + m) M (\hat{p}_0 + m) \bar{M},$$

and where the tilde over  $A$  denotes summation over the polarization  $e$ .

We shall carry out the calculation of  $\text{Tr } \tilde{A}$  in three stages. We write  $A$  as a sum of three terms:

$$A = A_1 + A_2 + A_3, \quad A_1 = \frac{1}{(pk)^2} \{(\hat{p} + m) a (\hat{p}_0 + m) \bar{a}\},$$

$$A_2 = \frac{1}{(p_0 k)^2} \{(\hat{p} + m) b (\hat{p}_0 + m) \bar{b}\}, \quad A_3 = \frac{1}{(pk)(p_0 k)} \{(\hat{p} + m) a (\hat{p}_0 + m) \bar{b} + (\hat{p} + m) b (\hat{p}_0 + m) \bar{a}\}.$$

We note that  $A_2$  may be obtained from  $A_1$  by carrying out the replacements

$$\begin{aligned} E &\rightarrow -E_0, & p_0 &\rightarrow -p, \\ E_0 &\rightarrow -E, & p &\rightarrow -p_0. \end{aligned}$$

Therefore, by computing the traces

$$\begin{aligned} \text{Tr } A_1 &= \frac{8}{(pk)^2} \{ (pk) (2E_0 k^0 - (p_0 k)) + 2(ep)^2 (2EE_0 - (pp_0) + m^2) + 2(ep)^2 (2E_0 k^0 - (p_0 k)) + \\ &\quad + 2(ep)(ep_0)(pk) \}, \\ \text{Tr } A_3 &= \frac{16(ep)(ep_0)}{(p_0 k)(pk)} \{ (p_0 k) - (pk) - 2m^2 - 4EE_0 + 2(pp_0) \} + \\ &\quad + \frac{16}{(p_0 k)(pk)} \{ k^0 [k^0 (pp_0) - E(p, k) - E_0(pk)] + (p_0 k)(pk) - m^2(k^0)^2 + (ep)^2(p_0 k) - (ep_0)^2(pk) \} \end{aligned}$$

and by summing over the photon polarization  $e$  and taking into account the relations

$$\left. \begin{aligned} \sum_e (ep)^2 &= p^2 \sin^2 \theta, \\ \sum_e (ep_0)^2 &= p_0^2 \sin^2 \theta_0, \\ \sum_e (ep_0)(ep) &= pp_0 \sin \theta \sin \theta_0 \cos \varphi, \end{aligned} \right\} \quad (26.17)$$

where  $\theta$  is the angle between  $k$  and  $p$ ,  $\theta_0$  is the angle between  $k$  and  $p_0$ ,  $\varphi$  is the angle between the planes  $(k, p)$  and  $(k, p_0)$ , we obtain

$$\begin{aligned} \text{Tr } \tilde{A} &= \frac{8p^2 \sin^2 \theta}{(pk)^2} (4E^2 - q^2) + \frac{8p_0^2 \sin^2 \theta_0}{(p_0 k)^2} (4E_0^2 - q^2) - \\ &\quad - \frac{16pp_0 \sin \theta \sin \theta_0 \cos \varphi}{(pk)(p_0 k)} [4EE_0 - q^2 + 2(k^0)^2] + 16(k^0)^2 \frac{p_0^2 \sin^2 \theta_0 + p^2 \sin^2 \theta}{(pk)(p_0 k)}. \end{aligned}$$

Substituting (26.16) into (26.14), we have

$$d\sigma = \frac{Z^2 e^6}{64 (2\pi)^3} \frac{p}{p_0} \frac{k^0 dk^0 d\Omega_k d\Omega_p}{q^2 q^2} \text{Tr } \tilde{A}.$$

By setting

$$d\Omega_k d\Omega_p = \sin \theta d\theta \sin \theta_0 d\theta_0 d\varphi d\varphi_0,$$

we obtain after integrating over  $d\varphi_0$  the formula for the differential cross section in the form

$$d\sigma = \frac{Z^2}{(2\pi)^2} \left( \frac{e^2}{4\pi} \right)^3 \frac{\rho}{\rho_0} \frac{k_0 dk_0 \sin \theta d\theta \sin \theta_0 d\theta_0 d\varphi}{q^2 q^2} \left\{ \frac{\rho^2 \sin^2 \theta}{(\rho k)^2} (4E^2 - q^2) + \frac{\rho_0^2 \sin^2 \theta_0}{(\rho_0 k)^2} (4E_0^2 - p^2) - \right. \\ \left. - \frac{2\rho\rho_0 \sin \theta \sin \theta_0 \cos \varphi}{(\rho k)(\rho_0 k)} (4EE_0 - q^2 + 2k_0^2) + \frac{2(k^0)^2 (\rho_0^2 \sin^2 \theta_0 + \rho^2 \sin^2 \theta)}{(\rho k)(\rho_0 k)} \right\}. \quad (26.18)$$

Since

$$\begin{aligned} (\rho k) &= k^0 (\rho_0 - |\rho| \cos \theta), \\ (\rho_0 k) &= k^0 (\rho_0 - |\rho| \cos \theta), \end{aligned}$$

the expression given by (26.18) is identical with the well-known formula [Heitler (1954)].





# Chapter 5

## REMOVAL OF DIVERGENCES FROM THE S-MATRIX

### §27. On the Divergences of the *S*-Matrix in Electrodynamics (Second Order)

Using the examples of Compton scattering, positron annihilation, and bremsstrahlung we have acquainted ourselves in detail with the type of matrix elements that correspond to one of the second-order diagrams.

The integrations of the corresponding matrix elements were carried out in a trivial fashion with the aid of the vertex  $\delta$ -functions (24.8) and the main labor consisted in the evaluation of the traces. When the diagram contains internal lines corresponding to virtual particles, the number of vertex  $\delta$ -functions is, in general, insufficient for contracting all the integrals. The necessary condition for this is the presence in the diagram of closed loops. The function  $F(p', p)$  given by (24.35) is then written in the form of a multiple integral over the four-momenta of the virtual particles. It then turns out that such integrals may diverge for large values of the virtual momenta. For example, matrix elements for Compton scattering and for pair annihilation diverge for higher orders in  $e^2$  ( $e^4$  and higher). Matrix elements of the second order, which correspond to the diagrams shown in Fig. 17, also diverge.

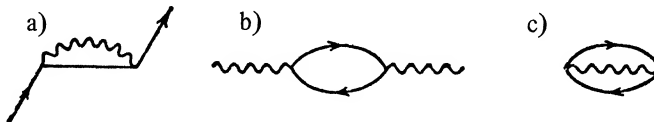


Fig. 17. Second-order divergent diagrams in electrodynamics.

27.1. *Divergent Diagram with Two External Electron Lines*  $\Sigma$ . We write the term in the scattering matrix which corresponds to the diagram of Fig. 17a

$$-e^2 : \bar{\psi}(x) \gamma_n S^c(x-y) \gamma_n D_0^c(x-y) \psi(y) :, \quad (27.1)$$

in the form

$$-i : \bar{\psi}(x) \Sigma(x-y) \psi(y) :, \quad (27.2)$$

where

$$\Sigma(x-y) = -ie^2 \gamma_n S^c(x-y) \gamma_n D_0^c(x-y). \quad (27.3)$$

The factor  $-i$  is separated out for convenience in subsequent analysis. With this normalization, the operator  $\Sigma$  turns out to be real in the momentum representation for  $k^2 < 0$  (see §25.2) and, when added to the electron mass  $m$ , forms the mass operator in the Dyson equations for Green's functions (see §38.4).

Simple calculation shows that the Fourier transform  $\Sigma(p)$  of the operator  $\Sigma(x-y)$

$$\Sigma(x-y) = \frac{1}{(2\pi)^4} \int e^{-ip(x-y)} \Sigma(p) dp \quad (27.4)$$

appears in the matrix element of the  $S$ -matrix in the following way:

$$F(p', p) = (2\pi)^4 \bar{v}(p') \Sigma(p) v(p) \quad (p' = p), \quad (27.5)$$

where  $\bar{v}$  and  $v$  are spinor amplitudes corresponding to the field functions  $\bar{\psi}$  and  $\psi$ . The convergence of the matrix element is therefore entirely determined by the convergence of the function  $\Sigma(p)$ . Recalling that in the momentum representation the causal functions have the form (compare (24.3) and (24.4))

$$D_0^c(k) = -\frac{1}{k^2 + i\varepsilon}, \quad (27.6)$$

$$S^c(p) = \frac{m + \hat{p}}{m^2 - p^2 - i\varepsilon}, \quad (27.7)$$

we find that

$$\Sigma(p) = \frac{e^2}{(2\pi)^4 i} \int dk D_0^c(k) \gamma_n S^c(p-k) \gamma_n = \frac{e^2}{(2\pi)^4 i} \int \frac{dk}{k^2 + i\varepsilon} \gamma_n \frac{\hat{p} - \hat{k} + m}{(p-k)^2 - m^2 + i\varepsilon} \gamma_n. \quad (27.8)$$

For large  $|k|$  the integrand falls off like  $|k|^{-3}$  and, therefore, the integral (27.8) diverges,

generally speaking, linearly. We thus see that the purely formal rules for dealing with products of causal functions, which we adopted earlier, lead to a *meaningless result* in this case.

This is essentially a manifestation of the fact that we did not define the product of singular functions as an integrable singular function. In order to solve the problem of determining the coefficients of the chronological product

$$T(\mathcal{L}(x_1)\mathcal{L}(x_2))$$

as integrable improper functions, we use the method of transition to the limit similar to that used in §18. In order to do this, we first consider an auxiliary fictitious case in which the field operator functions satisfy commutation relations in which the causal  $\Delta^c$ -functions are replaced by  $\text{reg}(\Delta^c)$ .

The functions

$$\text{reg } D_0^c(k) = -\frac{1}{k^2 + i\varepsilon} - \sum_M c_M \frac{1}{k^2 + i\varepsilon - M^2}, \quad (27.9)$$

$$\text{reg } S^c(p) = (m + \hat{p}) \left\{ \frac{1}{m^2 - p^2 - i\varepsilon} + \sum_M \frac{c_M}{M^2 - p^2 - i\varepsilon} \right\} \quad (27.10)$$

must be substituted into the expression for  $\Sigma(p)$  in place of (27.6) and (27.7).

It then turns out that a single auxiliary mass  $M$  is sufficient to regularize  $\Sigma(p)$ . By setting

$$c_M = -1,$$

we obtain

$$\begin{aligned} \text{reg } D_0^c(k) &= -\frac{1}{k^2 + i\varepsilon} + \frac{1}{k^2 - M^2 + i\varepsilon} = \frac{M^2}{(k^2 + i\varepsilon)(k^2 - M^2 + i\varepsilon)}, \\ \text{reg } S^c(p) &= \frac{m + \hat{p}}{m^2 - p^2 - i\varepsilon} - \frac{m + \hat{p}}{(M^2 - p^2 - i\varepsilon)} = \frac{(m + \hat{p})(M^2 - m^2)}{(m^2 - p^2 - i\varepsilon)(M^2 - p^2 - i\varepsilon)}. \end{aligned}$$

It is readily seen that  $\text{reg } D_0^c(k)$  falls off as  $|k|^{-4}$ , while  $\text{reg } S^c(p)$  falls off as  $|k|^{-3}$ , so that the integral in the expression for  $\text{reg } \Sigma(p)$

$$\text{reg } \Sigma(p) = \frac{e^2}{(2\pi)^4 i} \int dk \text{reg } D_0^c(k) \gamma^n \text{reg } S^c(p-k) \gamma_n \quad (27.11)$$

behaves like

$$\int \frac{d^4 k}{|k|^7} \sim \frac{1}{|k|^3}$$

for large values of  $|k|$ .

We now investigate the question of the behavior of  $\text{reg } \Sigma(p)$  as  $M \rightarrow \infty$  in the course of the process of removing the regularization. It is convenient to conduct this investigation by explicitly evaluating  $\text{reg } \Sigma(p)$ .

We shall utilize the following auxiliary device for the evaluation of the integral (27.11). We shall write the factors in the denominator of (27.11) in the form

$$\frac{1}{k^2 - m^2 + i\varepsilon} = \frac{1}{i} \int_0^\infty e^{i\alpha(k^2 - m^2 + i\varepsilon)} d\alpha. \quad (27.12)$$

Equation (27.12) will be referred to as the transformation to the *alpha-representation*. After this transformation, integration with respect to  $k$  in (27.11) reduces to Gaussian integrals.

The basic four-dimensional Gaussian integral has the form

$$\int e^{i(ak^2 + 2bk)} dk = \frac{\pi^2}{ia^2} e^{-\frac{ib^2}{a}} \quad (a > 0). \quad (27.13)$$

Formula (27.13) may be obtained by considering the typical integral

$$\int_{-\infty}^{\infty} e^{i(at^2 + 2bt)} dt \quad (a > 0),$$

which we shall always regard as the limit of the expression

$$\int_{-\infty}^{\infty} e^{i(at^2 + 2bt) - \eta t^2} dt \quad \text{for} \quad \eta \rightarrow +0.$$

In order to evaluate it, we introduce the change of variable

$$t = \frac{1+i}{\sqrt{2}} x - \frac{b}{a}, \quad i(at^2 + 2bt) = -ax^2 - \frac{ib^2}{a},$$

which consists of a rotation of the coordinate system through an angle  $-\pi/4$  and a shift of the origin. Then, in the limit as  $\eta \rightarrow 0$  we obtain:

$$\int_{-\infty}^{\infty} e^{i(at^2 + 2bt)} dt = \frac{1+i}{\sqrt{2}} e^{-\frac{ib^2}{a}} \int_{-\infty}^{\infty} e^{-ax^2} dx = \frac{1+i}{\sqrt{2}} \sqrt{\frac{\pi}{a}} e^{-\frac{ib^2}{a}} \quad (a > 0). \quad (27.14)$$

The value of this integral for  $a < 0$  may be obtained by taking the complex conjugate. We therefore have

$$\int_{-\infty}^{\infty} e^{i(at^2 + 2bt)} dt = \frac{1-i}{\sqrt{2}} \sqrt{\frac{\pi}{|a|}} e^{-\frac{ib^2}{a}} \quad (a < 0). \quad (27.15)$$

We can now begin the evaluation of the required four-dimensional integrals of the type

$$\int e^{i(ak^2 + 2bk)} dk,$$

where

$$k^2 = (k^0)^2 - \mathbf{k}^2, \quad bk = b^0 k^0 - \mathbf{b} \mathbf{k}, \quad dk = dk^0 d\mathbf{k}.$$

With the aid of formulas (27.14), (27.15) we arrive directly at (27.13):

$$\begin{aligned} \int e^{i(ak^2 + 2bk)} dk &\equiv \int_{-\infty}^{\infty} e^{i(ak^2 + 2b^0 k^0)} dk^0 \prod_{1 \leq \alpha \leq 3} \int_{-\infty}^{\infty} e^{-i(ak^{\alpha 2} + 2b^{\alpha} k^{\alpha})} dk^{\alpha} = \\ &= \frac{1+i}{\sqrt{2}} \left( \frac{1-i}{\sqrt{2}} \right)^3 \frac{\pi^2}{a^2} e^{-\frac{i}{a}(b^0 - b^2)} = \frac{\pi^2}{ia^2} e^{-\frac{ib^2}{a}} \quad (a > 0). \end{aligned}$$

The other integrals of this type, which we shall require, may be obtained from (27.13) by repeated differentiation with respect to  $b$ . Thus, for example,

$$\int e^{i(ak^2 + 2bk)} k^n dk = \frac{ib^n}{a} \left( \frac{\pi}{a} \right)^2 e^{-\frac{ib^2}{a}} \quad (a > 0) \quad (27.16)$$

$$\int e^{i(ak^2 + 2bk)} k^m k^n dk = \frac{ag^{mn} - 2ib^n b^m}{2a^2} \left( \frac{\pi}{a} \right)^2 e^{-\frac{ib^2}{a}} \quad (a > 0), \quad (27.17)$$

$$\int e^{i(ak^2 + 2bk)} k^2 dk = \frac{2a - ib^2}{a^2} \left( \frac{\pi}{a} \right)^2 e^{-\frac{ib^2}{a}} \quad (a > 0). \quad (27.18)$$

We now return to the evaluation of the integral given by (27.11). Substituting into it the integral representations of singular functions of the form of

$$\text{reg } D_0^e(k) = i \int_0^{\infty} e^{i\alpha k^2 - \varepsilon \alpha} (1 - e^{-i\alpha M^2}) d\alpha, \quad (27.19)$$

$$\text{reg } S^e(p) = i \left( m + \hat{p} \right) \int_0^{\infty} e^{i\alpha p^2 - \varepsilon \alpha} (e^{-i\alpha m^2} - e^{-i\alpha M^2}) d\alpha, \quad (27.20)$$

performing the integration over  $dk$  with the aid of (27.13) and (27.16), we obtain, keeping in mind that

$$\gamma^{\mu} \hat{p} \gamma_{\mu} = -2\hat{p} \quad \text{and} \quad \gamma^{\mu} \gamma_{\mu} = 4,$$

the following expression for  $\text{reg } \Sigma(p)$ :

$$\text{reg } \Sigma(p) = \frac{e^2}{8\pi^2} \int_0^{\infty} d\alpha \int_0^{\infty} d\beta \frac{e^{-\varepsilon(\alpha+\beta)}}{(\alpha+\beta)^3} e^{\frac{i\alpha\beta p^2}{\alpha+\beta}} \left( 2m - \hat{p} \frac{\alpha}{\alpha+\beta} \right) (1 - e^{-i\alpha M^2}) (e^{-i\beta m^2} - e^{-i\beta M^2}).$$

By going over to the new variables

$$\alpha = \xi \lambda, \quad \beta = (1 - \xi) \lambda,$$

and taking into account the Jacobian

$$\left| \frac{(\partial \alpha, \partial \beta)}{(\partial \xi, \partial \lambda)} \right| = \lambda$$

we obtain:

$$\text{reg } \Sigma(p) = \frac{e^2}{8\pi^2} \int_0^1 (2m - \hat{p}\xi) J_\varepsilon(\xi, M) d\xi, \quad (27.21)$$

where

$$J_\varepsilon(\xi, M) = \int_0^\infty \frac{d\lambda}{\lambda} e^{-\lambda\varepsilon + i\lambda\xi(1-\xi)p^2} (1 - e^{-i\xi\lambda M^2}) (e^{-i(1-\xi)\lambda m^2} - e^{-i(1-\xi)\lambda M^2}). \quad (27.22)$$

This integral can be evaluated with the aid of differentiation with respect to  $p^2$  which is taken as a parameter. Evaluation of integrals such as (27.22) can be performed with the aid of the formula

$$\int_0^\infty \frac{d\lambda}{\lambda} (e^{iA\lambda} - e^{iB\lambda}) e^{-\varepsilon\lambda} = \ln \frac{B + i\varepsilon}{A + i\varepsilon}. \quad (27.23)$$

Since the convergence of the integral at the upper limit is ensured by factor  $e^{-\lambda\varepsilon}$ , a degree of caution must be exercised when  $\varepsilon \rightarrow 0$ .

The presence of the infinitesimal quantity  $i\varepsilon$  in the logarithm  $\ln(a + i\varepsilon)$  ensures that it is complex for  $a < 0$ :

$$\ln(a + i\varepsilon) = \ln|a| + i\pi\theta(-a).$$

Henceforth we shall not explicitly show the terms  $i\varepsilon$  in the logarithm and will assume that (in accordance with the definition of the causal  $D^c$ -function) the squares of the masses  $M^2$ ,  $m^2$ , and so on, contain infinitesimal negative imaginary additions.

In the limit as  $\varepsilon \rightarrow 0$ , we have from (27.22)

$$J_0(\xi, M) = \ln \left[ \frac{M^2 - \xi p^2}{m^2} \right] + \ln \left[ \frac{m^2}{m^2 - \xi p^2} \frac{\xi M^2 + (1 - \xi) m^2 - \xi (1 - \xi) p^2}{M^2 - \xi (1 - \xi) p^2} \right]. \quad (27.24)$$

27.2. *Segregation of the Divergent Part of  $\Sigma$ .* Substituting the value obtained above for  $J_0(\xi, M)$  into (27.21), we obtain

$$\begin{aligned} \text{reg } \Sigma(p) = & \frac{e^2}{8\pi^2} \int_0^1 d\xi (2m - \hat{p}\xi) \ln \left[ \xi \frac{M^2 - \xi p^2}{m^2} \right] + \\ & + \frac{e^2}{8\pi^2} \int_0^1 d\xi (2m - \hat{p}\xi) \ln \left[ \frac{m^2}{m^2 - \xi p^2} \frac{\xi M^2 + (1 - \xi) m^2 - \xi (1 - \xi) p^2}{\xi [M^2 - \xi^2 (1 - \xi) p^2]} \right]; \end{aligned} \quad (27.25)$$

the second term of which will converge as  $M \rightarrow \infty$  to a definite limit equal to

$$\Sigma'(p) = \frac{e^2}{8\pi^2} \int_0^1 d\xi (2m - \hat{p}\xi) \ln \left[ \frac{m^2}{m^2 - \xi p^2} \right]. \quad (27.26)$$

The decomposition in (27.25) is chosen in such a way that

$$\Sigma'(0) = 0 \quad \text{and} \quad \left. \frac{\partial \Sigma'(p)}{\partial p^2} \right|_{p=0} = 0. \quad (27.27)$$

The first term in (27.25) is evidently logarithmically divergent as  $M \rightarrow \infty$ .

By going over to the configuration representation, we obtain for sufficiently large  $M$

$$\text{reg } \Sigma(x) = \frac{e^2}{(4\pi)^2} \left\{ \ln \frac{M^2}{m^2} \left( 4m - i \frac{\hat{\partial}}{\partial x} \right) + \left( \frac{i}{2} \frac{\hat{\partial}}{\partial x} - 4m \right) \right\} \delta(x) + \Sigma'_M(x), \quad (27.28)$$

with the Fourier transform of the function  $\Sigma'_M(\sigma)$  being given by the second term of expression (27.25). By repeating the argument of §18, we see that, as  $M \rightarrow \infty$ ,  $\Sigma'_M(x)$  converges in the improper sense to the integrable function

$$\lim \Sigma'_M(x) = \Sigma'(x),$$

whose Fourier transform is given by expression (27.26). Because of the factor  $\ln (M/m)^2$ , the function  $\text{reg } \Sigma(x)$  as a whole will not converge even in the improper sense. Since the first term of (27.28) vanishes for  $x \neq 0$ , we may write

$$\lim_{M \rightarrow \infty} \text{reg } \Sigma(x) = \Sigma'(x) \quad \text{everywhere for } x \neq 0.$$

We thus have carried out the segregation of the divergent part from the singular function  $\Sigma(x)$ . However, we emphasize that this operation is not unique. Indeed, we could have written (27.28), for example, in the form

$$\text{reg } \Sigma(x) = \frac{e^2}{16\pi^2} \left\{ \ln \left( \frac{M}{\mu} \right)^2 \left( 4m - i \frac{\hat{\partial}}{\partial x} \right) + \left( \frac{i}{2} \frac{\hat{\partial}}{\partial x} - 2m \right) \right\} \delta(x) + \Sigma''_M(x),$$

where



$$\Sigma_M^*(x) = \frac{e^2}{16\pi^2} \ln\left(\frac{\mu}{m}\right)^2 \left(4m - i \frac{\partial}{\partial x}\right) \delta(x) + \Sigma_M'(x),$$

and  $\mu$  is an arbitrary finite mass. We would have thus obtained for the regular part  $\Sigma_M''$  an expression differing from  $\Sigma_M'$  by terms proportional to  $\delta(x)$  and to its first derivatives.

We can obtain a change of a similar nature in the finite part  $\Sigma_M'$  by going over to any other method of regularization. For example, if  $\Sigma(p)$  is regularized by introducing into the integral (27.9) the *Feynman cutoff factor* [Feynman (1948b, 1949b)]

$$\frac{M^2}{M^2 - k^2},$$

which, in our case, is equivalent to regularizing only the photon  $D_0^C$ -function, then the result may be written in the form

$$\text{reg}_F \Sigma(x) = \frac{e^2}{16\pi^2} \left\{ \ln\left(\frac{M}{m}\right)^2 \left(4m - i \frac{\partial}{\partial x}\right) + \left(\frac{i}{2} \frac{\partial}{\partial x} - 4m\right) \right\} \delta(x) + \Sigma_F'(x),$$

where the regular function  $\Sigma_F'$  differs in the momentum representation from  $\Sigma'$  by the quantity

$$\frac{e^2}{16\pi^2} \int_0^1 d\xi (\hat{p}\xi - 2m) \ln(1 - \xi) = \frac{e^2}{16\pi^2} \left(2m - \frac{3}{4} \hat{p}\right).$$

In this way it may be seen that, when the regularization is removed, it is not  $\text{reg } \Sigma(p)$  which converges to a definite limit but, for example, the expression obtained by subtracting from it the first two terms of Maclaurin's series:

$$\text{reg } \Sigma(p) - \text{reg } \Sigma(0) - \frac{\partial \text{reg } \Sigma(p)}{\partial p^n} \Big|_{p=0} \cdot p^n. \quad (27.29)$$

This expression converges to a limit which does not depend on the method of regularization, since the addition to  $\text{reg } \Sigma(p)$  of any arbitrary polynomial of the first degree in  $p$  does not alter the "residual" term (27.29).

We now obtain the general expression for  $\Sigma'(p)$  by adding an arbitrary polynomial of the first degree in  $p$  to (27.29). From considerations of relativistic covariance, this polynomial must have the form

$$c_1 (\hat{p} - m) + c_2 m,$$

and, consequently, we obtain the general expression for  $\Sigma'(p)$  in the form

$$\Sigma'(p) = \frac{e^2}{8\pi^2} \left\{ \int_0^1 d\xi (2m - \hat{p}\xi) \ln \left[ \frac{m^2}{m^2 - \xi p^2} \right] + c_1 (\hat{p} - m) + c_2 m \right\}. \quad (27.30)$$

Accordingly, the expression for  $\Sigma'$  in the  $x$ -representation is defined to within the term

$$\frac{e^2}{8\pi^2} \left\{ c_1 \left( i \frac{\partial}{\partial x} - m \right) + c_2 m \right\} \delta(x),$$

which vanishes for  $x \neq 0$ , i.e., as was to be expected from general considerations, the arbitrariness in the given term of the  $T$ -product manifests itself only in an infinitesimal neighborhood of the point  $x = 0$ .

**27.3. Divergent Diagram with Two External Photon Lines II.** We now turn to the second divergent term in  $S_2(x, y)$ . The term of the scattering matrix which corresponds to the diagram of Fig. 17b may be represented by

$$-e^2 : \text{Tr} (\hat{A}(x) S^c(x-y) \hat{A}(y) S^c(y-x)) : = -i : A_m(x) \Pi^{mn}(x-y) A_n(y) :,$$

where

$$\Pi^{mn}(x-y) = -ie^2 \text{Tr} \gamma^m S^c(x-y) \gamma^n S^c(y-x) \quad (27.31)$$

is the so-called *polarization operator*.

In going over to the momentum representation

$$\Pi^{mn}(x-y) = \frac{1}{(2\pi)^4} \int e^{ik(x-y)} \Pi^{mn}(k) dk, \quad (27.32)$$

we find that the integral

$$\Pi^{mn}(k) = \frac{e^2}{(2\pi)^4 i} \int dp \text{Tr} \left( \gamma^m \frac{\hat{p} + m}{p^2 - m^2 + i\varepsilon} \gamma^n \frac{\hat{p} - \hat{k} + m}{(p-k)^2 - m^2 + i\varepsilon} \right) \quad (27.33)$$

diverges quadratically for large momenta.

For the explicit evaluation of  $\Pi^{mn}(k)$  we shall use the same methods that were applied to the evaluation of  $\Sigma(p)$ . Utilizing the regularized  $S^c$ -functions (27.20), we carry out the integration over  $dp$  with the aid of (27.13), (27.16), and (27.17). Also, by evaluating the trace in accordance with formula (6.16) we obtain

$$\begin{aligned} \text{reg } \Pi^{mn}(k) = & \frac{ie^2}{4\pi^2} \int_0^\infty d\alpha \int_0^\infty d\beta e^{-\varepsilon(\alpha+\beta) + \frac{i\alpha\beta k^2}{\alpha+\beta}} (e^{-i\alpha m^2} - e^{-i\alpha M^2}) (e^{-i\beta m^2} - e^{-i\beta M^2}) \times \\ & \times \frac{1}{(\alpha+\beta)^2} \left\{ \frac{i\alpha\beta}{(\alpha+\beta)^2} (2k^n k^m - k^2 g^{mn}) - g^{mn} \left( \frac{1}{\alpha+\beta} + im^2 \right) \right\}. \end{aligned}$$

By introducing the new variables

$$\alpha = \xi \lambda, \quad \beta = (1 - \xi) \lambda$$

we shall transform this expression into the form

$$\text{reg } \Pi^{mn}(k) = \frac{ie^2}{4\pi^2} \int_0^1 d\xi \int_0^\infty \frac{d\lambda}{\lambda} e^{-\varepsilon\lambda} (e^{-i\xi\lambda m^2} - e^{-i\xi\lambda M^2}) (e^{-l(1-\xi)\lambda m^2} - e^{-l(1-\xi)\lambda M^2}) \times \\ \times e^{i\lambda\xi(1-\xi)k^2} \left\{ i\xi(1-\xi)(2k^m k^n - g^{mn}k^2) - g^{mn} \left( \frac{1}{\lambda} + im^2 \right) \right\}.$$

In order to carry out the integration with respect to  $\lambda$  we shall write the term containing  $\lambda^2$  in the denominator in the form

$$\int_0^\infty \frac{d\lambda}{\lambda^2} e^{-\varepsilon\lambda} f(\lambda),$$

where

$$f(\lambda) = e^{i\lambda\xi(1-\xi)k^2} (e^{-i\xi\lambda m^2} - e^{-i\xi\lambda M^2}) (e^{-l(1-\xi)\lambda m^2} - e^{-l(1-\xi)\lambda M^2}),$$

and, after integrating by parts and taking into account the fact that  $f(\lambda)$  remains finite as  $\lambda \rightarrow \infty$  and vanishes like  $\lambda^2$  as  $\lambda \rightarrow 0$ , we find that

$$\int_0^\infty \frac{d\lambda}{\lambda^2} f(\lambda) e^{-\varepsilon\lambda} = \int_0^\infty \frac{d\lambda}{\lambda} \frac{\partial}{\partial \lambda} (e^{-\varepsilon\lambda} f(\lambda)) \xrightarrow{\varepsilon \rightarrow 0} \int_0^\infty \frac{d\lambda}{\lambda} e^{-\varepsilon\lambda} \frac{\partial f(\lambda)}{\partial \lambda}.$$

We therefore have

$$\text{reg } \Pi^{mn}(k) = \frac{ie^2}{4\pi^2} \int_0^1 d\xi F_\varepsilon(\xi, M),$$

where the integral

$$F_\varepsilon(\xi, M) = \int_0^\infty \frac{d\lambda}{\lambda} e^{-\varepsilon\lambda} \left\{ f(\lambda) [i\xi(1-\xi)(2k^m k^n - g^{mn}k^2) - im^2 g^{mn}] - g^{mn} \frac{\partial f}{\partial \lambda} \right\}$$

may be evaluated by differentiation with respect to the parameter  $\varepsilon$ . In this way we obtain

$$\lim_{\varepsilon \rightarrow 0} F_\varepsilon(\xi, M) = F_0(\xi, M) = [2i\xi(1-\xi)(k^m k^n - g^{mn}k^2) - im^2 g^{mn}] \times \\ \times \left\{ \ln \left[ \frac{(1-\xi)M^2 + \xi m^2 - \xi(1-\xi)k^2}{m^2 - \xi(1-\xi)k^2} \right] - \ln \left[ \frac{M^2 - \xi(1-\xi)k^2}{\xi M^2 + (1-\xi)m^2 - \xi(1-\xi)k^2} \right] \right\} + \\ + ig^{mn}\xi \left\{ m^2 \ln \left[ \frac{(1-\xi)M^2 + \xi m^2 - \xi(1-\xi)k^2}{m^2 - \xi(1-\xi)k^2} \right] - M^2 \ln \left[ \frac{M^2 - \xi(1-\xi)k^2}{\xi M^2 + (1-\xi)m^2 - \xi(1-\xi)k^2} \right] \right\} + \\ + ig^{mn}(1-\xi) \left\{ m^2 \ln \left[ \frac{\xi M^2 + (1-\xi)m^2 - \xi(1-\xi)k^2}{m^2 - \xi(1-\xi)k^2} \right] - \right. \\ \left. - M^2 \ln \left[ \frac{M^2 - \xi(1-\xi)k^2}{(1-\xi)M^2 + \xi m^2 - \xi(1-\xi)k^2} \right] \right\}.$$

In the limit as  $M \rightarrow \infty$ , this expression takes on the form

$$\begin{aligned} \lim_{M \rightarrow \infty} F_0(\xi, M) = & ig^{mn} (M^2 - m^2) [\xi \ln \xi + (1 - \xi) \ln (1 - \xi)] + \\ & + 2i\xi (1 - \xi) \ln \frac{M^2}{m^2} (k^n k^m - g^{mn} k^2) + 2i\xi (1 - \xi) (k^n k^m - g^{mn} k^2) \ln \left[ \frac{\xi (1 - \xi) m^2}{m^2 - \xi (1 - \xi) k^2} \right] + \\ & + ig^{mn} [m^2 - \xi (1 - \xi) k^2]. \end{aligned}$$

27.4. *Segregation of the Divergences from  $\Pi$  and Gauge Invariance.* By integrating over  $\xi$  we find that, in the limit of large  $M$ ,

$$\text{reg } \Pi^{mn}(k) = \frac{e^2}{8\pi^2} g^{mn} (M^2 - m^2) - \frac{e^2}{4\pi^2} \ln \frac{M^2}{m^2} \frac{k^n k^m - g^{mn} k^2}{3} + \Pi'^{mn}(k), \quad (27.34)$$

where

$$\Pi'^{mn}(k) = \frac{e^2}{2\pi^2} (k^n k^m - g^{mn} k^2) \int_0^1 d\xi \cdot \xi (1 - \xi) \ln \left[ \frac{m^2 - \xi (1 - \xi) k^2}{\xi (1 - \xi) m^2} \right] \quad (27.35)$$

is a regular function.

By going over to the configuration representation, we obtain for sufficiently large  $M$

$$\begin{aligned} \text{reg } \Pi^{mn}(x) = & \frac{e^2}{8\pi^2} g^{mn} (M^2 - m^2) \delta(x) + \\ & + \frac{e^2}{12\pi^2} \ln \frac{M^2}{m^2} \left( \frac{\partial}{\partial x_n} \frac{\partial}{\partial x_m} - g^{mn} \square \right) \delta(x) + \Pi_M'^{mn}(x): \quad (27.36) \end{aligned}$$

In the limit as  $M \rightarrow \infty$ , the term  $\Pi_M^{mn}(x)$  converges in the improper sense to the integrable function  $\Pi^{mn}(x)$ :

$$\lim_{M \rightarrow \infty} \Pi_M'^{mn}(x) = \Pi'^{mn}(x),$$

whose Fourier transform is given by (27.35). This completes the segregation of the singularities from  $\Pi^{mn}(x)$ . We note that, just as in the preceding case, the decomposition of  $\text{reg } \Pi^{mn}(x)$  into a singular and a finite part is not unique. Consequently, the finite part  $\Pi^{mn}(x)$  is *not unique*. An arbitrary expression which is a polynomial in the components of  $k^n$  of degree not higher than the second may be added to  $\Pi^{mn}(x)$ , since the singular part in this case is a polynomial of the second degree in  $k$ .

In order to complete our analysis of  $\Pi^{mn}$ , we shall also formulate the condition of *gauge invariance*, which must be satisfied by  $\Pi^{mn}$ . It is readily seen that the function  $\Pi^{mn}(k)$  appears in the matrix elements, similarly to the function  $\Sigma(p)$ , in the following combination with the potentials  $A_m(k)$ :

$$(2\pi)^4 : A_m(k) \Pi^{mn}(k) A_n(k) : . \quad (27.37)$$

As has been pointed out in §5, the potentials of the electromagnetic field  $A_m$  are introduced from the beginning in such a way that all physically observable quantities do not change their value under the gauge transformation

$$A_m(x) \rightarrow A'_m(x) = A_m(x) + \frac{\partial f(x)}{\partial x^m}$$

or, in the momentum representation,

$$A_m(k) \rightarrow A'_m(k) = A_m(k) + ik_m f(k).$$

Therefore, the requirement of invariance of the matrix elements of the operators such as (27.37)

$$: A'_m(k) \Pi^{mn}(k) A'_n(k) : = : A_m(k) \Pi^{mn}(k) A_n(k) :$$

leads us to the condition

$$k_m \Pi^{mn}(k) = 0, \quad (27.38)$$

from which it follows that the function  $\Pi^{mn}$  must have the form

$$\Pi^{mn}(k) = (k^m k^n - g^{mn} k^2) \pi(k^2). \quad (27.39)$$

Since the singular components in (27.34) and (27.35) will subsequently be compensated by the quasilocal operator  $\Lambda_2(x_1, x_2)$  (see §27.5), and do not contribute to  $S_2$ , the condition of gauge invariance can be imposed on the regular part  $\Pi'$  which, according to (27.35) satisfies condition (27.38). It is also clear that the expression

$$c_3 (k^n k^m - g^{mn} k^2)$$

can be added to (27.35), i.e., the decomposition (27.36) should be replaced by the following one:

$$\begin{aligned} \text{reg } \Pi^{mn}(x) = & \frac{e^2}{8\pi^2} g^{mn} (M^2 - m^2) \delta(x) + \left( \frac{e^2}{12\pi^2} \ln \frac{M^2}{m^2} + \frac{e^2}{2\pi^2} c_3 \right) \left( \frac{\partial}{\partial x_n} \frac{\partial}{\partial x_m} - g^{mn} \square \right) \delta(x) + \Pi'_{\text{inv}}{}^{mn}(x), \end{aligned}$$

where the finite part  $\Pi'_{\text{inv}}{}^{mn}(x)$  is invariant under gauge transformation and has in the momentum representation the form (24.39):

$$\Pi_{\text{inv}}{}^{mn}(k) = \frac{e^2}{2\pi^2} (k^m k^n - g^{mn} k^2) \left( c_3 + \int_0^1 d\xi \cdot \xi (1 - \xi) \ln \left[ \frac{m^2 - \xi (1 - \xi) k^2}{\xi (1 - \xi) m^2} \right] \right). \quad (27.40)$$

The terms in the  $S$ -matrix corresponding to the vacuum diagram of Fig. 17c can be investigated in a similar way. Without going into details, we note that the corresponding function  $R(x_1 - x_2)$  contains a singular part which, after regularization, diverges as  $M^4$  in the limit as  $M \rightarrow \infty$ . However, the contribution of the diagram of Fig. 17c does not contain any dependence on the momenta of the physical particles and is simply the number  $iR_2$ . It can be shown (by including higher-order diagrams) that this number appears in the  $S$ -matrix in the form of the phase factor  $e^{iR_2}$  and has no physical consequences.

27.5. *Construction of an Integrable  $S_2$ .* We thus arrive at the conclusion that the second-order term in the scattering matrix

$$S_2(x_1, x_2) = i^2 T(\mathcal{L}(x_1), \mathcal{L}(x_2))$$

in the regularized case ( $M < \infty$ ) under consideration may be written in the form

$$\begin{aligned} i^2 T(\mathcal{L}(x_1), \mathcal{L}(x_2)) = & i^2 T'_M[\mathcal{L}(x_1), \mathcal{L}(x_2)] - \\ & - i \left\{ a_2^M : \bar{\psi}(x_1) \left( i \frac{\partial}{\partial x_1} - m \right) \delta(x_1 - x_2) \psi(x_2) : - \delta m : \bar{\psi}(x_1) \delta(x_1 - x_2) \bar{\psi}(x_2) : + \right. \\ & + \delta \mu^2 : A^n(x_1) \delta(x_1 - x_2) A_n(x_2) : + a_3^M : A^n(x_1) \left( \frac{\partial}{\partial x_1^n} \frac{\partial}{\partial x_1^m} - \square_1 g^{mn} \right) \delta(x_1 - x_2) A^m(x_2) : + \\ & \left. + R_{\text{sing}} \left( M, \frac{\partial}{\partial x} \right) \delta(x_1 - x_2) + \text{same terms with } x_1 \text{ replaced by } x_2 \right\}. \end{aligned} \quad (27.41)$$

Moreover, the constants  $a_2^M, a_3^M, \delta m, \delta \mu$  diverge for large  $M$ :

$$\left. \begin{aligned} a_2 & \sim \frac{e^2}{16\pi^2} \left[ \ln \frac{M^2}{m^2} - \frac{1}{2} - c_1 \right], & \delta m & \sim \frac{e^2}{16\pi^2} \left[ 3m \ln \frac{M^2}{m^2} - \frac{7m}{2} + c_2 \right], \\ a_3 & \sim \frac{e^2}{2\pi^2} \left[ \frac{1}{6} \ln \frac{M^2}{m^2} - c_3 \right], & \delta \mu^2 & \sim \frac{e^2}{8\pi^2} [M^2 - m^2], \end{aligned} \right\} \quad (27.42)$$

while in the construction of the coefficient functions of the expression  $T'_M$  the rules of correspondence must be altered: for the ordinary lines in Feynman diagrams the functions  $\text{reg } \Delta^c$  must be used in place of  $\Delta^c$ , while for closed diagrams the corresponding finite functions  $\Sigma'_M, \Pi'^{mn}_{M, \text{inv}}$  and  $R'_M$  must be used. It is then evident that all the coefficient functions of the expression  $T'_M(\mathcal{L}(x_1), \mathcal{L}(x_2))$  converge to a finite limit when the regularization is removed.

We also see that all the divergences in  $S_2(x_1, x_2)$  come from terms proportional to  $\delta(x_1 - x_2)$  and to its derivatives which differ from zero only in the infinitesimal neighborhood of the point  $x_1 = x_2$ . As was already mentioned in §22, it is just in the neighborhood of the point  $x_1 = x_2$  that the  $T$ -product  $T(\mathcal{L}(x_1), \mathcal{L}(x_2))$  is not completely defined.

Therefore, the possibility arises of defining this product in the neighborhood of the point  $x_1 = x_2$  by the limit

$$T'(\mathcal{L}(x_1), \mathcal{L}(x_2)) = \lim_{M \rightarrow \infty} T'_M(\mathcal{L}(x_1), \mathcal{L}(x_2)),$$

which will guarantee the *integrability* of  $S_2(x_1, x_2)$ .

There is also another completely equivalent possibility. As was established in §21, the most general form of  $S_2$  includes an arbitrary quasilocal operator:

$$S_2(x_1, x_2) = i^2 T(\mathcal{L}(x_1) \mathcal{L}(x_2)) + i\Lambda_2(x_1, x_2). \quad (27.43)$$

It is therefore possible, without going to the limit as  $M \rightarrow \infty$ , to define the quasilocal operator  $i\Lambda_2(x_1, x_2)$  by

$$i\Lambda_2(x_1, x_2) = T(\mathcal{L}(x_1) \mathcal{L}(x_2)) - T'_M(\mathcal{L}(x_1) \mathcal{L}(x_2)), \quad (27.44)$$

so that it compensates all the singular terms due to  $T(\mathcal{L}(x_1) \mathcal{L}(x_2))$  in (27.43). Then, after the transition to the limit as  $M \rightarrow \infty$ , we obtain  $S_2(x_1, x_2)$  in the form of an integrable polylocal operator:

$$S_2(x_1, x_2) = -T'(\mathcal{L}(x_1) \mathcal{L}(x_2)).$$

As has been shown in §22, the quasilocal operator  $i\Lambda_2$  may be included in the total effective interaction Lagrangian. To accomplish this, one must add to  $\mathcal{L}(x)$  the additional terms  $\mathcal{L}^{(2)}(x)$  which will compensate the singular terms in  $S_2$  after integration over  $dx$ , i.e.,

$$i \int \mathcal{L}^{(2)} dx + \frac{i^2}{2} \int \{T(\mathcal{L}(x_1) \mathcal{L}(x_2)) - T'_M(\mathcal{L}(x_1) \mathcal{L}(x_2))\} dx_1 dx_2 = 0.$$

By using (27.41), these *counterterms* may be written after integration by parts in the form

$$\begin{aligned} \mathcal{L}^{(2)}(x) = & \frac{1}{2} \int \Lambda_2(x, y) dy = a_2^M \left\{ \frac{i}{2} : \left( \bar{\psi}(x) \gamma^n \frac{\partial \psi}{\partial x^n} - \frac{\partial \bar{\psi}}{\partial x^n} \gamma^n \psi(x) \right) : - m : \bar{\psi} \psi : \right\} - \\ & - \delta m : \bar{\psi}(x) \psi(x) : + \delta \mu^2 : A_n(x) A^n(x) : + a_3^M \left\{ : \frac{\partial A_n}{\partial x^m} \frac{\partial A^n}{\partial x_m} : - : \left( \frac{\partial A}{\partial x} \right)^2 : \right\} + R, \end{aligned} \quad (27.45)$$

where the constants  $a_2$ ,  $a_3$ ,  $\delta m$ , and  $\delta \mu$  are of the order  $e^2$  and diverge as  $M \rightarrow \infty$ .

Thus, after redefining  $T(\mathcal{L}(x_1) \mathcal{L}(x_2))$  in the neighborhood of the point  $x_1 = x_2$  or after adding the quasilocal operator (27.44), which is equivalent to the addition to the interaction Lagrangian of counterterms (27.45) of order  $e^2$ , and after going to the limit as  $M \rightarrow \infty$ , we obtain for the terms of the  $S$ -matrix of the order  $e^2$  the regularized expression

$$\frac{i^2}{2!} \int dx_1 dx_2 T'(\mathcal{L}(x_1) \mathcal{L}(x_2)).$$

### §28. Divergences of the $S$ -Matrix in Electrodynamics (Third Order)

We have now completed our examination of the problem of regularization of the matrix to the second order in  $e$ . However, it is readily seen that a completely analogous procedure may be used to regularize higher-order terms of the  $S$ -matrix.

Consider the third-order term in the scattering matrix

$$S_3(x, y, z) = i^3 T(\mathcal{L}(x) \mathcal{L}(y) \mathcal{L}(z)). \quad (28.1)$$

Some of the divergent terms in  $S_3$ , which contain divergences of the second order corresponding, for example, to the diagrams shown in Fig. 18, are compensated simultaneously with the removal of the divergences from  $S_2$  by counterterms of the second order, i.e., by a redefinition of the  $T$ -product (28.1) at the points at which any two arguments are equal.

Specific third-order divergences may therefore be contained only in those terms of  $S_3$  that correspond to the diagrams of Figs. 19 and 20. However, terms corresponding to Fig. 19a and those corresponding to Fig. 19b always have their sum equal to zero. Indeed, these terms are proportional to  $e^3$  and do not contain factors corresponding to free electrons and positrons. Therefore, under the transformation of charge conjugation (13.23), the corresponding matrix elements will change by the factor  $(-1)^3 = -1$  and, due to the absence of real electrons or positrons, will describe the same processes. Consequently, they are equal to zero.\*

**28.1. Vertex Diagram of the Third Order.** It is therefore necessary to investigate only the third-order term corresponding to the diagram shown in Fig. 20. This term may be taken to be

$$(ie)^3 : \bar{\psi}(x) \hat{A}(x) \psi(x) \bar{\psi}(y) \hat{A}(y) \psi(y) \bar{\psi}(z) \hat{A}(z) \psi(z) : = ie : \bar{\psi}(x) \Gamma^n(x, z|y) \psi(z) A_n(y) :,$$

where we have introduced the *vertex function* of the third order

$$\Gamma^n(x, z|y) = ie^2 \gamma^k S^c(x-y) \gamma^n S^c(y-z) \gamma_k D_0^c(x-z). \quad (28.2)$$

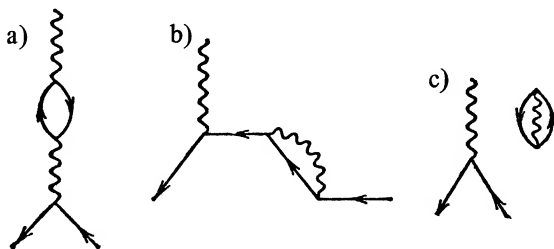


Fig. 18.

\*This statement is a special case of Furry's theorem (see § 33).



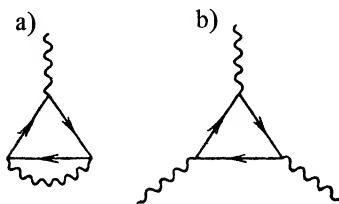


Fig. 19.

By going over to the momentum representation

$$\Gamma^n(x, z, y) = \frac{1}{(2\pi)^3} \int e^{ip(z-x) + ik(y-x)} \Gamma^n(p, k) dp dk, \quad (28.3)$$

we obtain the following expression for the vertex function:

$$\begin{aligned} \Gamma^n(p, k) &= \frac{ie^2}{(2\pi)^4} \int dq D_0^c(p-q) \gamma^k S^c(q+k) \gamma^n S^c(q) \gamma_k = \\ &= \frac{e^2}{(2\pi)^4} i \int \frac{dq}{(p-q)^2 + i\epsilon} \gamma^k \frac{\hat{q} + \hat{k} + m}{(q+k)^2 - m^2} \gamma^n \frac{\hat{q} + m}{q^2 - m^2} \gamma_k. \end{aligned} \quad (28.4)$$

We note that the choice of the momentum variables in (28.3) and (28.4) corresponds to a diagram in which the incident electron line has momentum  $p$ , the photon line has momentum  $k$ , the emerging electron line has momentum  $p + k$ , and the variable of integration  $q$  in (28.4) has been chosen equal to the difference of the four-momenta of the incident electron and the virtual photon as shown in Fig. 21.

The integral (28.4) diverges logarithmically for large  $q$ . In order to evaluate it, we shall make use of the regularization procedure (27.18), (27.19), adopted earlier. Substituting the expressions for  $S^c$  and  $D_0^c$  into (28.3), we find, after some rearrangement of the Dirac matrices,

$$\begin{aligned} \text{reg } \Gamma^n(p, k) &= \\ &= -\frac{e^2}{8\pi^4} \int_0^\infty \int_0^\infty \int_0^\infty d\alpha d\beta d\gamma e^{i\alpha p^2 + i\beta k^2 - \epsilon(\alpha + \beta + \gamma)} (1 - e^{-i\alpha M^2}) (e^{-i\beta m^2} - e^{-i\beta M^2}) \times \\ &\times (e^{-i\gamma m^2} - e^{-i\gamma M^2}) \int dq e^{iq^2(\alpha + \beta + \gamma) + 2iq(k\beta - p\alpha)} [\gamma^n m^2 + \hat{q} \gamma^n (\hat{k} + \hat{q}) - 2m(k^n + 2q^n)]. \end{aligned}$$

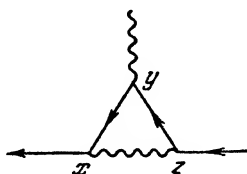


Fig. 20. Third-order divergent diagrams.

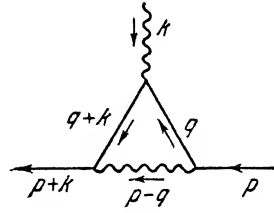


Fig. 21. Choice of notation for the momenta in Fig. 20.

Carrying out the integration over  $q$  with the aid of (27.13), (27.16), and (27.17), and introducing the new variables  $x, y, \lambda$  through

$$\alpha = \lambda x, \quad \beta = \lambda y, \quad \gamma = \lambda (1 - x - y), \quad (28.5)$$

we obtain after integration over  $\lambda$  in the limit of large  $M$

$$\begin{aligned} \text{reg } \Gamma^n(p, k) = & \frac{e^2}{8\pi^2} \gamma^n \int_0^1 dx \int_0^{1-x} dy \left\{ \ln \frac{M^2}{m^2} + \ln \frac{xy(1-x-y)}{(1-x)(1-y)(x+y)} + \right. \\ & \left. + \ln \left[ \frac{m^2}{m^2(1-x) - x(1-x)p^2 - y(1-y)k^2 - 2xy(pk)} \right] \right\} + \\ & + \frac{e^2}{8\pi^2} \int_0^1 dx \int_0^{1-x} dy \frac{\gamma^n m^2 - 2mk^n + (\hat{p}x - \hat{k}y) \gamma^n \hat{k} + 4m(k^ny - p^nx) + (\hat{p}x - \hat{k}y) \gamma^n (\hat{p}x - \hat{k}y)}{m^2(1-x) - x p^2 - y k^2 + (xp - yk)^2}. \end{aligned}$$

**28.2. Segregation of the Divergence from  $\Gamma$  and Gauge Invariance.** We thus see that  $\text{reg } \Gamma$  may be written in the form of the sum

$$\text{reg } \Gamma^n(p, k) = \frac{e^2}{16\pi^2} \gamma^n \left[ \ln \frac{M^2}{m^2} - \frac{1}{2} \right] + \Gamma'_M{}^n(p, k), \quad (28.6)$$

and that, in the limit of large  $M$ , the term  $\Gamma'_M{}^n$  converges to a finite limit which after some transformations may be written in the form

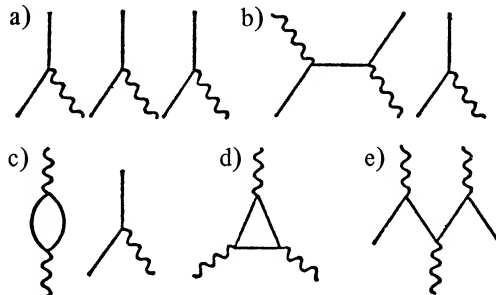


Fig. 22.

$$\begin{aligned}
\Gamma'^n(p, k) = & \frac{e^2}{8\pi^2} \gamma^n \left\{ \frac{(p+k)^2 - m^2}{2(p+k)^2} - \frac{3}{4} - \frac{[(p+k)^2 - m^2]^2}{2[(p+k)^2]^2} \ln \left( \frac{(p+k)^2 - m^2}{m^2} \right) - \right. \\
& \left. - \int_0^1 dx \int_0^{1-x} dy y \frac{k^2(1-2y) + 2xkp}{m^2(1-x) - xp^2 - yk^2 + (xp-yk)^2} \right\} + \\
& + \frac{e^2}{8\pi^2} \int_0^1 dx \int_0^{1-x} dy \frac{\gamma^n m^2 - 2mk^n + (\hat{p}x - \hat{k}y) \gamma^n \hat{k} + 4m(k^n y - p^n x) + (\hat{p}x - \hat{k}y) \gamma^n (\hat{p}x - \hat{k}y)}{m^2(1-x) - xy(p+k)^2 - x(1-x-y)p^2 - y(1-x-y)k^2}.
\end{aligned} \quad (28.7)$$

In going over to the configuration representation, we find that the following relation holds for such values of  $M$ :

$$\text{reg } \Gamma^n(x, z|y) = \frac{e^2}{16\pi^2} \gamma^n \left( \ln \frac{M^2}{m^2} - \frac{1}{2} \right) \delta(x-z) \delta(x-y) + \Gamma_M'^n(x, z|y), \quad (28.8)$$

where in the limit as  $M \rightarrow \infty$ , the term  $\Gamma_M'^n$  converges to an integrable function  $\Gamma'^n$  which in the momentum representation is defined by (28.7). The first term in (28.8) diverges logarithmically as  $M \rightarrow \infty$ .

The decomposition (28.6) is, of course, *not unique*. It was chosen so that

$$\Gamma'^n(0, 0) = 0. \quad (28.9)$$

The degree of ambiguity is determined by the structure of the singular term. Therefore, a constant proportional to the matrix  $\gamma^n$  may be added to (28.7). However, this constant is not arbitrary and is determined by the condition of *gauge invariance*.

Let us investigate this condition as applied to the third-order terms of the scattering matrix. These terms may be divided into two groups. One of these consists of terms containing the three operators of the electromagnetic field  $A(x)$ ,  $A(y)$ ,  $A(z)$  and not containing electromagnetic pairings  $D_0^c$ , while the other includes terms containing one operator  $A$  and one pairing  $D_0^c$ . The terms of the first group correspond to the diagrams shown in Fig. 22. The terms which correspond to the diagrams of Figs. 22a-c are normal products of the lowest-order terms and are therefore manifestly gauge invariant. The term corresponding to Fig. 22d vanishes in accordance with Furry's theorem, and, finally, the gauge invariance of the term corresponding to Fig. 22e may be established by a direct calculation.

The sum of the terms of the second group which contain divergences of the third order may be written as a whole in the form

$$:A_n(x) J^n(x, y, z): + :A_n(y) J^n(y, z, x): + :A_n(z) J^n(z, x, y):.$$

The requirement of gauge invariance imposes on each term of this expression conditions of the form

$$\int \frac{\partial f(x)}{\partial x^n} J^n(x, y, z) dx = 0,$$

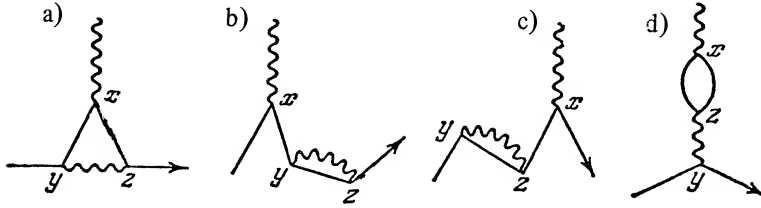


Fig. 23.

which in view of the arbitrariness of the function  $f$  give

$$\frac{\partial}{\partial x^n} J^n(x, y, z) = 0. \quad (28.10)$$

We now turn to the structure of the function  $J^n(x, y, z)$ . This function contains terms corresponding to the four diagrams (Fig. 23) and to four additional diagrams which differ from those shown by an interchange of the points  $z$  and  $y$ .

We note first of all that the terms  $J^n$  which correspond to the diagram of Fig. 23d

$$J_d^m(x, y, z) \sim \{ \Pi^{mn}(x-z) : \bar{\psi}(y) \gamma_n \psi(y) : D_0^c(y-z) + \\ + \text{term with } z \text{ and } y \text{ interchanged} \}$$

automatically give after the removal of divergences and in virtue of condition (27.38)

$$\frac{\partial}{\partial x^n} J_d^n(x, y, z) = 0.$$

Therefore, only the terms of  $S_3$  which correspond to the first three diagrams of Fig. 23 remain to be considered:

$$\{ : \bar{\psi}(z) \gamma^m S^c(z-x) \gamma^n S^c(x-y) \gamma_m \psi(y) : + : \bar{\psi}(z) \gamma^m S^c(z-y) \gamma_m S^c(y-x) \gamma^n \psi(x) : + \\ + : \bar{\psi}(x) \gamma^n S^c(x-z) \gamma^m S^c(z-y) \gamma_m \psi(y) : \} D_0^c(y-z). \quad (28.11)$$

Differentiating formally with respect to  $x^n$  and summing them after taking into account the equations

$$\left\{ \begin{aligned} \left( i \frac{\partial}{\partial x} - m \right) \psi(x) &= 0, \quad i \frac{\partial \bar{\psi}}{\partial x^n} \gamma^n + m \bar{\psi}(x) = 0, \\ \left( i \frac{\partial}{\partial x} - m \right) S^c(x) &= i \frac{\partial S^c(x)}{\partial x^n} \gamma^n - m S^c(x) = -\delta(x), \end{aligned} \right\} \quad (28.12)$$

we obtain

$$D_0^c(y-z) \{ : \bar{\psi}(z) \gamma^m \delta(z-x) S^c(x-y) \gamma_m \psi(y) : - : \bar{\psi}(z) \gamma^m S^c(z-x) \delta(x-y) \gamma_m \psi(y) : + \\ + : \bar{\psi}(z) \gamma^m S^c(z-y) \gamma_m \delta(y-x) \psi(x) : - : \bar{\psi}(x) \delta(x-z) \gamma^m S^c(z-y) \gamma_m \psi(y) : \} = 0.$$

Thus the sum of the terms of the  $S$ -matrix which correspond to the diagrams of Figs. 23a, b, c is indeed gauge invariant.

The verification of condition (28.10) carried out above was of a purely formal nature, since each of the terms of expression (28.11) is actually divergent. In fact, it is only necessary to verify that its finite part is gauge invariant. To make this verification more convenient, it is advantageous first to use Feynman's method of regularization and only then to go over to the method of regularization adopted by us. If we therefore regularize only the photon function

$$D_0^c(y-z) \rightarrow \text{reg } D_0^c(y-z),$$

we obtain the expression

$$\begin{aligned} \text{reg } D_0^c(y-z) \{ & : \bar{\psi}(z) \gamma^m S^c(z-x) \gamma^n S^c(x-y) \gamma_m \psi(y) : + \\ & + : \bar{\psi}(z) \gamma^m S^c(z-y) \gamma_m S^c(y-x) \gamma^n \psi(x) : + : \bar{\psi}(x) \gamma^n S^c(x-z) \gamma^m S^c(z-y) \gamma_m \psi(y) : \}, \end{aligned} \quad (28.13)$$

which as a whole clearly satisfies condition (28.10), since the factors being differentiated

$$\bar{\psi}(x), \quad S^c(x-w), \quad S^c(w-x), \quad \psi(x) \quad (w=y, z)$$

are not changed.

The singular part of (28.13) for  $M < \infty$  and after taking into account the expansions

$$\left. \begin{aligned} \text{reg}_F \Sigma(x-y) &= \Sigma'_F(x-y) - a_2^F \left( i \frac{\partial}{\partial x} - m \right) \delta(x-y) - \delta m \delta(x-y), \\ \text{reg}_F \Gamma^n(x, y | \xi) &= a_1^F \gamma^n \delta(x-y) \delta(y-\xi) + \Gamma_F'^n(x, y | \xi) \end{aligned} \right\} \quad (28.14)$$

may be written in the form

$$\begin{aligned} a_1^F : \bar{\psi}(z) \gamma^n \delta(z-x) \delta(x-y) \psi(y) : &+ \\ + a_2^F : \bar{\psi}(z) \left[ \left( i \frac{\partial}{\partial z} - m \right) \delta(z-y) \right] S^c(y-x) \gamma^n \psi(x) : &- \\ - \delta m : \bar{\psi}(z) \delta(z-y) S^c(y-x) \gamma^n \psi(x) : &+ \\ + a_3^F : \bar{\psi}(x) \gamma^n S(x-z) \left[ \left( i \frac{\partial}{\partial z} - m \right) \delta(z-y) \right] \psi(y) : &- \\ - \delta m : \bar{\psi}(x) \gamma^n S^c(x-z) \delta(z-y) \psi(y) : &. \end{aligned}$$

Differentiating this combination with respect to  $x^n$ , summing over  $n$ , and taking equations (28.12) into account we obtain the expression

$$(a_1^F - a_2^F) \left\{ : \bar{\psi}(z) \left( \frac{\partial}{\partial x} \delta(z-x) \right) \delta(x-y) \psi(y) : + : \bar{\psi}(z) \delta(z-x) \left( \frac{\partial}{\partial x} \delta(x-y) \right) \psi(y) : \right\}.$$

28.3. *Ward's Identity.* Thus in order to guarantee the gauge invariance of the singular part of expression (28.13) it is necessary that

$$a_1^F = a_2^F. \quad (28.15)$$

Since (28.13) is gauge invariant as a whole, the above condition guarantees the invariance of its finite part. Then by going to the limit as  $M \rightarrow \infty$  after a rearrangement of the effective Lagrangian, we obtain for  $S_3$  a gauge invariant expression not containing any infinities. The identity of the singular constants (28.15), which guarantees gauge invariance, was first established in a somewhat more general form by Ward (1950) and is known as *Ward's identity*.

We have obtained Ward's identity by using Feynman's regularization. We shall now show that it also holds in the case of the method of regularization usually employed by us. To do this, we introduce for the sake of brevity the notation

$$\text{reg}_F \Sigma \equiv \Sigma_F \quad \text{and} \quad \text{reg}_F \Gamma^n \equiv \Gamma_F^n,$$

and rewrite (28.14) in the form

$$\begin{aligned} \Sigma_F' &= \Sigma_F(p) + a_2^F \hat{p} - m a_2^F + \delta m = \\ &= \Sigma_F(p) - \Sigma_F(0) - \left. \frac{\partial \Sigma_F(p)}{\partial p^n} \right|_{p=0} \cdot p^n + c_0^F + c_2^F \hat{p}, \end{aligned} \quad (28.16)$$

$$\Gamma_F'^n(p, k) = \Gamma_F^n(p, k) - a_1^F \gamma^n = \Gamma_F^n(p, k) - \Gamma_F^n(0, 0) - c_1^F \gamma^n, \quad (28.17)$$

from which it follows that

$$a_2^F \gamma^n = - \left. \frac{\partial \Sigma_F(p)}{\partial p_n} \right|_{p=0} + c_2^F \gamma^n = \Gamma_F^n(0, 0) + c_1^F \gamma^n. \quad (28.18)$$

Direct calculation will readily show that

$$\left. \frac{\partial \Sigma_F(p)}{\partial p_n} \right|_{p=0} = - \Gamma_F^n(p, 0). \quad (28.19)$$

It therefore follows from (28.18) that

$$c_1^F = c_2^F.$$

Now recalling that in accordance with (27.19), the combination

$$\Sigma_F(p) - \Sigma_F(0) - \left. \frac{\partial \Sigma_F(p)}{\partial p^n} \right|_{p=0} \cdot p^n$$

does not depend on the method of regularization and, in accordance with (27.27), is equal to expression given by (27.26), and also taking into account analogous considerations for the function  $\Gamma^n$ , we conclude that expressions

$$\Sigma'(p) + c_0 + c_2 \hat{p} \quad \text{and} \quad \Gamma'^n(p, k) - c_1 \gamma'$$

satisfy the requirement of gauge invariance under the condition

$$c_1 = c_2.$$

Therefore, the singular parts of the functions  $\Sigma(x)$  and  $\Gamma(x, y | \xi)$ , which are subtracted off in our usual method of regularization, may be written in the form

$$a_1 \left( i \frac{\partial}{\partial x} - m \right) \delta(x) + \delta m \delta(x), \quad (28.20)$$

$$- a_1 \gamma^n \delta(x - y) \delta(x - \xi), \quad (28.21)$$

where

$$a_1 = \frac{e^2}{16\pi^2} \left[ \ln \frac{M^2}{m^2} - \frac{1}{2} - c_1 \right], \quad (28.22)$$

$$\delta m = \frac{e^2}{16\pi^2} \left\{ 3m \ln \frac{M^2}{m^2} - \frac{7m}{2} + c_0 \right\}. \quad (28.23)$$

Ward's identity thus also holds in the case of our method of regularization. In view of the fact that in the preceding argument we have used only the relations

$$\Sigma(0) = 0, \quad \left. \frac{\partial \Sigma}{\partial p^n} \right|_{p=0} = 0, \quad \Gamma^n(0, 0) = 0,$$

this shows that Ward's identity does not in general depend on the method of regularization.

**28.4. Construction of an Integrable Function  $S_3$ .** We conclude that the third-order term in the scattering matrix may be written in the form

$$S_3(x_1, x_2, x_3) = i^3 T'_M(\mathcal{L}(x_1) \mathcal{L}(x_2) \mathcal{L}(x_3)) - i^2 T(\mathcal{L}(x_1) \Lambda_2(x_2, x_3)) - \\ - i^2 T(\mathcal{L}(x_2) \Lambda_2(x_1, x_3)) - i^2 T(\mathcal{L}(x_3) \Lambda_2(x_1, x_2)) - i \Lambda_3(x_1, x_2, x_3), \quad (28.24)$$

where  $T'_M$  is a regular operator function in the coefficient functions of which the function  $\text{reg } \Delta^c$  should be used for the ordinary lines, the functions  $\Pi'_M$  and  $\Sigma'_M$  should be used for the closed diagrams shown in Figs. 17a, b and the function  $\Gamma''_M$  should be used for the vertex part (Fig. 20).

It is then clear that when the regularization is removed, all the coefficient functions of

the operator  $T'_M$  converge to finite limits which depend on  $\Delta^c$ ,  $\Pi'$ ,  $\Sigma'$ , and  $\Gamma'$ . The terms in expression (28.24) containing the functions

$$i\Lambda_2(x_1, x_2) = T(\mathcal{L}(x_1)\mathcal{L}(x_2)) - T'_M(\mathcal{L}(x_1)\mathcal{L}(x_2)),$$

take into account divergences of the second order corresponding to diagrams of the type of Fig. 18, while the quasilocal operator

$$\Lambda_3(x_1, x_2, x_3) = -a_1 e \{ \bar{\psi}(x_1) \hat{A}(x_2) \psi(x_3) : \delta(x_1 - x_2) \delta(x_2 - x_3) + \\ + \text{terms which differ by interchanges of arguments} \} \quad (28.25)$$

corresponds to the divergences of the vertex parts (Fig. 20).

After integration over  $x_1, x_2, x_3$  the terms containing the functions  $\Lambda_2$  give the contribution to the  $S$ -matrix

$$\frac{1}{2} \int dx_1 dx_2 dx_3 T(\mathcal{L}(x_1) \Lambda_2(x_2, x_3)),$$

which is compensated by the terms in  $S_2$ , containing the counterterms  $\mathcal{L}^{(2)}(x)$ :

$$\frac{i^2}{2!} \int T(\mathcal{L}(x_1) \mathcal{L}^{(2)}(x_2)) dx_1 dx_2 + \frac{i^2}{2!} \int T(\mathcal{L}(x_2) \mathcal{L}^{(2)}(x_1)) dx_1 dx_2 = \\ = -\frac{1}{2} \int dx_1 dx_2 dx_3 T(\mathcal{L}(x_1) \Lambda_2(x_2, x_3)).$$

The last divergent term of (28.24) must be compensated by adding to the interaction Lagrangian a new counterterm of the third order in  $e$ :

$$\mathcal{L}^{(3)}(x) = -ea_1 : \bar{\psi}(x) \hat{A}(x) \psi(x) : = \frac{1}{3!} \int \Lambda_3(x, x_2, x_3) dx_2 dx_3. \quad (28.26)$$

Thus, after introducing into the Lagrangian the additional counterterms  $\mathcal{L}^{(2)}(x)$  and  $\mathcal{L}^{(3)}(x)$ , and going to the limit as  $M \rightarrow \infty$ , we obtain for the second- and third-order terms of the  $S$ -matrix the integrable expressions

$$T'(\mathcal{L}(x_1)\mathcal{L}(x_2)) \quad \text{and} \quad T'(\mathcal{L}(x_1)\mathcal{L}(x_2)\mathcal{L}(x_3)). \quad (28.27)$$

We emphasize that the integrable expressions (28.27) for  $S_2$  and  $S_3$  may be obtained, without making use of counterterms, by a suitable redefinition of  $T$ -products in the region in which their arguments coincide. This last possibility is preferable, since it turns out that *the structure of the counterterms depends not only on the specific form of the auxiliary regularization, but also on "the region of switching on" the interaction described by the function  $g(x)$ .*

Up until now we have restricted ourselves to the consideration of the case where the



interaction is completely switched on over all space-time and  $g(x) = 1$ . It is just this case that is important for the evaluation of the matrix elements for the processes of scattering and of mutual transformation of particles when the interaction is switched on and switched off in remotely distant past and future.

However, it turns out that when we are interested in the properties of systems of particles existing in bound levels (energy levels, lifetimes, transition probabilities between bound states (for details see Chapter VII), we have to examine the situation for which the interaction is switched on only over a certain part of four-space and the function  $g(x)$  increases from zero to unity within small regions near the surfaces bounding this part of four-space.

Bearing this last case in mind, we shall investigate the structure of the counterterms in the effective Lagrangian  $\mathcal{L}(x; g)$ , which guarantee that the operator

$$S(g) = T \left( \exp i \int \mathcal{L}(x; g) dx \right)$$

will be integrable.

Let us consider the second-order terms. The counterterm  $\mathcal{L}^{(2)}(x; g)$  is determined by the condition which is a natural generalization of (27.43):

$$\int \mathcal{L}^{(2)}(x; g) dx = \frac{i}{2i} \int dx_1 dx_2 g(x_1) g(x_2) [T(\mathcal{L}(x_1) \mathcal{L}(x_2)) - T'_M(\mathcal{L}(x_1) \mathcal{L}(x_2))].$$

Substituting into the above equation the explicit expression for the difference  $T - T'_M$  from (27.41), and integrating by parts, we find that

$$\begin{aligned} \mathcal{L}^{(2)}(x; g) = & \mathcal{L}^{(2)}(x) g^2(x) - i a_2 : \bar{\psi}(x) \gamma^n \frac{\partial g(x)}{\partial x^n} \psi(x) : g(x) + \\ & + a_3 \left\{ : A^m(x) A^n(x) : \frac{\partial^2 g(x)}{\partial x^n \partial x^m} + : A^n(x) \frac{\partial A^m}{\partial x^n} : \frac{\partial g}{\partial x^m} - \right. \\ & - : A^m(x) \frac{\partial A^n}{\partial x^n} : \frac{\partial g(x)}{\partial x^m} + : A^m(x) A_m(x) : \frac{\partial^2 g(x)}{\partial x^n \partial x_n} \left. \right\} g(x) + \\ & + \{ \text{terms containing } R \text{ and its derivatives} \}, \quad (28.28) \end{aligned}$$

where the operator  $\mathcal{L}^{(2)}(x)$  is given by (27.45). It is also clear that, in the third order,

$$\mathcal{L}^{(3)}(x; g) = \mathcal{L}^{(3)}(x) g^3(x),$$

where  $\mathcal{L}^{(3)}(x)$  is given by (28.26).

Thus, in the process of integration by parts, terms appear in  $\mathcal{L}(x; g)$  which contain derivatives of the function  $g(x)$  and which differ from the usual counterterms in their operator structure. This will turn out to be essential for the removal of divergences from Schroedinger's equation.

## §29. General Rules for the Removal of Divergences from the S-Matrix

*29.1. Formulation of the Problem.* In §27 and §28, the integrable expressions for  $S_2$  and  $S_3$  were constructed by using spinor electrodynamics as an example. We now undertake the formulation of the general method for the removal of divergences from the terms of the S-matrix of arbitrary order, which is based on the same principle as the examples considered above (see Bogolyubov (1952)). We shall first of all replace the true causal functions  $\Delta^c$  by

the regularized expressions  $\text{reg } \Delta^c$  with an appropriate number of auxiliary masses  $M_i$ . For finite values of these masses, the usual  $T$ -product of the Lagrangians

$$T(\mathcal{L}(x_1) \dots \mathcal{L}(x_n)) \quad (29.1)$$

is quite definite, and its coefficient functions are continuous. However, as we have just seen, it is not always possible to make the limiting transition  $M \rightarrow \infty$  in (29.1) even if it were to be understood in the improper sense.

More precisely, the coefficient functions of the operator (29.1) will converge in the improper sense only in those regions of space-time in which each of the arguments  $x_1, x_2, \dots, x_n$  differs from all the others. To segregate the "convergent part" from expression (29.1), it is necessary to apply to it, just as in the cases considered above, a certain subtraction procedure. To formulate this procedure, it is convenient to start with the general formula (21.41) which expresses  $S_n(x_1, \dots, x_n)$  in terms of the interaction Lagrangian  $\mathcal{L}(x) = \Lambda_1(x)$  and the quasilocal operators  $\Lambda_\nu (\nu \geq 2)$ :

$$\begin{aligned} S_n(x_1, \dots, x_n) = & i^n T(\mathcal{L}(x_1) \dots \mathcal{L}(x_n)) + \\ & + \sum_{\substack{2 \leq m \leq n-1 \\ \sum \nu_i = n}} i^m P'(x_1, \dots, x_{\nu_1} | \dots | \dots, x_n) T(\Lambda_{\nu_1}(x_1, \dots, x_{\nu_1}) \dots \Lambda_{\nu_m}(\dots, x_n)) + \\ & + i \Lambda_n(x_1, \dots, x_n). \end{aligned} \quad (29.2)$$

As has been shown, this expression is the most general expression satisfying all the conditions (of symmetry, covariance, unitarity, and causality) imposed on  $S_n$  for an arbitrary choice of the Hermitian covariant quasilocal operators  $\Lambda_\nu$ .

If we succeed in choosing the quasilocal operators  $\Lambda_\nu$  in such a way that the expressions  $S_n$  will turn out to be convergent (we shall always understand the convergence of expressions of this type to be in the improper sense), then their limit as  $M \rightarrow \infty$  will be, first, an integrable operator function (in the sense of the definition given in §18), and, second, it will satisfy all the conditions imposed on  $S_n$ . Indeed, the condition of covariance (21.4) is of a linear nature and, therefore, the limiting transition for it is trivial, while the possibility of going to the limit in the conditions of unitarity (21.9) and of causality (21.13) is governed by the theorem of §18 which states that the limit of the usual product of two operator functions is equal to the corresponding product of their limits.

**29.2. General Method for Removing Divergences.** Stepanov (1963, 1965) has shown that a convergent expression can be constructed for  $S_n$  by using mathematical induction and the Hahn-Banach theorem on the expansion of the linear functional without recourse to the immediate Pauli-Villars regularization.

We rewrite the causality condition for operator functions (21.13) in the form

$$\begin{aligned} S_{n+1}(y, x_1, \dots, x_n) = & R_{n+1}(y, x_1, \dots, x_n), \\ & \text{if } y \geq x_j \text{ for at least one } j, \end{aligned} \quad (29.3)$$

where

$$R_{n+1}(y, x_1, \dots, x_n) = - \sum_{0 \leq k \leq n-1} P \left( \frac{x_1, \dots, x_k}{x_{k+1}, \dots, x_n} \right) S_{k+1}(y, x_1, \dots, x_k) \overset{\uparrow}{S}_{n-k}(x_{k+1}, \dots, x_n). \quad (29.4)$$

The relationship given by (29.3) provides a formal definition of the operator expression  $S_{n+1}$  in terms of the preceding  $S_k$ ,  $k \leq n$ . In §28, the operator functions  $S_2$  and  $S_3$  were defined constructively as integrable generalized functions, i.e., the kernels of certain linear functionals. We now use the operator relationships given by (29.3) and (29.4) to express  $S_{n+1}$  in terms of the preceding  $S_k$ , assuming that they are specified as kernels of linear functionals

$$\int S_k(x_1, \dots, x_k) F(x_1, \dots, x_k) dx_1 \dots dx_k$$

for the functions  $F(x_1, \dots, x_2)$  from a certain class  $C(q, r, k)$ . Using the theorem of §19.3 on the multiplication of commutation functions of the same frequency, we can now find numbers  $q_n, r_n$  such that the right-hand side of (29.4) defines the functional

$$\int R_{n+1}(x_1, \dots, x_{n+1}) F(x_1, \dots, x_{n+1}) dx_1 \dots dx_{n+1}$$

for functions  $F$  from the class  $C(q_n, r_n, n+1)$ . If we now segregate from  $C(q_n, r_n, n+1)$  the subclass  $C_1$  of functions with the property

$$F_1(x_1, \dots, x_{n+1}) = 0, \quad \text{if} \quad x_j > x_1 \quad \text{for all} \quad j > 1, \quad (\tilde{C}_1)$$

then we can define the operator  $S_{n+1}$  on this subclass with the aid of

$$\begin{aligned} \int S_{n+1}(x_1, \dots, x_{n+1}) F_1(x_1, \dots, x_{n+1}) dx_1 \dots dx_{n+1} = \\ = \int R_{n+1}(x_1, \dots, x_{n+1}) F_1(x_1, \dots, x_{n+1}) dx_1 \dots dx_{n+1}, \end{aligned} \quad (29.5)$$

which is the rigorous formulation of the causality condition (29.3). Neither (29.5) nor (29.3) reflect the symmetry of the operator  $S_{n+1}$ . Since the interchange of variables  $x_1 \leftrightarrow x_m$  leads to the replacement  $F_1 \rightarrow F_m$ , where

$$F_m(x_1, \dots, x_m, \dots, x_{n+1}) = 0, \quad \text{if} \quad x_j > x_m \quad \text{for all} \quad j \neq m,$$

the symmetry of  $S_{n+1}$  ensures that the operator  $S_{n+1}$  is defined on a broader class  $\tilde{C}$ :

$$F(x_1, \dots, x_{n+1}) = \sum_m F_m(x_1, \dots, x_{n+1}). \quad (\tilde{C})$$

It is clear that since the functions  $F_m$  are continuous, their sum must vanish at  $x_1 = x_2 = \dots = x_{n+1}$ . It can be shown [Stepanov (1965)] that the converse of this statement is also true, i.e., any function from  $C(q_n, r_n, n+1)$  which has a zero of sufficiently high order for equal arguments can be expressed by the sum  $(\tilde{C})$ .

We have thus verified that the symmetric operator  $S_{n+1}$ , initially defined by the

nonsymmetric causality condition on the subclass  $\tilde{C}_1$ , is defined on all functions  $F \in \tilde{C}$  with sufficiently high-order zeros when all the arguments are equal. It may be shown that this operator is given by the usual formula (21.31) in terms of the chronological products of Lagrangians.

To determine  $S_{n+1}$  when all the arguments are equal, i.e., to execute the transition from the class  $\tilde{C}$  to the entire class  $C(q, r, n+1)$ , we can use the Hahn-Banach theorem [see, for example, Vladimirov (1964)] on the extension of functionals. When this extension is performed, a certain finite quasilocal operator  $i\Lambda_{n+1}$ , which by virtue of the unitarity condition turns out to be also anti-Hermitian, is found to remain arbitrary.

Thus, it is sufficient for us to set up a method of constructing from the given Lagrangian  $\mathcal{L}(x) = \Lambda_1(x)$  a sequence of quasilocal operators  $\Lambda_\nu$  ( $\nu \geq 2$ ) which guarantee the convergence of (29.2). All the conditions imposed on  $S_n$ , including the condition of integrability, are automatically fulfilled for the limiting expressions obtained by the above method. To solve this problem, it is obviously sufficient that we should choose for  $\Lambda_\nu(x_1, \dots, x_\nu)$  quasilocal operators of the same operator type as (29.1) (with  $n = \nu$ ). In saying that  $\Lambda_\nu$  belongs to the same operator type as (26.1), we mean that  $\Lambda_\nu$  consists of the same operator terms

$$\vdots \dots u_\alpha(x_j) \dots,$$

as (29.1), and differs from it only by its coefficient functions.

It now follows from (29.2) that  $S_n(x_1, \dots, x_n)$  will also be an expression of the same operator type as (29.1). To simplify the formulas, it is convenient to set

$$i^{1-\nu} \Lambda_\nu(x_1, \dots, x_\nu) = \Delta_\nu(x_1, \dots, x_\nu), \quad \mathcal{L}(x) = \Lambda_1(x) = \Delta_1(x),$$

so that

$$S_n = i^n T'_n,$$

where

$$\begin{aligned} T'_n = & T(\mathcal{L}(x_1) \dots \mathcal{L}(x_n)) + \\ & + \sum_{\substack{2 \leq m \leq n-1 \\ \sum_{i=1}^m \nu_i = n}} P'(x_1, \dots, x_{\nu_1} | \dots | \dots, x_n) T(\Delta_{\nu_1}(x_1, \dots, x_{\nu_1}) \dots \Delta_{\nu_m}(\dots, x_n)) + \\ & + \Delta_n(x_1, \dots, x_n). \end{aligned} \quad (29.6)$$

Since  $T'_n$  belongs to the same operator type as (29.1), we may evidently develop a method for the actual construction of  $T'_n$  by means of a certain transformation of the coefficient functions of the operator  $T$ .

**29.3. Graphical Representation of the Subtraction Procedure and the R-Operation.** Our discussion so far has largely been in the nature of an existence analysis, since we have not

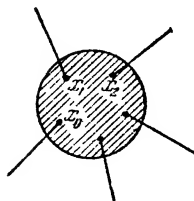


Fig. 24.

indicated the specific procedure for constructing the operators  $\Lambda_n$  and, consequently,  $T'_n$ . We shall now give the constructive form of the Hahn-Banach theorem, i.e., the recipe for constructing  $\Lambda_n$  and finite  $T'_n$ . We shall find it convenient to work with graphical representations.

The coefficient function for the  $T$ -product which corresponds to some particular diagram with  $\nu$  vertices and with given internal lines is represented by a product of regularized causal functions of the type

$$\prod_l \{\text{reg } \Delta_l^c(x_a - x_b)\}. \quad (29.7)$$

The coefficient functions of the quasilocal operator  $\Delta_\nu$  represent, for the same diagram, a product of  $\delta$ -functions and of their derivatives. Therefore, the whole set of points  $x_1, \dots, x_\nu$  behaves as a single entity in the diagram, as a result of which it is convenient, when working with  $\Delta_\nu$ , to introduce the concept of the *generalized vertex*  $G$  (see Fig. 24). We shall denote the coefficient function of the operator  $\Delta_\nu$ , which corresponds to a given generalized vertex  $G$ , by  $d_G(x_1, \dots, x_\nu)$ . We now consider an expression of the form

$$T(\Delta_\nu(x_1, \dots, x_\nu) \mathcal{L}(x_{\nu+1}) \dots \mathcal{L}(x_n)). \quad (29.8)$$

It is clear that its coefficient functions may be obtained from the coefficient functions of the operator (29.1) by the following procedure. Consider a diagram corresponding to the operator (29.8). By combining the points  $x_1, \dots, x_n$  in this diagram into the generalized vertex  $G$  we shall replace the product (29.7) which corresponds to the lines  $l$  internal with respect to  $G$  by the coefficient function  $d_G(x_1, \dots, x_\nu)$ . We shall formally denote such a replacement operation by the symbol  $\Delta(G)$ . Then the coefficient functions of the operator (29.8) will be obtained from the coefficient functions of the operator (29.1) by the operation  $\Delta(G)$ .

In the more general case, the arbitrary term after the summation sign on the right-hand side of (29.6) can be associated with the decomposition of the entire diagram  $G$  into the subdiagrams  $G_1 \dots G_m$ :

$$G = G_1 * G_2 * \dots * G_m$$

(The diagram  $G$  appears as the topological product  $G_1, \dots, G_m$ .) The coefficient functions in the expression

$$T(\Delta_{v_1}(x_1, \dots, x_{v_1}) \dots \Delta_{v_m}(\dots, x_n))$$

can be obtained from the coefficient functions of the operator (29.1) with the aid of the operation

$$\Delta(G_1) \dots \Delta(G_m),$$

which consists of combining the points  $x_1, \dots, x_{v_1}, x_{v_1+1}, \dots, x_{v_1+v_2}, \dots, x_n$  into generalized vertices  $G_1, G_2, \dots, G_m$ , and of replacing the factors in the product (29.7), which correspond to the internal lines of  $G_1, \dots, G_m$ , by the corresponding  $d_{G_1}, \dots, d_{G_m}$ , leaving unchanged the factors that correspond to lines connecting different generalized vertices.

Thus  $T'_n$  may be obtained from  $T$  by applying the operation

$$R(G) = 1 + \sum_{\substack{2 \leq m \leq n-1 \\ G = G_1 * G_2 * \dots * G_m}} \Delta(G_1) \dots \Delta(G_m) + \Delta(G). \quad (29.9)$$

Here the summation is carried out over all possible ways of dividing the set of points  $x_1, \dots, x_n$  of the diagram  $G$  into the generalized vertices  $G_1, G_2, \dots, G_m$ . The symmetry of  $T'_n$  associated with the symmetrization operator  $P$  which is present in (29.2) is taken into account in this case by making sure that, in such a decomposition, the points  $x_1, \dots, x_n$  appear in a perfectly symmetric way. We also emphasize that if in the decomposition of the diagram any of the generalized vertices  $G_k$  coincides with an ordinary vertex  $x_k$ , then in the sum on the right-hand side of (29.9) the corresponding operation  $\Delta(G_k)$  must be regarded as the unit operation because we agreed, just before (29.6), that the quasilocal operator  $\Delta_1(x_k)$  is identical with the Lagrangian.

So far, the operation  $R(G)$  has been defined in a purely formal fashion. It will take on a concrete meaning after rules have been established for the actual determination of the function  $d_{G_a}$  for a given diagram  $G_a$ . By specifying  $d_G$  we thereby define  $\Delta$ , and, consequently,  $T'_n$ . We shall choose  $d_G$  in such a way as to make  $T'_n$  integrable when  $M \rightarrow \infty$ .

Before we proceed to the formulation of the method for constructing  $d_G$  we must introduce some further concepts. We shall say that a given diagram is *connected* if it cannot be decomposed into parts that are not connected by lines. If, on the other hand, the diagram  $G$  can be decomposed into connected subdiagrams between which there are no connections, then we shall say that  $G$  is a disconnected diagram and its "pieces"  $G_1, G_2, \dots$  will be said to be the *components of connectedness*. A connected diagram will be called *weakly connected* if it can be transformed into a disconnected diagram by removing one line, and *strongly connected* if this cannot be done.

We now note that, for a disconnected diagram, the coefficient function of a  $T$ -product has the form of a product of two coefficient functions with different arguments. But the

product of two functions with different arguments is convergent when each function is convergent separately. From this it follows that in a disconnected diagram, the divergences are automatically removed after they have been removed in its connected parts. Therefore, the operator  $\Delta(G)$  should be taken equal to zero for disconnected diagrams.

With this choice of  $\Delta(G)$ , the operation  $R(G)$  applied to a coefficient function of the  $T$ -product which corresponds to a diagram  $G$  consisting of two disconnected parts  $G_1$  and  $G_2$  decomposes into the product  $R(G_1)R(G_2)$  of two operations which operate separately on the coefficient functions corresponding to the diagrams  $G_1$  and  $G_2$ .

For weakly connected diagrams, we obtain coefficient functions of the form

$$K_M(x_1, \dots, x_k) \operatorname{reg} \Delta^c(x_a - y_b) Q_M(y_1, \dots, y_s).$$

Since  $K$  and  $Q$  are invariant under translation, by setting

$$x - x_a = x', \quad y - y_b = y', \quad x_a - y_b = \xi,$$

we obtain a product of coefficient functions with independent arguments

$$K_M(x'_1, \dots, x'_k) \operatorname{reg} \Delta^c(\xi) Q_M(y'_1, \dots, y'_s). \quad (29.10)$$

This expression will also converge as a whole if  $K$  and  $Q$  are convergent individually. Therefore, for weakly connected diagrams, we should also set  $\Delta(G) = 0$ .

Here, as in the preceding case, it is clear that, with the above choice of the operator  $\Delta(G)$ , the coefficient functions of the operator  $T'_n$  for weakly connected diagrams have the same structure as (29.10). Thus, in (29.9) we need only discuss the strongly connected decompositions of  $G$  into generalized vertices  $G_a$ .

**29.4. Index of the Diagram  $\omega(G)$  and the Degree of Divergence.** In order to make the operator  $\Delta(G)$  specific, we also introduce the concept of the *index of the diagram*. To do this, we go over to the momentum representation. The coefficient functions of the  $T$ -product in the  $p$ -representation will obviously have the form

$$J_M(k) = \int \prod_{(1 \leq q \leq n)} \delta(\Sigma p - k_q) \prod_l \{ \operatorname{reg} \Delta^c_l(p_l) dp_l \}. \quad (29.11)$$

Here the arguments of the  $\delta$ -functions contain algebraic sums of the momenta of the internal lines of the diagram meeting at the vertex  $q$  to which the external momenta  $k_q$  have been added.

In accordance with the procedure of regularizing the  $\Delta^c$ -functions, which we have adopted, we also have

$$\operatorname{reg} \Delta^c(p) = Z(p_l) \left\{ \frac{1}{m^2 - p_l^2 - i\varepsilon} - \sum_j c_j \frac{1}{M_j^2 - p_l^2 - i\varepsilon} \right\},$$

where  $Z(p)$  is the same polynomial that appears in the unregularized  $\Delta^c$ -function.

If we go to the limit as  $M_j \rightarrow \infty$  in (29.11), then the whole integral will, in general, turn out to be divergent for large momenta. We now compute its total degree of divergence. Since we are investigating connected diagrams,  $4(n-1)$  integration may be carried out with the aid of the  $\delta$ -functions (the one remaining  $\delta$ -function expresses the law of conservation of the total four-momentum) and we are left with  $4(L-n+1)$  independent variables of integration, where  $L$  denotes the total number of internal lines.

Just as in the case of integration over three-dimensional space where one uses the radius as a variable of integration, we shall introduce the corresponding "radial" momentum  $p$  in the integration over the  $4(L-n+1)$ -dimensional space. The product of the independent differentials contained in  $\prod_l dp_l$  then gives the factor  $P^{4(L-n+1)} dP/P$ . Taking into account only the terms of the highest degree in the function  $\Delta^c(p)$ , we obtain the factor

$$P^{\sum_l r_l - 2L} = P^{\sum_l (r_l - 2)},$$

and, therefore, in carrying out the integration over  $P$ , the factor multiplying  $dP/P$  will for large  $P$  either increase or decrease as

$$P^{\sum_l (r_l + 2) - 4(n-1)}.$$

Thus the integral over  $P$  will turn out to be divergent if

$$\sum_l (r_l + 2) - 4(n-1) \geq 0,$$

and convergent if

$$\sum_l (r_l + 2) - 4(n-1) < 0.$$

We shall call the number

$$\omega(G) = \sum_l (r_l + 2) - 4(n-1) \quad (29.12)$$

the *index of the diagram  $G$* .

Naturally, the convergence of the integral over  $P$  does not yet imply that the integral (29.11) converges as a whole.

Here a situation may arise similar to the one occurring when in the course of evaluating the integral



$$\int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dy \frac{x}{(y^2 + 1)^2}$$

the integral

$$\int \frac{Q^2 dQ \cos^2 \varphi d\varphi}{(Q^2 \sin^2 \varphi + 1)^2}$$

with respect to the radial variable  $\rho (\rho = x/\cos \varphi = y/\sin \varphi)$  converges, while the remaining integration over  $\varphi$  turns out to be divergent because of singularities at  $\varphi = 0, \varphi = \pi$ .

The index  $\omega(G)$  of the diagram may also be related to the *conditional rate of increase* with increasing momentum. To evaluate it, we multiply all the external momenta and masses by some number  $a$  and determine the factor by which the integral (29.11) will be multiplied, without taking regularization into account, and paying attention only to the highest power of  $a$ . It is readily seen that this factor is just equal to

$$a^{\omega(G)}. \quad (29.13)$$

Thus, the *index of the diagram is exactly equal to the above conditional rate of increase*.

We note that this rate of increase is referred to as conditional because the calculation leading to (29.13) is carried out in a purely formal fashion making a careful analysis of the convergence of the integral, and does not take into account logarithmically divergent factors.

We now note that by breaking up  $G$  into  $s$  generalized vertices

$$G = G_1 * G_2 * \dots * G_m, \quad (*_m)$$

we obtain

$$\omega(G) = \sum_{1 \leq i \leq m} \omega(G_i) + \sum_i (r_i + 2) - 4(m - 1), \quad (29.14)$$

We now consider the coefficient function  $d_{G_j}$ . In the momentum representation, it has the form

$$\delta(\sum p_j) Z_{G_j}(\dots p \dots),$$

where  $Z_{G_j}(\dots p \dots)$  is a certain polynomial in the components of  $p$ .

We shall see later that, in order to compensate the divergences in the  $T$ -product, it is sufficient to choose for  $Z_{G_j}(\dots p \dots)$  a polynomial of degree  $\omega(G_j)$ . It follows from (29.14) that, with this choice of  $d_{G_j}$ , neither the total degree of divergence nor the conditional rate of increase with increasing momentum will be made larger by applying the operation

$\Delta(G_1) \dots \Delta(G_m)$ , and that, consequently,  $\omega(G)$  is not increased by applying the operation  $R(G)$  as a whole.

We have already seen in the examples just considered that it is convenient for the analysis and the evaluation of integrals such as (29.11) to make use of the integral representation of causal functions (we shall refer to it below as the “ $\alpha$ -representation”)

$$\Delta^c(p) = Z(p) i \int_0^\infty e^{i\alpha(p^2 - m^2 + i\varepsilon)} d\alpha,$$

$$\text{reg } \Delta^c(p) = Z(p) i \int_0^\infty e^{i\alpha(p^2 - m^2 + i\varepsilon)} I(\alpha) d\alpha,$$

where

$$I(\alpha) = 1 + \sum_M c_M e^{-i\alpha(M^2 - m^2)}.$$

It is convenient to write the factor  $Z(p)$  in exponential form. To do this we shall make use of the relation

$$Z(p) = Z(-i\nabla_q) e^{ipq} |_{q=0}.$$

With the aid of this representation, the integration over the internal momenta in (29.11) is reduced to Gaussian quadratures:

$$\prod_l Z(-i\nabla_{q_l}) \int e^{i \sum_l \alpha_l p_l^2 + i \sum_l p_l q_l} \prod_{1 \leq q \leq n} \delta(\sum p + k_q) \prod_l dp_l |_{q_l=0} =$$

$$= \delta(\sum k) f(\dots k \dots \alpha \dots) \quad (29.15)$$

and only integrations over the variables  $\alpha$  remain:

$$J_M(k) = \delta(\sum k) \int_0^\infty d\alpha_1 \dots \int_0^\infty d\alpha_L f(\dots k \dots \alpha \dots) e^{-i \sum_l \alpha_l m_l^2 - \varepsilon \sum_l \alpha_l} \prod_{1 \leq l \leq L} I(\alpha_l). \quad (29.16)$$

After carrying out the integration over  $p_l$  we find that

$$f(\dots k \dots \alpha \dots) = F(\dots k \dots \alpha \dots) e^{i \sum_{a,b} A_{ab}(\dots \alpha \dots) (k_a k_b)}, \quad (29.17)$$

where  $F$  is a polynomial in  $k$  and a rational function of  $\alpha$ , which has nonintegrable poles when some of the  $\alpha$  vanish. Since the convergence of the integral (29.16) is guaranteed for

large  $\alpha$  by the factors  $\exp(-\epsilon \Sigma_l \alpha_l)$ , its possible divergences in the unregularized case are determined in the given representation just by the existence of these nonintegrable poles.

To determine the nature of the singularity, we introduce the new variables

$$\alpha_i = \lambda \xi_i, \quad \sum \xi_j = 1$$

and after fixing  $\xi_j \neq 0$  we evaluate the order of the pole at the point  $\lambda = 0$ . By going over in (29.15) to the new "momenta"

$$p_l \sqrt{\lambda} = P_l; \quad q_l \frac{1}{\sqrt{\lambda}} = Q_l; \quad k_q \sqrt{\lambda} = K_q,$$

we can write the left-hand side of (29.15) in the form

$$\lambda^{2(n-L)} \prod_i Z \left( -\frac{i \nabla_{Q_i}}{\sqrt{\lambda}} \right) \int \exp i \left\{ \sum_i (\xi_i P_i^2 + P_i Q_i) \right\} \prod_q \delta \left( \sum P + K_q \right) \prod_i dP_i |_{Q=0}.$$

Integrating with respect to  $P_l$ , and cancelling  $\delta(\Sigma k)$ , we obtain for small  $\lambda$

$$\begin{aligned} f(\dots k \dots a \dots) &= \lambda^{-\frac{\omega(G)}{2} - L} F'(\dots K \dots) e^{i \sum_{a,b} A'_{ab}(k_a k_b)} = \\ &= \lambda^{-\frac{\omega(G)}{2} - L} F'(\dots \sqrt{\lambda} k) e^{i \sum_{a,b} A'_{ab}(\xi)(k_a k_b)}. \end{aligned} \quad (29.18)$$

Hence the effective order of the pole in  $\lambda$  for  $\lambda = 0$  is, in fact, determined by the index  $\omega(G)$  if we take into account the determinant value

$$\frac{(\partial \alpha_1, \dots, \partial \alpha_L)}{(\partial \xi_1, \dots, \partial \xi_{L-1}, \partial \lambda)} = \lambda^{L-1}$$

*29.5. Structure of the Exponential Quadratic Form.* Consider the structure of the quadratic form

$$A(\dots \alpha \dots; \dots k \dots) \equiv A_{ab}(\dots \alpha \dots) k^a k^b. \quad (29.19)$$

First, we note that this does not depend on the form of the polynomials  $Z(-i \nabla_q)$ . We shall therefore assume, for the sake of simplicity, that  $Z = 1$ . This situation corresponds to the case of a purely scalar theory, and there is no difficulty in obtaining explicit expressions not only for the quadratic form  $A$ , but also for the pre-exponential factor  $F(\dots \alpha \dots)$  in (29.17). The corresponding formulas can be proved by induction.

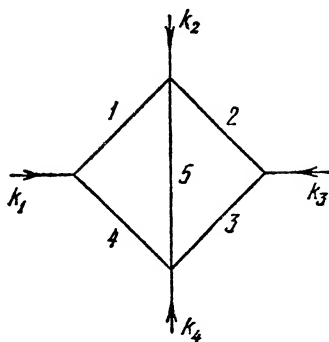


Fig. 25.

It will be useful to introduce a number of definitions at this point.\*

Suppose that, as before,  $G$  is a connected diagram with  $L$  internal lines and  $n$  vertices. Any connected subdiagram of  $G$  containing all the vertices of  $G$  but free of cycles will be called a *tree* of  $G$ . It is clear that each tree contains exactly  $n - 1$  lines. Similarly, a *two-tree* will be defined as any subdiagram of  $G$  containing all the vertices of the original diagram, but free of cycles, and consisting of exactly two connectedness components. It is clear, that each two-tree will contain  $n - 1$  lines. Finally, a *chord* of a tree (two-tree) will be defined as any line not belonging to this tree (two-tree). It is clear that  $L - n + 1$  chords correspond to each tree and  $L - n + 2$  chords to each two-tree. As an example, consider Fig. 25. The sets of all trees corresponding to this diagram is shown in Fig. 26 and its two-trees are shown in Fig. 27.

We now introduce a statement about the quadratic form  $A$  and the factor  $F$ , which is fundamental for this section:

$$A(\dots\alpha\dots, \dots k\dots) = \frac{Q(\dots\alpha\dots; \dots k\dots)}{D(\dots\alpha\dots)}, \quad (29.20)$$

$$F(\dots\alpha\dots) = \frac{i^{n+1}}{\pi^{2n-2L-2}} \frac{1}{D^2(\dots\alpha\dots)}, \quad (29.21)$$

where  $D$  and  $Q$  are homogeneous functions of the parameters  $\alpha$ , constructed in accordance with the formulas given below, which are universal for all diagrams.

*Construction of the form of  $D$ .* To each tree of the diagram we assign the product of  $L - n + 1$  parameters  $\alpha_j$  associated with all the  $L - n + 1$  chords of this tree. The sum of such products over all the trees for the diagram gives the form of  $D$ . Symbolically,

\*The decomposition of a diagram into "trees" and "chords" (which look a little artificial in the theory of the  $S$ -matrix) appears quite naturally in the case of perturbation-theory calculations of field operators in the Heisenberg representation, where the trees and chords correspond to different singular functions [see Källén (1951)].

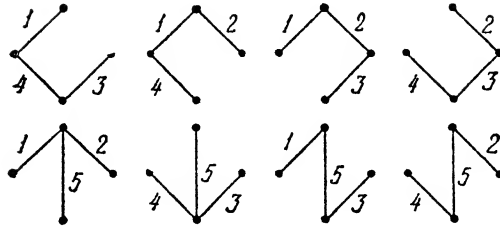


Fig. 26. The trees of Fig. 25.

$$D(\dots\alpha\dots) = \sum_{\text{over trees}} \left( \prod_{\text{over chords}} \dots\alpha_j\dots \right). \quad (29.22)$$

*Construction of the form of  $Q$ .* To each two-tree of the diagram we assign the product of  $L - n + 2$  parameters  $\alpha_j$  associated with all the  $L - n + 2$  chords of this two-tree. We then multiply the resulting product by  $(\Sigma k_i)^2$ , i.e., the square of the sum of all the momenta  $k_i$  in the vertex of one component of the chosen two-tree (because of consideration of momentum, it is immaterial which component). The form of  $Q$  is the sum of such expressions over all the possible two-trees of the diagram. Symbolically,

$$Q(\dots\alpha\dots; \dots k\dots) = \sum_{\text{over 2-tr.}} \left( \prod_{\text{over ch. of 2-tr.}} \dots\alpha_j\dots \right) \left( \sum_{\text{over comp. of 2-tr.}} k_{j_a} \right)^2. \quad (29.23)$$

It is clear that  $Q$  is a quadratic form in all the external momenta  $k$ .

Recalling (29.16), we arrive at the following general  $\alpha$ -parametric representation for regularized coefficient function of an arbitrary diagram in scalar theory:

$$J_M(k) = \frac{i^{n+1}\delta(\Sigma k)}{\pi^{2n-2L-2}} \int_0^\infty d\alpha_1 \dots \int_0^\infty d\alpha_L D^{-2}(\dots\alpha\dots) \times \\ \times \exp \left\{ i \frac{Q(\dots\alpha; \dots k\dots)}{D(\dots\alpha\dots)} - i \sum_l \alpha_l (m_l^2 - i\varepsilon) \right\} \prod_{1 \leq l \leq L} I(\alpha_l). \quad (29.24)$$

We now return to Fig. 25 and use it to illustrate the recipes defined by (29.32) and (29.33). According to Fig. 26, we have for  $D$ :

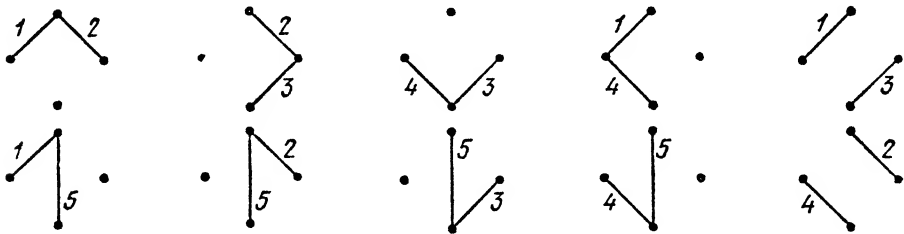


Fig. 27. Two-tree diagrams of Fig. 25.

$$D(\dots\alpha\dots) = \alpha_2\alpha_5 + \alpha_3\alpha_5 + \alpha_4\alpha_5 + \alpha_1\alpha_5 + \alpha_3\alpha_4 + \alpha_1\alpha_2 + \alpha_2\alpha_4 + \alpha_1\alpha_3.$$

and, similarly, for  $Q$ :

$$Q = k_4^2\alpha_3\alpha_4\alpha_5 + k_1^2\alpha_1\alpha_4\alpha_5 + k_3^2\alpha_2\alpha_3\alpha_5 + k_2^2\alpha_1\alpha_2\alpha_5 + (k_1 + k_2)^2\alpha_2\alpha_4\alpha_5 + \dots,$$

where we have written out only those terms that correspond to the two-trees shown in the upper half of Fig. 27.

Of course, the rules given by (29.22) and (29.23) lead to relatively complicated expressions for the quadratic form  $A$  and the pre-exponential factor  $F$  in the case of higher-order diagrams. Nevertheless, the representation given by (29.24) is very convenient in general studies because it exhibits many of the properties of the integrand, which are independent of the diagram order and structure.

However, in the ensuing analysis, we shall need only the following two statements regarding the quadratic form  $A = QD^{-1}$ :

$$\sum A_{ab}x^ax^b \geq 0, \quad \left(\sum x^2\right) \neq 0, \quad (29.25)$$

where  $x$  is a Euclidean vector and

$$\sum_{a,b} A_{ab}x^ax^b \leq \left(\sum_l \alpha_l\right) \left(\sum_a x^a\right)^2, \quad (29.26)$$

where the sum is evaluated over all vertices except one. The first of these is a trivial statement because the fact that  $Q$  and  $D$  are nonnegative can be seen directly from (29.22) and (29.23). To prove the second statement, we note that for each term in  $Q$ , the square of the sum of "external momenta"  $x$  satisfies the following inequality:

$$\left(\sum_{\substack{\text{over} \\ \text{comp.} \\ \text{of 2-tr.}}} x_j\right)^2 \leq \left(\sum_{\substack{\text{over} \\ \text{comp.} \\ \text{of 2-tr.}}} |x_j|\right)^2 \leq \left(\sum_a |x_a|\right)^2,$$

and, therefore,

$$Q(\dots\alpha\dots|\dots x\dots) \leq \left(\sum_a |x_a|\right)^2 \sum_{\substack{\text{over} \\ \text{2-tr.}}} \left(\prod_{\substack{\text{over ch.} \\ \text{of 2-tr.}}} \alpha_j\right).$$

On the other hand, each two-tree of the diagram can be obtained from some tree by removing one, say, the  $m$ -th, line. Consequently, for a given two-tree,

$$\left(\prod_{\substack{\text{over ch.} \\ \text{of 2-tr.}}} \alpha_j\right) = \alpha_m \left(\prod_{\substack{\text{over} \\ \text{chords}}} \alpha_j\right) \leq \left(\sum_l \alpha_l\right) \left(\prod_{\substack{\text{over} \\ \text{chords}}} \alpha_j\right).$$

Thus, it follows from the last two relationships that

$$Q \leq \left( \sum_a |x_a| \right)^2 \left( \sum_l \alpha_l \right) \sum_{\substack{\text{over} \\ \text{tr.}}} \left( \prod_{\substack{\text{over} \\ \text{ch.}}} \alpha_f \right),$$

where the sum is evaluated over a set of trees of the diagram. If we extend the summation over all the trees, we merely strengthen the inequality. We therefore have

$$Q \leq \left( \sum_a |x_a| \right)^2 \left( \sum_l \alpha_l \right) D,$$

and hence (29.26) follows for  $A = Q/D$ .

We now consider the changes that will take place in the coefficient functions of the  $T$ -products as a result of the operation  $\Delta(G_j)$  in the integral “ $\alpha$ -representation” (29.15) which we have adopted. We recall that in accordance with its definition, the operation  $\Delta(G_j)$  in the  $x$ -representation consists of replacing the part of the product  $\Pi_l \Delta_l^c(x_a - x_b)$  that corresponds to the internal lines of  $G_j$  by the coefficient function of a quasilocal operator  $d_{G_j}(\dots x_a \dots)$  which, in the  $p$ -representation, has the form of a polynomial multiplied by a  $\delta$ -function:

$$\delta \left( \sum p \right) Z_{G_j}(\dots p \dots).$$

In going over to the integral representation (29.15), we shall use for the reduction of the polynomial  $Z_G$  to exponential form the relation

$$Z_G(\dots p \dots) = Z_G(\dots -i\nabla_q \dots) e^{i\Sigma(p_a q_a)} \Big|_{q=0}, \quad (29.27)$$

which is a natural generalization of the formula for  $Z_l(p)$  from  $\Delta_l^c(p)$ . It is evident that the quadratic form in the exponential will depend neither on  $Z_G$  nor on  $Z_l$ . Thus, by applying the operation  $\Delta(G_j)$  we shall again obtain an expression of type of (29.17) not containing any variables corresponding to the internal lines of  $G_j$ .

The “new” quadratic exponential form is obtained from the “old” one by setting equal to zero all the  $\alpha_l$  corresponding to the internal lines of  $G_j$ . At the same time the momenta  $k_a$  which correspond to the vertices of the diagram combined into a generalized vertex  $G_j$  automatically drop out of the quadratic form. It is also evident that the preceding  $\alpha_l$  will also not appear in the factor  $F(\dots k \dots \alpha \dots)$  multiplying the exponential.

To analyze the singularity, we shall now reduce all the  $\alpha_l$  by the same factor by setting

$$\alpha_l = \xi_l \lambda, \quad \sum \xi_l = 1, \quad \xi_l > 0, \quad \lambda \rightarrow 0,$$

and determine the maximum effective order of the pole at  $\lambda = 0$  in the resulting integral. It was established earlier that the effective order of the pole is equal to half the conditional

rate of increase plus unity, and that the conditional rate of increase is not altered by the operation  $\Delta(G_j)$ . Consequently, in the present case, the effective order of the pole is

$$\frac{\omega(G)}{2} + 1.$$

Similarly, it may be concluded that the effective pole order is not changed by the operation  $\Delta(G_1) \dots \Delta(G_m)$ .

**29.6. Choice of the Operation  $\Delta(G)$ .** Having established the above important property, we now undertake the specific choice of the operation  $\Delta(G)$ . We shall first investigate the case where the masses  $m_i$  of the particles of all the fields under consideration are greater than zero. As we have just shown, the effective pole order of the expression  $\Delta(G_1) \dots \Delta(G_m) J_M(k)$  is equal to  $\omega(G)/2 + 1$  in the  $\alpha$ -representation.

On the other hand, it follows from (29.16) and (29.18) that each differentiation with respect to the components of the momenta  $k_a$  lowers the pole order by  $\frac{1}{2}$ . Therefore, by taking the partial derivative of the  $N$ th order with respect to the components  $k_a$

$$\frac{\partial^N J_M(\dots k \dots)}{\partial^{N_1} k_1 \dots \partial^{N_m} k_m} \quad (\sum N_i = N),$$

we obtain an expression in which the effective pole order will be reduced by  $N/2$  and will turn out to be equal to  $(\omega(G) - N + 2)/2$ . On choosing  $N = \omega(G) + 1$ , we obtain in the integrand the factor  $d\lambda \cdot \lambda^{-1/2}$ . However, since all our functions are rational functions of the variables  $\alpha$ , they must also be rational with respect to  $\lambda$ . Therefore, the factor  $\lambda^{-1/2}$  is in fact absent.

We now note that if from the function  $J(\dots k \dots)$  we subtract the sum of all the leading terms of its expansion into Maclaurin's series up to terms of order  $\omega(G)$  inclusive, then the remaining term

$$J(\dots k \dots) - \{J(\dots k \dots)\}_{\omega(G)}$$

may be expressed in accordance with the well-known Schlömilch formula in the form of an integral over partial derivatives of order  $\omega(G) + 1$ . Thus, in the remaining term the effective pole order at  $\lambda = 0$  is equal to zero.

We now define the operation  $\Delta G$  by the following recurrence relation:

$$\Delta(G) = 1, \quad (29.28a)$$

where the generalized block  $G$  coincides with the ordinary vertex and

$$\Delta(G) = -M(G) \sum_{2 \leq m \leq n; (*_m)} \Delta(G_1) \dots \Delta(G_m) \quad (29.28b)$$



in other cases. The symbol  $(*_m)$  under the summation sign reminds us about the decomposition of  $G$  into  $m$  generalized vertices [see the formula before (29.14)].

The operation  $M(G)$  is defined by

$$M(G) [\delta (\sum k) F(k)] = \delta (\sum k) \{F(k)\}_{\omega(G)}, \quad (29.29)$$

where  $G$  is a strongly connected divergent generalized block and  $M(G) = 0$  in all other cases. Using (29.28a), we can now rewrite (29.9) in the following form:

$$R(G) = \sum_{1 \leq m \leq n; (*_m)} \Delta(G_1) \dots \Delta(G_m). \quad (29.30)$$

It follows from these formulas that

$$R(G) J_M(k) = \{1 - M(G)\} J'_M(k),$$

where

$$J'_M(k) = \sum_{2 \leq m \leq n; (*_m)} \Delta(G_1) \dots \Delta(G_m) J_M(k)$$

and, since the singularity in  $J'_M$  is no higher than in  $J_M$ , we conclude that  $R(G)J_M$  does not, in general, have a singularity at the point  $\lambda = 0$ .

It follows that the application of the last operation  $\Delta(G)$  removes the singularities as all the  $\alpha$  are proportionally reduced to zero. Since, at the same time, the operations  $\Delta(G_1) \dots \Delta(G_m)$  compensate the singularities in the region where only some of the  $\alpha$  tend to zero, there are, in general, *no singularities* in  $R(G)J_M$ . This last conclusion constitutes an important theorem whose proof has been given by Parasyuk (1925) [see also Bogolyubov and Parasyuk (1955a, b, 1956, 1957), Hepp (1966), and Anikin, Zav'yalov, and Polivanov (1973)].

The above method of constructing the operation  $\Delta(G)$  is arbitrary to the extent that the choice of the center of expansion at the point  $k = 0$  is arbitrary. The operation  $\Delta(G)$  must therefore be generalized.

To do this, consider a system of finite polynomials  $Z'_G(\dots k \dots)$  of degree not higher than  $\omega(G)$  and such that the expressions

$$\delta (\sum k) Z'_G(\dots k \dots)$$

are the momentum representations of the coefficient functions of certain covariant Hermitian quasilocal operators  $\Lambda'(\dots x \dots)$ . The most general expression for  $\Delta(G)$  can be obtained by defining it for the strongly-connected generalized vertices as the sum of the previously introduced operation (29.28b) and the operation of adding the polynomial  $Z'_G$ , i.e.,

$$\Delta(G) = -M(G) \sum_{2 \leq m \leq n; (*_m)} \Delta(G_1) \dots \Delta(G_m) + Z(G), \quad (29.28c)$$

where  $Z(G)$  is the operator for the corresponding replacement.

The most general expression for  $S_n$  corresponding to (29.28c) differs from that obtained earlier by the inclusion of the covariant Hermitian operators  $\Lambda'_n$ . However, the addition of the  $\Lambda'_n$  to  $S_n$  can be taken into account by modifying the effective interaction Lagrangian. It follows that the change in the recipe for constructing the integrable coefficient functions is equivalent to the addition of new counterterms to the effective Lagrangian. We may therefore suppose that the  $T'$ -product is always defined, for example, by (29.28c), but there is an arbitrariness in the choice of the Lagrangian. This arbitrariness lies in the fact that terms corresponding to quasilocal operators such as (21.37), which are connected with generalized strongly connected vertices of  $G$  with nonnegative index  $\omega(G)$ , can always be included in  $\mathcal{L}(x)$ .

### §30. Structure of the $R$ -Operation

*30.1. Factorization of the  $R$ -Operation.* Because of their recurrent nature, the formulas given by (29.28) are inconvenient for practical operations with the diagrams. We shall therefore transform them to a simpler form and then, in the next section, we shall obtain an explicit expression for  $\lim_{M \rightarrow \infty} R(G) J_M(k)$ , which will be a generalization of the  $\alpha$ -representation (29.24) to the case of divergent diagrams [see Shcherbina (1964), Zav'yalov and Stepanov (1965), and Zav'yalov (1975)].

Consider the general situation where the polynomials  $Z'_G$  corresponding to "finite renormalization" and, consequently, the operators  $Z(G)$  in (29.28), are, in general, nonzero. Let  $G_1, G_2, \dots, G_m$  represent arbitrary generalized blocks of the diagram, and  $A(G_2)$  either the operator  $M(G_i)$  or the operator  $Z(G_i)$ . The "three-point product"  $:A(G_1) \dots A(G_m):$  will then be defined as the quantity satisfying the following rules:

(a) If among the  $G_1, \dots, G_m$  there is at least one pair  $G_i, G_j$  of partially intersecting generalized blocks (i.e., blocks not contained in each other but having common vertices), then  $:A(G_1) \dots A(G_m):$  is assumed equal to zero.

(b) If among the  $G_1, \dots, G_m$  there is a pair  $G_i, G_j$  such that  $G_j$  is contained in  $G_i$  and  $A(G_i) = Z(G_i)$ , then  $:A(G_1) \dots A(G_m):$  is again assumed equal to zero.

(c) In all other cases,  $A(G_1) \dots A(G_m)$  is assumed equal to the ordinary product of all the cofactors  $A(G_{i_1}) \dots A(G_{i_m})$ , taken in the "natural" order. The natural order is taken to be the order in which the fact that  $G_j$  is contained in  $G_i$  implies that  $A(G_i)$  lies to the left of  $A(G_j)$ .

We can now formulate the following result:

$$R(G) = :[1 - M(G_1) + Z(G_1)] \dots [1 - M(G_k) + Z(G_k)]:, \quad (30.1)$$

where  $G_1, \dots, G_k$  are all the divergent strongly connected generalized blocks of the diagram  $G$ .

Thus, to express  $R(G)$  directly in terms of the operators  $M$  and  $Z$ , we must remove the brackets on the right-hand side of (30.1) and, from the resulting sum, subtract all components satisfying conditions (a) and (b) and perform the "natural" rearrangement of the factors in the remaining components. In particular, if the diagram does not contain partially intersecting divergences, and all  $Z(G_i) = 0$ , then (30.1) reduces to the ordinary product

$$R(G) = [1 - M(G_1)] \dots [1 - M(G_k)]. \quad (30.2)$$

We must now prove (30.1). We begin by noting that if  $G_i$  does not coincide with a single vertex, then

$$\Delta(G_i) = [Z(G_i) - M(G_i)] \prod_k [1 - M(G_{ik}) + Z(G_{ik})], \quad (30.3)$$

where the product is evaluated over all the  $G_{ik}$  strictly contained in  $G_i$ . To show this, it will be sufficient to verify that the operators (30.3) satisfy the recurrence relations (29.28c).

Using the properties of the "three-point product," we rewrite (30.3) in the form

$$\Delta(G_i) = -M(G_i) \prod_k [1 - M(G_{ik}) + Z(G_{ik})] + Z(G_i). \quad (30.4)$$

Consequently, we must now establish that the "three-point product" on the right-hand side of (30.4) is identical with the sum on the right-hand side [of formula (29.28c)] applied to  $G_i$ .

For the sake of brevity, we now substitute  $B(G) \equiv Z(G) - M(G)$ , and evaluate the brackets in the three-point product

$$: \prod_k [1 + B(G_{ik})] : \quad (30.5)$$

and from the resulting sum we remove all terms containing the products  $B(G_{ik}) \dots B(G_{ij})$  with partially intersecting  $G_{ik}$ ,  $G_{ij}$ . Each of the remaining terms defines a certain decomposition of the vertices  $G_i$  into nonintersecting groups of vertices  $G_{i1}, \dots, G_{im}$ . It is precisely each such term that specifies some equivalence relation between the vertices of the diagram  $G_i$ , where two vertices are regarded as equivalent if the corresponding term contains a factor  $B(G_{ik})$  such that the generalized block  $G_{ik}$  contains these vertices. In this respect, the vertex equivalence classes are in fact the groups  $G_{i1}, \dots, G_{im}$ .

On the other hand, for each decomposition  $G_i = G_{i1} * \dots * G_{im}$  of the diagram into nonempty nonintersecting groups of vertices  $G_{i1}, \dots, G_{im}$  we can find terms in the sum (30.5) which generate precisely this decomposition. These terms are such that the maximum cofactors  $B(G_{ik})$  in them are precisely the  $B(G_{ik})$  corresponding to the groups  $G_{ik}$  in the given decomposition [the cofactor  $B(G_{ik})$  will be called the maximum cofactor if the corresponding term does not contain other  $B(G_i)$  such that  $G_{ik}$  are contained in  $G_{ij}$ .]

The product (30.5) can therefore be treated as a sum over all the possible decompositions  $G_i = G_{i1} * \dots * G_{im}$  ( $G_{ik} \neq G_i$ ). The given decomposition  $G_i = G_{i1} * \dots * G_{i,k-1} * G_{ik} * \dots * G_{im}$ , where  $G_{i1}, \dots, G_{i,k-1}$  are single-vertex generalized blocks and  $G_{ik}, \dots, G_{im}$  contain not less than two vertices, then corresponds to the term

$$: B(G_{ik}) \dots B(G_{im}) \left\{ \prod_{jk} [1 + B(G_{ik, jk})] \dots \prod_{jm} [1 + B(G_{im, jm})] \right\} :,$$

where each of the generalized blocks  $G_{ir, j}$  is strictly contained in  $G_{ir}$ . In accordance with the rules of "three-point ordering," the last expression can be written in the form

$$\Delta(G_{ik}) \dots \Delta(G_{im}),$$

where  $\Delta(G_{ir})$  has the form given by (30.3). Since, in addition,  $\Delta(G_{ir}) = 1$  for single-vertex generalized blocks, we find that

$$: \prod_k [1 - M(G_{ik}) + Z(G_{ik})] := \sum_{\substack{2 \leq m \leq n \\ (G_i = G_{i1} * \dots * G_{im})}} \Delta(G_{i1}) \dots \Delta(G_{im}).$$

We have thus verified that the recurrence relations (29.28c) are satisfied for the above operators  $\Delta$ , and this proves (30.3).

Now consider the expression for  $R'(G)$ :

$$R'(G) = : \prod_i [1 + B(G_i)] :, \quad (30.6)$$

where the product is evaluated over all the generalized vertices  $G_i$  of the diagram  $G$ , containing not less than two vertices. If we repeat the argument used above in connection with the vertex equivalence classes, we again conclude that  $R'(G)$  is the sum over all the possible decompositions of  $G$  into nonempty nonintersecting groups of vertices  $(G_1, \dots, G_m)$  ( $G = G_1 * \dots * G_m$ ). The given decomposition  $G = G_1 * \dots * G_{k-1} * G_k * \dots * G_m$ , where  $G_1, \dots, G_k$  are the single-vertex points, will correspond to the term

$$: B(G_k) \prod_k [1 + B(G_{ik})] : \dots : B(G_m) \prod_m [1 + B(G_{im})] : = \\ = \Delta(G_k) \dots \Delta(G_m) = \Delta(G_1) \dots \Delta(G_{k-1}) \Delta(G_k) \dots \Delta(G_m),$$

in this sum. In other words,  $R'(G)$  is identical with the right-hand side of (29.30) and, consequently,  $R'(G) = R(G)$ . This proves (30.2).

**30.2. Parametric Representation.** The relationship given by (30.1) enables us to present the renormalized coefficient function of the diagram  $R(G)J_M(k)$  in the form of the  $\alpha$ -parametric integral such as (29.16) and (29.24). We shall confine our attention to the case where the normalization is such that all the  $Z(G_i) = 0$  in (30.1).

Consider, to begin with, a scalar diagram which has a nonnegative index  $\omega(G)$  but does not contain divergent generalized blocks that do not coincide with the diagram itself. It is clear that in this situation, the limit of the integral in (29.24) as the regularizing masses turn to infinity will not exist because the factor  $D^{-2}(\alpha)$  contains a singularity for  $\alpha \rightarrow 0$ . In accordance with (30.1), instead of  $J_M(k)$  we must consider the function  $R(G)J_m(k) = [1 - M(G)]J_M(k)$ . We have

$$R(G) J_M(k) = \frac{i^{n+1} \delta(\sum k)}{\pi^{2(n-L-1)}} \int_0^\infty \prod_l \{I(\alpha_l) d\alpha_l\} \frac{e^{-i \sum \alpha_l (m_l^2 - i\epsilon)}}{D^2(\alpha)} \left[ \exp i \frac{Q(\alpha, k)}{D(\alpha)} - \sum_{v=0}^{\omega(G)/2} \frac{i^v}{v!} \left( \frac{Q(\alpha, k)}{D(\alpha)} \right)^v \right]. \quad (30.7)$$

We now recall that each term of  $D(\alpha)$  includes exactly  $A - n + 1$  parameters  $\alpha_j$ , i.e.,

$$D(\kappa^2 \alpha) = \kappa^{2L-2n+2} D(\alpha).$$

Similarly,

$$Q(\kappa^2 \alpha, k) = \kappa^{2L-2n+4} Q(\alpha, k).$$

Therefore,

$$\frac{\kappa^{4(L-n+1)}}{D^2(\kappa^2 \alpha)} e^{i \frac{Q(\kappa^2 \alpha, k)}{D(\kappa^2 \alpha)}} = \frac{1}{D^2(\alpha)} e^{i \kappa^2 \frac{Q(\alpha, k)}{D(\alpha)}}. \quad (30.8)$$

The last factor in the integrand of (30.7) is now none other but the residual term (for  $\kappa = 1$ ) of the Taylor expansion of the left-hand side of (30.8) around the point  $\kappa = 0$ . In view of this, we write

$$R_G J_M(k) = \frac{i^{n+1} \delta(\sum k)}{\pi^{2(n-L-1)}} \int_0^\infty \prod_l \{I(\alpha_l) d\alpha_l\} \hat{R}_\kappa^{(\omega)} \times \\ \times \left[ \kappa^{4(L-n+1)} \frac{1}{D^2(\kappa^2 \alpha)} \exp \left\{ i \frac{Q(\kappa^2 \alpha)}{D(\kappa^2 \alpha)} - i \sum_l \kappa^2 \alpha_l (\mu_l^2 - i\epsilon) \right\} \right], \quad (30.9)$$

where  $\mu_l = m_l \kappa$  and the application of  $\hat{R}_\kappa^{(\omega)}$  to the function  $f(\kappa)$  is defined by

$$\hat{R}_\kappa^{(\omega)} f(\kappa) = f(1) - f(0) - \dots - \frac{1}{\omega!} \left( \frac{d^\omega f(\kappa)}{d\kappa^\omega} \right)_{\kappa=0}. \quad (30.10)$$

It is now clear that to obtain a correct form for the  $\alpha$ -parametric integrals for  $R(G)J_M(k)$  we must replace all the  $\alpha_l$  in the integrand of (29.24) (except for those in  $I(\alpha_l)$ ) with  $\beta_l = \kappa^2 \alpha_l$ . Moreover, we must replace all the masses  $m_l$  by  $\mu_l = m_l/\kappa$ , multiply the integrand by  $\kappa^{4(L-n+1)}$ , and apply the operator  $\hat{R}_\kappa^{(\omega)}$  to it. Recalling (30.2), we conclude that for an arbitrary scalar diagram without partially overlapping divergences, the renormalized expression is obtained by the successive application of this type of procedure to all the divergent generalized blocks, since this expression can be obtained from diagrams such as (30.9) and the resulting integrations over the internal momenta do not involve the parameters  $\alpha_j$ . Moreover, it turns out that the same rule is also valid for diagrams with overlapping divergences, since the "extra" terms which distinguish the ordinary product such as (30.2)

from the "three-point product" will automatically vanish in the  $\alpha$ -parametric integral. Before we write out the corresponding general formula, let us represent the operator  $\hat{R}_\kappa^{(\omega)}$  in the integral form:

$$\hat{R}_\kappa^{(\omega)} = \frac{1}{\omega!} \int_0^1 d\kappa (1-\kappa)^\omega \left( \frac{d}{d\kappa} \right)^{\omega+1}. \quad (30.11)$$

To summarize: the coefficient function of an arbitrary renormalized scalar diagram containing divergent generalized blocks  $\Gamma_2, \dots, \Gamma_k$  with indices  $\omega_1, \omega_2, \dots, \omega_k$ , respectively, has the form

$$\begin{aligned} R(G) J_M(k) = & \frac{i^{n+1} \delta(\Sigma k)}{\pi^{2(n-L-1)} \omega_1! \dots \omega_k!} \int_0^\infty \prod_l \{I(\alpha_l) d\alpha_l\} \times \\ & \times \prod_{1 \leq p \leq k} \left\{ \int_0^1 d\kappa_p (1-\kappa_p)^{\omega_p} \left( \frac{\partial}{\partial \kappa_p} \right)^{\omega_p+1} \kappa_p^{4(L_p-n_p+1)} \times \right. \\ & \times \left. \frac{1}{D^2(\beta)} \exp \left\{ i \frac{Q(\beta, k)}{D(\beta)} - i \sum_l (\mu_l - i\varepsilon) \beta_l \right\} \right\}. \end{aligned} \quad (30.12)$$

Here  $\beta_j = \alpha_j$  if the  $j$ th line does not enter any of the divergent generalized blocks  $\Gamma_p$ , and  $\beta_j = \kappa_q^2 \dots \kappa_r^2 \alpha_j$  if the  $j$ th line does enter the divergent generalized blocks  $\Gamma_q, \dots, \Gamma_r$ . Similarly,  $\mu_l = m_l$  if the line  $l$  does not enter any of the  $\Gamma_p$  and  $\mu_l = m_l(\kappa_q \dots \kappa_r)^{-1}$  if the  $l$ th line enters  $\Gamma_q, \dots, \Gamma_r$ .

The forms  $Q$  and  $D$  are constructed (now out of the parameters  $\beta_j$  and not  $\alpha_j$ ) in accordance with the same rules (29.22) and (29.23) as before.

In precisely the same way, we can write down the parametric representation for the diagrams containing spinor lines. To do this, in the integrand of the nonrenormalized amplitude (29.16)

$$J_M(k) = \delta\left(\sum k\right) \int_0^\infty \prod_l \{I(\alpha_l) d\alpha_l\} f(\dots k \dots \alpha \dots) e^{-i \sum \alpha_l (m_l^2 - i\varepsilon)}$$

we must use the above rules to replace all the parameters  $\alpha_l$  (other than those in  $I(\alpha_l)$ ) with  $\beta_l$ , replace all the masses  $m_l$  by  $\mu_l$ , introduce the factors  $\kappa_p^{4(L_p-n_p+1)}$ , and apply the operators  $\hat{R}_{\kappa_p}^{(\omega_p)}$  defined by (30.11) to the integrand. We can also readily modify the representation (30.12) so that it corresponds to nonzero points of subtraction in the external momenta of the divergent generalized vertices [see Zav'yalov (1975)].

In (30.12) we can now directly pass to the limit as  $M \rightarrow \infty$  for fixed  $\varepsilon > 0$ . After differentiating with respect to  $\kappa$  in the integrand (30.12),  $\lim_{M \rightarrow \infty} R(G) J_M$  will be given by a finite sum of absolutely convergent integrals in the parameters  $\alpha$  and  $\kappa$ .

Here it is necessary to emphasize the arbitrariness in the choice of the operation  $R(G)$ . Indeed, in accordance with (29.25), the operation  $M(G)$  amounts to the subtraction from  $F$  of a Maclaurin polynomial of degree equal to the conditional rate of increase, which turns out to be sufficient for the removal of infinities. It is quite evident that if, for example, we were to define the operation  $M(G)$  by the subtraction from  $F$  of a Maclaurin polynomial of degree  $n > \omega(G)$ , we would also arrive at expressions containing no divergences. It may be shown that the arbitrariness connected with the choice of  $n$  actually amounts to a change in the Lagrangian. The minimum values  $n = \omega(G)$  that we have chosen lead to counterterms which differ least in their operator structure from the terms of the initial Lagrangian. However, such a choice is not inevitable.

*30.3. Transition to the Limit  $\epsilon \rightarrow 0$ .* We now consider the analytic properties of the expression obtained above and the possibility of going to the limit as  $\epsilon \rightarrow 0$ . By rotating the coordinate axes of the variables  $\alpha$  by  $90^\circ$  in the complex plane, i.e., by introducing the change of variables

$$\alpha_k = -i\beta_k, \quad (30.13)$$

we write the integral (30.12) in the form

$$\int_0^\infty d\beta_1 \dots \int_0^\infty d\beta_L F(\dots k \dots \beta) \exp \left( - \sum_l \beta_l m_l^2 + i\epsilon \sum_l \beta_l + \sum A_{ab}(\dots \beta \dots) k_a k_b \right). \quad (30.14)$$

By writing the exponent in the following form:

$$A = \sum A_{ab}(\dots \beta \dots) k_a^0 k_b^0 - \sum A_{ab}(\dots \beta \dots) k_a k_b - \sum \beta_l m_l^2 + i\epsilon \sum \beta_l,$$

we find with the aid of (29.20)

$$A < \sum_l \beta_l \left( \sum_a k_a^0 \right)^2 - \sum_l \beta_l m_l^2 + i\epsilon \sum_l \beta_l.$$

It is now evident that when the condition

$$\left( \sum_a k_a^0 \right)^2 < \min m_l^2 \quad (30.15)$$

is satisfied, the form  $A$  turns out to be negative, the integral (30.14) absolutely convergent, and the transformation (30.13) permissible. In the resultant expression one may go to the limit  $\epsilon \rightarrow 0$  and obtain a function which is analytic in the region defined by the inequality (30.15). This inequality may be written in a relativistically invariant form:

$$\left| \sum_a (k_a \xi) \right|^2 < \min m_i^2,$$

where  $\xi$  is a time-like unit vector directed towards the future:

$$\xi^2 = 1, \quad \xi^0 > 0.$$

In the procedure described above, we first made  $M \rightarrow \infty$  and then carried out the limiting transition  $\epsilon \rightarrow 0$ . However, the rotation (30.13) could have been carried out before going to the limit  $M \rightarrow \infty$ . We would have then obtained a linear combination of integrals such as (30.14), which contain the masses  $M_i$ . If the transitions to the limits  $M \rightarrow \infty$  and  $\epsilon \rightarrow 0$  are made simultaneously, the terms containing the masses  $M_i$  tend to zero, and we obtain the same results. Therefore, in the case under consideration, the point  $k_a = 0$  turns out to be regular, and expansions into Maclaurin's series are permissible. This is due to the fact that we have assumed that all the masses  $m_i$  are real and positive, and that

$$\min m_i^2 > 0.$$

When some of the masses are equal to zero, the point  $k_a = 0$  need not be regular. It may then turn out that it is not permissible in defining the operation  $\Delta(G)$  to carry out the expansion about the point  $k_a = 0$ , and the point about which the expansion is made may have to be chosen at a certain point  $(k^0, k)$  with a purely imaginary time component  $k^0 = i\omega$ . However, since the choice of a particular point in momentum space (with the exception of the point  $k = 0$ ) is not invariant from the point of view of four-dimensional rotations, the corresponding polynomial must in addition be averaged over the sphere

$$\omega^2 + k^2 = \mu^2, \quad \text{where } \mu^2 = \text{const.}$$

With such a choice of the operation  $\Delta(G)$ , the conclusions which we have reached above with respect to the properties of the coefficient functions obtained as a result of the operation  $R(G)$  will evidently remain valid, with the one difference that the function will be analytic only for points  $k$  with purely imaginary  $k^0$ .

We now assume that the momenta  $k$  are completely arbitrary. As long as the masses have finite purely imaginary negative additional terms  $-i\epsilon$ , the integrals will contain cutoff factors  $\exp(-\epsilon \Sigma_i \alpha_i)$ , and the functions will be regular. As  $\epsilon$  tends to zero, these functions will converge only in the improper sense. The limiting expressions obtained in this process, which represent the true coefficient functions of  $T'$ -products, turn out to be improper and may contain singularities for certain ranges of values of their arguments.

However, in view of their integrability properties, the operator integrals

$$\int T'(\mathcal{L}(x_1) \dots \mathcal{L}(x_n)) g(x_1) \dots g(x_n) dx_1 \dots dx_n$$

turn out to converge for sufficiently regular functions  $g(x)$  that fall off sufficiently rapidly at infinity, and difficulties arise in the case of the limiting transition  $g(x) \rightarrow 1$ : the



corresponding matrix elements of  $S(1)$  turn out to diverge. In such cases it is customary to speak of divergences of the *infrared catastrophe* type or of *resonance denominators*.

Singularities of the former type arise, as is well known, because of the failure of perturbation theory in the description of processes involving quanta of very low energy, and can be eliminated from the results by the Bloch-Nordsieck (1937) method, or by introducing as small constant playing the role of "photon mass" into the photon  $D_0^c$ -function (see §35.4 below).

Singularities of the second type arise, for example, when a high-order scattering process can be reduced for given momenta to simpler independent lower-order processes.

It is important to note that a divergence of one of these kinds arises in ordinary quantum mechanics whenever perturbation theory fails. We take this opportunity to emphasize that the condition of integrability of the operator functions  $S_n(x_1, \dots, x_n)$  guarantees only the absence of divergences specific for quantum field theory, i.e., "high-momentum divergences."

We have formulated the prescription for constructing integrable coefficient functions for the operators  $S_n(x_1, \dots, x_n)$ . We now note that the introduction of the regularized causal functions  $\Delta^c \rightarrow \text{reg } \Delta^c$  was purely auxiliary in our discussion and, essentially, was needed only to establish that the resulting expressions for  $S_n$  satisfied all the conditions imposed on the terms of the scattering-matrix expansion. In practice, for example, in the evaluation of the above coefficient functions, it is fully possible to operate with the "pure"  $\Delta^c$ -functions. If we then go over to the  $\alpha$ -representation, we can apply the operation  $R(G)$ , having excluded only a small region near the point  $\alpha = 0$  from the region of integration.

**30.4. An Illustration.** As an example of the application of the above general rules, consider the diagram  $G$  corresponding to the scalar meson theory (Fig. 28). This diagram contains three divergent vertices  $\Gamma_i$  (Fig. 29) for each of which  $\omega(\Gamma_i) = 0$ . The vertices  $\Gamma_{123}$  and  $\Gamma_{234}$  partially intersect. The diagram  $G = \Gamma_{1234}$  is therefore a diagram belonging to the second class. There are three decompositions containing divergent vertices:

$$G_1 * \Gamma_{23} * G_4, \quad \Gamma_{123} * G_4, \quad G_1 * \Gamma_{234},$$

where  $G_1$  and  $G_4$  coincide with simple vertices.

According to (29.9) and (29.28), we have

$$R(G) = (1 - M(G)) (1 + \Delta_{23} + \Delta_{123} + \Delta_{234}), \quad (30.16)$$

where, for the sake of simplicity, we have substituted  $\Delta(\Gamma_{23}) \equiv \Delta_{23}$ , and so on. Next, applying (29.28) to  $\Gamma_{123}$ ,  $\Gamma_{234}$ , and  $\Gamma_{23}$ , we obtain

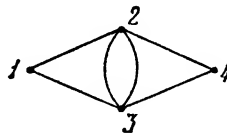
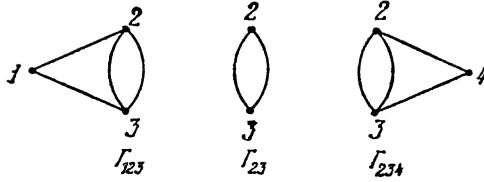


Fig. 28. Diagram of  $G$ .

Fig. 29. Divergent vertices in the diagram of  $G$ .

$$\Delta_{123} = -M_{123}(1 + \Delta_{23}), \quad \Delta_{234} = -M_{234}(1 + \Delta_{23}), \quad \Delta_{23} = -M_{23}.$$

The final result is therefore

$$R(G) = (1 - M_{1234})(1 - M_{123} - M_{234})(1 - M_{23}). \quad (30.17)$$

We now consider the counterterms that regularize the diagram of Fig. 28. Since for all the divergent vertices  $\omega(\Gamma_i) = 0$ , it follows that, according to (29.29), the operation  $(-M)$  reduces to the subtraction from  $\Gamma_i$  of the value of the corresponding coefficient function at the point where all the external (relative to  $\Gamma_i$ ) momenta are zero. This subtraction results in the final expression. As noted in §29.6, the point of subtraction is actually arbitrary, and this is reflected in the present case, in the fact that we can add an arbitrary constant to the residual expression (this is the generalized  $R$ -operation). Thus,

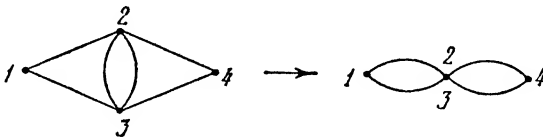
$$(1 - M_{\omega=0}) \delta \left( \sum k \right) F(\dots k \dots) = \delta \left( \sum k \right) \{ F(\dots k \dots) - F(\dots 0 \dots) + C \}.$$

It was shown in §29.5 that the subtracted term is equivalent to the contribution of (30.12) in the limit as  $\kappa_j \rightarrow 0$ . We now note that in this limit,

$$D_G(\beta) \rightarrow \kappa_i^{L_i - n_i + 1} D_{\Gamma_i}(\alpha) D_{H_i}(\beta), \quad A_G(\beta, k) \rightarrow A_{H_i}(\beta, k), \quad (30.18)$$

where the diagram  $H_i$  is obtained from the diagram  $G$  by shrinking the divergent vertex to a point (Fig. 30).

The counterterms in which we are interested can now be readily written out with the aid of (30.18). We shall do this graphically, without writing out long analytic expressions. The application of the operation  $(1 - M_{23})$  to the contribution of the diagram of Fig. 28 can, according to (30.18), be represented by the graphical operation shown in Fig. 31 where the diagram in the broken oval corresponds to a constant  $c$  and the value of its contribution at the point subtraction

Fig. 30. Operation of "contracting to a point" of the divergent vertex  $\Gamma_{23}$ .

$$(1 - M_{23}) \left( \text{diagram with vertices 1, 2, 3, 4} \right) = \left( \text{diagram with vertices 1, 2, 3, 4} - \text{diagram with vertices 1, 2, 3, 4 and a loop on 2,3} \right)$$

Fig. 31. Effect of the operation  $(1 - M_{23})$ .

$$\text{diagram with vertices 2, 3 in a dashed circle} = \text{diagram with vertices 2, 3 connected by a line} \Big|_{K=0} + \mathcal{C}, \quad (30.19)$$

which diverges logarithmically for  $M \rightarrow \infty$ . We now apply the operator  $(1 - M_{123} - M_{234})$  to the right-hand side of the relation shown in Fig. 31 and obtain the expression corresponding to Fig. 31.

Finally, the application of the last subtraction  $(1 - M_{1234})$  gives, similarly, the expression shown in Fig. 33.

It is readily verified that the application of the operator  $M_{123}M_{234}$  to the expression in Fig. 31 yields zero, so that instead of (30.17) we have

$$R(G) = (1 - M(G)) (1 - M(\Gamma_{123})) (1 - M(\Gamma_{234})) (1 - M(\Gamma_{23})). \quad (30.20)$$

This is a consequence of the general property (30.2) of the  $R$ -operation, mentioned above.

### §31. Analytic Properties of the Coefficient Functions in the Momentum Representation

*31.1. Analytic Properties of  $S_n$ .* In the preceding section it was established that the coefficient functions  $S_n$  of the scattering matrix in the momentum representation are analytic functions within the range of values of the momenta  $k_a$  for which

$$\sum_a k_a^0 < \min m_l \quad (31.1)$$

after carrying out the transition to the limit  $\epsilon \rightarrow 0$ .

We may therefore consider that the functions  $S_n$  obtained after the divergences have been removed are defined as analytic functions of their arguments in the region determined by the inequality (31.1), while the values of the functions  $S_n$  outside this region may be

$$\text{diagram with vertices 1, 2, 3, 4} - \text{diagram with vertices 1, 2, 3, 4 and a loop on 2,3 in a dashed circle} - \text{diagram with vertices 1, 2, 3, 4 and a loop on 2,3 in a dashed circle} + \text{diagram with vertices 1, 2, 3, 4 and a loop on 2,3 in a dashed circle} + \text{diagram with vertices 1, 2, 3, 4 and a loop on 2,3 in a dashed circle} + \text{diagram with vertices 1, 2, 3, 4 and a loop on 2,3 in a dashed circle} + \text{diagram with vertices 1, 2, 3, 4 and a loop on 2,3 in a dashed circle}$$

Fig. 32. Effect of the operation  $(1 - M_{123} - M_{234})(1 - M_{23})$ .



$$K^H(x, \dots, x_a, \dots, x_b, \dots) = \sum_i Q_i(x, \dots, x_a, \dots) Q'_i(\dots, x_b, \dots),$$

where  $Q_i$  and  $Q'_i$  are invariant under translation. By displacing the arguments of  $Q_i$  by an amount  $\xi$  such that  $x^0 + \xi^0 > x_b^0$ , we obtain, by virtue of (21.13),

$$K^H(x, \dots, x_a, \dots, x_b, \dots) = K^H(x + \xi, \dots, x_a + \xi, \dots, x_b, \dots) = 0.$$

Therefore, we shall in future consider only connected diagrams.

We go over to the momentum representation

$$\int K(x, x_1, \dots, x_n) \exp i \left( px + \sum_{j=1}^n p_j x_j \right) dx dx_1 \dots dx_n = \delta \left( p + \sum_j p_j \right) K(p_1, \dots, p_n),$$

where

$$K(p_1, \dots, p_n) = \int K(x, x_1, \dots, x_n) \exp i \left( \sum_j p_j (x_j - x) \right) dx_1 \dots dx_n \quad (31.4)$$

is a certain improper function of the momenta  $p_1, \dots, p_n$ .

*31.3. Analytic Properties of the Functions  $H_n$ .* Let us suppose that the variables  $p_l$  are complex and let us represent them in the form

$$p_l = v_l + i\Gamma_l,$$

where  $v$  and  $\Gamma$  are real four-vectors. Then in place of (31.4) we may evidently write

$$K(p_1, \dots, p_n) = \int K(x, x_1, \dots, x_n) g'(x_1) \dots g'(x_n) \exp i \left( \sum_l v_l (x_l - x) \right) dx_1 \dots dx_n,$$

where

$$g'(x_l) = \exp [-\Gamma_l (x_l - x)].$$

It may now be seen that if one chooses for  $\Gamma_l$  a four-vector directed along the positive time-axis:

$$\Gamma = (\Gamma^0, 0, 0, 0), \quad \Gamma^0 > 0,$$

then for  $x_l^0 \rightarrow \infty$  the function  $g$  will tend to zero exponentially, and since the integration is carried out over the region  $x_l^0 > x^0$  the integral (31.4) will turn out to be convergent and the function  $K(p_1, \dots, p_n)$  will turn out to be regular for  $p_l = v_l + i\Gamma_l$ . It is clear that the same will be true if for  $\Gamma_l$  we choose a time-like four-vector directed towards the future. In the preceding the function  $K(p_1, \dots, p_n)$  may be regarded for real values of  $p_l$  as the

improper limit of a regular function for  $\Gamma_l \rightarrow 0$ . In other words the function  $K(p_1, \dots, p_n)$  turns out to be analytic in the region

$$p_l = v_l + i\epsilon_l \Gamma,$$

where  $v_l$  is a real four-vector,  $\epsilon_l$  is a real positive parameter, while  $\Gamma$  is a time-like four-vector directed towards the future.

We shall now show that, when condition (31.1) is satisfied, the coefficient functions of the operators  $H_n$  will also be analytic and will coincide with the coefficient functions of  $S_{n+1}$ . It was shown in §30 that the function  $S_{n+1}$  is regular in the region defined by (31.1). Therefore, it is sufficient to show that the products (31.1) will give no contribution to  $H_n$  in this region.\* The corresponding coefficient functions have the form

$$\int Q(\dots, k_a - \sum \lambda, \dots) \delta(\sum k_a - \sum \lambda) Q'(\dots, k_b + \sum \lambda, \dots) \times \\ \times \delta(\sum k_b + \sum \lambda) \prod \{Z_{\alpha\beta}(\lambda_{\alpha\beta}) \theta(-\lambda_{\alpha\beta}^0) \delta(\lambda_{\alpha\beta}^2 - m_{\alpha\beta}^2) d\lambda_{\alpha\beta}\}.$$

From the conditions  $\lambda_{\alpha\beta}^0 < 0$  and  $\lambda_{\alpha\beta}^2 = m_{\alpha\beta}^2$  it follows that

$$-\sum \lambda_i^0 \geq m_i;$$

on the other hand we have

$$\sum |k| \leq \min m,$$

therefore one of the two requirements:

$$\sum k_a^0 - \sum \lambda^0 = 0, \quad \sum k_b^0 + \sum \lambda^0 = 0,$$

will always turn out not to be satisfied, the contribution of (31.3) will in fact turn out to be zero, and  $H_n$  will turn out to coincide with  $S_{n+1}$  which is analytic. Remembering that, by adding to the masses small purely imaginary negative terms (31.2), we obtained functions  $S_n$  analytic in the whole plane of the momentum variables, we arrive at the following prescription for the coefficient functions of the operator  $H_n$  in terms of the coefficient functions of the operator  $S_{n+1}$ .

Consider the operator  $S_{n+1}$  which corresponds to a certain (necessarily connected) diagram. The coefficient functions of this operator will be analytic for small momenta.

\*When the minimum mass is zero (the electromagnetic field), it may be taken equal to some small quantity  $\kappa$  for purposes of this section. Such an operation does not lead to any contradictions.

Analytic continuation of these expressions for large momenta with the aid of the replacement (31.2) will again lead us to the coefficient functions of the operator  $S_{n+1}$ . To obtain the coefficient functions of the operator  $H_n$ , we must carry out the analytic continuation for large momenta by the substitution

$$p_i^0 \rightarrow p_i^0 + i\Gamma^0. \quad (31.5)$$

We have considered the connection between the coefficient functions of the operators

$$S(g) \quad \text{and} \quad H(x; g) = i \frac{\delta S(g)}{\delta g(x)} \overset{\dagger}{S}(g).$$

As we pointed out in §20, in the case when the auxiliary functional argument is introduced not in the form of an “intensity of interaction”  $g$  but in the form of a classical external field  $u$ , the operator

$$i \frac{\delta S(u)}{\delta u(x)} \overset{\dagger}{S}(u)$$

will also possess the property of causality (20.31) in view of which its coefficient functions will be related in a similar manner to the coefficient functions of the matrix  $S(u)$ . It turns out that the analytic relationship between  $S(u)$  and  $i[\delta S(u)/\delta u(x)]\overset{\dagger}{S}(u)$  in the case of arbitrary introduction of the field  $u$  may be established for these operators as a whole without making use of their functional decomposition. This will be shown in Chapter X.

## §32. Classification of the Renormalizability of Different Theories

*32.1. Interactions of the First and Second Kinds.* As we have shown in the preceding section, the change in the effective interaction Lagrangian due to the introduction into it of counterterms depends on the presence of strongly connected diagrams  $G$  with non-negative index  $\omega(G)$ . Let us consider the relation between the structure of such diagrams and the structure of the corresponding counterterms in the  $x$ -representation.

If the above diagram unites  $n$  vertices and contains  $s$  external lines then the corresponding quasilocal operator will have the form

$$: u_{\alpha_1}(x_1) \dots u_{\alpha_s}(x_s) : Z \left( \dots \frac{\partial}{\partial x} \dots \right) \delta(x_1 - x_2) \dots \delta(x_{n-1} - x_n),$$

with the degree of the polynomial  $Z$  being equal to the index  $\omega(G)$  of the diagram. Integrating this over all the variables  $x_i$  except one, we obtain the counterterm of the Lagrangian. In carrying out these trivial integrations, the derivatives will be transferred from the  $\delta$ -functions to the field operators  $u$ , and the result of the integration will take the form of the normal product of a certain number of operator field functions and their derivatives. The total order of all the derivatives then turns out to be equal to the index  $\omega(G)$  of the diagram,

while the “degree” of the whole expression with respect to the operator functions turns out to be equal to the number  $s$  of external lines.

Therefore, if a given theory (completely determined by the fundamental (“initial”) term of the interaction Lagrangian and by the structure of the causal functions) leads to strongly connected diagrams of nonnegative index for which the numbers  $\omega(G)$  and  $s$  turn out to be bounded then, for the complete removal of all the divergences, such a theory will require the introduction of counterterms of a finite number of types. In speaking of the counterterm type we have in mind its operator type and the order of the derivatives of each field operator. In the opposite case, the number of different types of counterterms turns out to be infinite.

We shall now analyze the dependence of the indices  $\omega(G)$  on the number of external and internal lines of a diagram. To do this, we introduce the concept of the *index of a vertex*, which we define by

$$\omega_i = \frac{1}{2} \sum_{l_{\text{int}}} (r_l + 2) - 4, \quad (32.1)$$

where the sum is evaluated over all the internal lines entering the  $i$ th vertex. It is readily seen that the index of the diagram is expressed in terms of the indices of the vertex parts of the diagram in the following way:

$$\omega(G) = \sum_{1 \leq i \leq n} \omega_i + 4, \quad (32.2)$$

since each internal line enters simultaneously two different vertices. For a given vertex type, the index  $\omega_i$  will take on its maximum value  $\omega_i^{\text{max}}$  when all the lines entering the vertex turn out to be internal lines. If

$$\omega_i^{\text{max}} \leq 0, \quad (32.3)$$

then it follows from (32.2) that

$$\omega(G) \leq 4.$$

Conversely, if for some vertex types

$$\omega_i^{\text{max}} > 0, \quad (32.4)$$

then it is always possible to construct a diagram  $G$  containing a sufficiently large number of vertices of this type so that  $\omega(G)$  is larger than any number specified in advance. Thus, the index of a diagram either does not exceed four, or it may be made arbitrarily large.

Taking into account the fact that



$$\omega_l = \omega_l^{\max} - \frac{1}{2} \sum_{l_{\text{ext}}} (r_l + 2),$$

where  $l_{\text{ext}}$  are the indices of the external lines entering a given vertex, the dependence of  $\omega(G)$  on the number of external lines may be written in the form

$$\omega(G) = \sum_i \omega_i^{\max} + 4 - \frac{1}{2} \sum_{l_{\text{ext}}} (r_l + 2); \quad (32.5)$$

where the sum in the last term is evaluated over all the external lines of the given diagram.

Therefore, for (32.3), the number of external lines in a diagram with a positive index does not exceed four. In this case, both  $\omega(G)$  and  $s$  are bounded by the number 4, and the number of types of corresponding counterterms turns out to be finite and may be subjected to detailed classification. When (32.4) holds, both sums on the right-hand side of (32.5) may be made arbitrarily large for nonnegative  $\omega(G)$ . Both  $\omega(G)$  and  $s$  turn out to be unbounded and, in order to compensate for the divergences of increasing order, it is necessary to introduce counterterms with an increasing degree of "linearity" and an increasing number of derivatives. It is not possible to obtain a closed expression for the total effective Lagrangian.

In accordance with the above properties, the types of interaction may be divided into two classes: (a) interactions of the *first kind* (all  $\omega_i \leq 0$ ), and (b) interactions of the *second kind* (some  $\omega_i > 0$ ). The corresponding theories are said to be *renormalizable* and *nonrenormalizable*, respectively.

One essential qualification must be made in respect to the above definition. The point is that, in certain cases, the individual vertex factors may mutually compensate one another and thereby lower the effective value of  $\omega_i$ . Consider, for example, the interaction of the ordinary fermion field (spin  $\frac{1}{2}$ ) with a neutral vector meson field of the type

$$:\bar{\psi}(x) \gamma^\mu \psi(x) \varphi_\mu(x):. \quad (32.6)$$

Direct calculation of  $\omega_i$ , making use of the fact that the pairing of the vector field in the  $p$ -representation has the form

$$\frac{g^{\mu\nu} - k^\mu k^\nu / m^2}{k^2 - m^2}$$

and that, consequently, the  $r_l$  in this case are equal to 2, will evidently give us  $\omega_l = 1$  and will lead us to classify the Lagrangian (32.6) as nonrenormalizable. However, it may be shown that, in actual fact, this Lagrangian describes an interaction of the first type. In order to show this, following Stueckelberg (1938), we shall decompose the vector field into a transverse and longitudinal part:

$$\varphi_n(x) = \Phi_n(x) + \frac{1}{m} B;_n$$

with the corresponding pairings

$$\overline{\Phi_n(x)} \Phi_m(y) \sim i g^{mn}, \quad \overline{B;_n B;_m} \sim k_n k_m.$$

Hence it is clear that the renormalizability of the Lagrangian (32.6) is due to the longitudinal part of the scalar field. It can now be shown that the corresponding term in the interaction Lagrangian

$$\int : \bar{\psi}(x) \gamma^a \psi(x) B;_n(x) : dx,$$

has the structure (conserved current)  $\times$  (gradient), and does not contribute to matrix elements, so that the longitudinal field  $B$  is no longer present in the  $S$ -matrix, the effective values  $r_i$  and vertex indices  $\omega_i$  are reduced to zero, and the Lagrangian (32.6) turns out to be renormalized.

Similar questions of the "effective change of pairings" under the gauge-type canonical transformations will be specially discussed below after we have examined the conditions for the conservation of charge and for gauge invariance which we have not touched upon so far.

Without at present going into details of these very special cases of compensation of singularities that are due to a certain group of transformations, we shall now give a general classification of the simplest Lagrangians similar to those introduced in §8.

**32.2. Summary of Interactions of the First Kind.** In determining the corresponding indices  $\omega_i^{\max}$  we shall take into account the fact that, in accordance with the general structure of the commutation and causal functions, the degree of the polynomial  $r_l$  for the scalar field and for the zero-mass vector field (the electromagnetic field) with scalar coupling is equal to zero, for a spinor field of spin  $\frac{1}{2}$  is equal to unity, and for a vector field (with  $m \neq 0$ ) with scalar coupling is equal to two. In the case of gradient coupling, we find with the aid of (22.11) that  $r_l$  is equal to two for the scalar and for the electromagnetic fields, and to four for the vector field.

As it was previously noted, the number of external lines in strongly connected diagrams with a nonnegative index cannot exceed four. The maximum degree of linearity of an interaction Lagrangian of the first kind is therefore equal to four. From the formula

$$\omega_i^{\max} = \frac{1}{2} \sum_l (r_l + 2) - 4, \quad (32.7)$$

where the sum is evaluated over all the lines leaving a given vertex, we find that all four

lines must have the index  $r_l = 0$ , i.e., only the four-fold product of scalar fields and of electromagnetic fields

$$:\varphi_1\varphi_2\varphi_3\varphi_4:,\quad :\varphi_1\varphi_2 A_k A^k:,\quad :A_k A^k A_l A^l: \quad (32.8)$$

describes an interaction of the first kind. All other four-fold interactions, for example, the four-fold Fermi interaction of spinor operator functions

$$(\bar{\psi}_1 O \psi_2) (\bar{\psi}_3 O \psi_4) \quad (\omega_l = 2),$$

and the four-fold interactions including derivatives and vector field functions, for example,

$$:\varphi_1\varphi_2 \frac{\partial \varphi_3}{\partial x^k} A^k: \quad (\omega_l = 1), \quad :\varphi_1\varphi_2 U_k U^k: \quad (\omega_l = 2),$$

represent interactions of the second kind.

Cubic terms in a Lagrangian of the first kind may evidently have the following structure: (a) a product of three scalar and electromagnetic functions without derivatives:

$$:\varphi_1\varphi_2\varphi_3:,\quad :\varphi_1 A_k A^k: \quad (\omega_l = -1) \quad (32.9)$$

(the other combinations do not form a scalar); (b) a product of three scalar and electromagnetic functions with one first derivative:

$$:\varphi_1 \frac{\partial \varphi_2}{\partial x^k} A^k: \quad (\omega_l = 0); \quad (32.10)$$

the interaction of charged scalar mesons with the electromagnetic field (scalar electrodynamics) is of this type; (c) a product of one scalar, one vector, and one electromagnetic function:

$$:\varphi_\alpha \varphi_\beta^k A_k: \quad (\omega_l = 0); \quad (32.11)$$

and (d) a product of two spinor functions and one scalar or electromagnetic function:

$$:\bar{\psi}_\alpha M_{\alpha\beta} \psi_\beta \varphi:,\quad :\bar{\psi}_\alpha \Gamma_{\alpha\beta}^k \psi_\beta A_k: \quad (\omega_l = 0). \quad (32.12)$$

All the other cubic interactions, such as the interaction of a spinor field with the scalar field of the gradient coupling type

$$:\bar{\psi}_\alpha \Gamma_{\alpha\beta}^k \psi_\beta \frac{\partial \varphi}{\partial x^k}: \quad (\omega_l = 1),$$

lead, generally speaking, to nonrenormalizable theories.

The nine Lagrangians (32.8) and (32.12) listed above exhaust all possible types of interactions of the first kind, since the quadratic forms which satisfy equation (32.3) correspond to vertices that are entered by two lines. Such forms do not describe processes of mutual transformation of particles and, therefore, represent only the possible types of counterterms.

The division of Lagrangians into interactions of the first and second kind, which we have carried out above, is in fact not sufficiently consistent. It was made from the point of view of the convergence of the individual terms of the expansion of the  $S$ -matrix into a series in powers of the coupling constant. It is quite possible that, after the summation of such a series has been carried out, the analytic nature of the functions under investigation will be altered, and this will affect the above classification.

Without making use of perturbation theory we must therefore determine which interactions belong to the first kind and which to the second kind. In other words, the question arises as to what kind of local Lagrangian will enable us to construct a closed theory. This problem is important because, as we shall now show, there is an essential physical difference between theories of the first and of the second kind.

**32.3. Nature of Interactions of the Second Kind.** We shall now consider in greater detail the properties of interactions of the second kind. As we have already seen, among the infinite number of types of counterterms arising in such theories, there are groups of terms of the same operator type but with infinitely increasing orders of derivatives. Such series in fact represent expansions of certain nonlocal expressions and may therefore represent *nonlocal interactions*.

For example, the Lagrangian for the pseudovector meson-nucleon coupling

$$: \bar{\psi} \gamma^5 \gamma^k \psi \frac{\partial \varphi}{\partial x^k} :$$

requires the introduction into the effective Lagrangian of an infinite number of counterterms of the form

$$C_{\alpha_1}, \dots, \alpha_n : \bar{\psi} \gamma^5 \gamma^k \psi \frac{\partial^n}{(\partial x^{\alpha})^n} \left( \frac{\partial \varphi}{\partial x^k} \right) :,$$

whose sum may be regarded as the expansion of the nonlocal expression

$$\int : \bar{\psi}(x) \gamma^5 \gamma^k \psi(x) \frac{\partial \varphi(y)}{\partial y^k} : K(x-y) dy$$

into a series involving derivatives of ever increasing order of the function  $\varphi$ .

Thus, in the case of interactions of the second type, the localizability of the effective Lagrangian in fact disappears and the Lagrangian begins to depend on the behavior of the field functions not only within an infinitesimal neighborhood of the point  $x$ . It then turns out that, independently of the smallness of the coupling constant, higher-order terms become more important for sufficiently large momenta. Indeed, it follows from dimensional

considerations that the counterterm containing  $n$  derivatives is proportional to the factor  $l^n$  where  $l$  is a small parameter of the dimensions of length ("universal length") polynomially related to the coupling constant. In the  $p$ -representation, the derivatives  $\partial/\partial x$  are changed into momenta and we obtain an expansion in powers of

$$pl = l/\lambda, \quad (32.13)$$

where

$$\lambda = 1/p$$

is the Compton wavelength. For sufficiently large  $p$ , the quantity  $\lambda$  becomes comparable with the universal length  $l$  and the parameter (32.13) of the expansion of the nonlocal Lagrangian into a series involving derivatives ceases to be small. In this case, the universal length  $l$  is a characteristic of the physical "smearing-out" of the particle, and its appearance indicates the importance of the influence of the inner structure of the particle.

We conclude that the interaction Lagrangians of the second kind apparently represent "fragments" of nonlocalized interactions represented in localized form. For a consistent development of such theories it is necessary to start from the outset with a nonlocal Lagrangian which takes into account the internal structure of the elementary particles. From this it follows that there is a deep physical difference between interactions of the first and the second kind.

The following question is then of interest: do all the interactions which occur in nature belong to the first kind? We emphasize that a sufficiently well-founded answer to this question may be obtained only by comparing experimental results with calculations that are not based on perturbation theory.

The above hypothetic description of the nonlocal nature of interactions of the second kind is not indisputable. There is at least one further alternative that is quite widely discussed in the literature. It is connected with the possibility of nonanalytic quantum field-theoretic expansions in terms of the coupling constant [i.e., violation of hypothesis (b) in the discussion of the expansion property in §20.1]. The sufficiently strong nonanalyticity at  $g = 0$  may produce the result that "the degree of divergence" of the coefficients in the expansion in powers of  $g$  will rapidly increase.

Details can be found in the papers by Redmond and Uretski (1958) and Bogolyubov, Logunov, and Shirkov (1959). As an example, we reproduce the model expression for the contribution of fermion-fermion scattering in the theory based on the Fermi-type fourfold interaction of the form

$$T(p^2, G) = G + G^2 p^2 K(p^2, G),$$

$$K(p^2, G) = \int_{4m^2}^{\infty} \frac{dz}{(z - p^2 - i\epsilon) [(1 + Gz \ln(z/m^2))^2 + \pi^2 G^2 z^2]}. \quad (32.14)$$

This integral in a way corresponds to the sum of contributions of iterations of one-loop diagrams (see below Fig. 49). It converges and can be evaluated explicitly:

$$K(p^2, G) = \frac{1}{1 + Gp^2 \ln \frac{p^2}{m^2}} - \frac{p^2}{Gp_0^2 (p^2 - p_0^2) \left[ 1 + \ln \left( \frac{p_0^2}{m^2} \right) \right]}. \quad (32.15)$$

where  $p_0^2$  is a root of the equation

$$1 + Gp_0^2 \ln \frac{p_0^2}{m^2} = 0. \quad (32.16)$$

However, if we attempt to expand the integrand into a series in powers of  $G$ , and integrate term by term, we obtain the series of increasing powers

$$K \sim \ln \Lambda + G\Lambda \ln^2 \Lambda + G^2 \Lambda^2 \ln^3 \Lambda + \dots \quad (32.17)$$

where  $\Lambda$  is the square of the momentum cutoff or Pauli-Villars mass, which simulates sufficiently well the structure of the main divergences of the perturbation-theory series in theories of the second kind.

This property reflects the nonanalyticity of the integral (32.14) in  $G$  at  $G = 0$ . Analysis of (32.15) and (32.16) shows that the singularity in the integral  $K$  has the form

$$K \sim \frac{1}{1 - \ln Gm^2}. \quad (32.18)$$

The function

$$f(x) = \frac{1}{1 - \ln x} \quad (32.19)$$

has the properties

$$f(0) = 0, \quad f'(0) = f''(0) = \dots f^{(n)}(0) = \dots = \infty,$$

which are reflected in the structure of the series given by (32.17).

For comparison, we note that constructions analogous to (32.14) can be developed for theories of the second kind as well. The corresponding model expression for the photon propagator in spinor electrodynamics is

$$D(k^2, \alpha) = -\frac{d(k^2, \alpha)}{k^2}; \quad d(k^2, \alpha) = 1 - \frac{\alpha}{3\pi} L(k^2, \alpha),$$

$$L(k^2, \alpha) = \int_{4m^2}^{\infty} \frac{dz}{(z - k^2) \left[ \left( 1 - \frac{\alpha}{3\pi} \ln \frac{z}{m^2} \right)^2 + \frac{\alpha^2}{9} \right]}. \quad (32.20)$$

The integral  $L$  is similar to  $K$  in that it converges, but it does not admit of an expansion in powers of  $\alpha$ . The divergence of the coefficients of the expansion has a purely logarithmic structure. The singularity at  $\alpha = 0$  is of the form

$$L \sim \exp\left(-\frac{3\pi}{\alpha}\right).$$

The function

$$\varphi(z) = \exp\left(-\frac{1}{z}\right),$$

differs from the function  $f$  in (32.19) in that it is zero at  $z = 0$  and so are all its derivatives.

**32.4. Specification of a Theory of the First Kind by a Finite Number of Constants.** Let us now perform a more careful analysis of the various possibilities for constructing theories of the first kind. As we established earlier, the possible types of terms in the effective interaction Lagrangian are restricted to a finite number of terms of fourth, third, and second order in the field operators. Therefore, in theories of the first kind, the choice of the interaction Lagrangian reduces to the choice of a *finite number* of “coupling constants.” The number of independent constants is reduced by demanding Hermitian and gauge invariance, the conservation of electrical and baryon charges, and so on. To define the theory completely, it is, of course, necessary to specify the masses of the particles in the absence of the interaction. Any theory of the first kind is thus completely characterized by a finite set of numbers, i.e., the particle masses and the coupling constants.

In our variant of the theory, the fundamental quantity in addition to the Lagrangians of the free particles, which fix the properties of the noninteracting field, is the interaction Lagrangian  $\mathcal{L}(x)$ . As we have seen, by choosing  $\mathcal{L}(x)$  in the usual way, and by redefining the  $T$ -product, it is possible to obtain integrable expressions for the terms of the  $S$ -matrix:

$$S_n(x_1, \dots, x_n) = i^n T'(\mathcal{L}(x_1) \dots \mathcal{L}(x_n)).$$

Although the set of prescriptions for the construction of the operator  $T'$  is not unique, the arbitrariness contained in it corresponds to a finite change in the coupling constants of the interaction Lagrangian, i.e., to the addition to the initial Lagrangian  $\mathcal{L}(x)$  of finite counterterms of the same type as the terms occurring in the permissible effective Lagrangian of the first kind. Therefore, in order to make the calculations completely unambiguous, it is necessary to specify  $\mathcal{L}(x)$  in a manner appropriate to the fixed set of prescriptions for the construction of the  $T'$ -product.

On the other hand, as we have seen, a completely equivalent result is obtained if instead of redefining the  $T$ -product one uses the usual  $T$ -product (with some form of auxiliary regularization in the intermediate stages of the argument), but instead of  $\mathcal{L}(x)$  one employs a certain effective interaction Lagrangian  $\mathcal{L}_{\text{eff}}$  which contains, in addition to the initial interaction Lagrangian, divergent counterterms which compensate divergences in

the usual  $T$ -products. From the point of view of the  $S$ -matrix, the situation is as if, in place of the initial complete Lagrangian

$$\mathcal{L}_0(x) + \mathcal{L}(x)$$

we had the Lagrangian

$$\mathcal{L}_{\text{tot}} = \mathcal{L}_0 + \mathcal{L}_{\text{eff}} .$$

The “masses” and “charges” occurring in it, i.e., the coefficients of the appropriate operator combinations, are found to diverge; nevertheless the observable quantities calculated with their aid (including masses, charges, and so on) have finite values.

In this way, we arrive at the so-called “*renormalization*” point of view where, to obtain finite values for the observables being calculated, one introduces into the Lagrangian infinite “bare” masses, charges, and so on. It is then said that the removal of infinities from the theory is attained by means of a “renormalization” of the fundamental constants. The renormalizing factors contain divergent expressions.

However, this point of view *is not carried through sufficiently consistently*. This is so because the introduction of the additional counterterms into  $\mathcal{L}_{\text{tot}}$  does *not* reduce to the renormalization of the fundamental quantities (for example, in the case of spinor electrodynamics, the clearly gauge-noninvariant terms involving the photon mass\* and the term  $(\partial A)^2$ , the fourfold terms in meson theories, and so on). Moreover, as we have seen in §28, the structure of the counterterms changes when we investigate the generalized scattering matrix  $S(g)$  and turns out to depend on the behavior of the function  $g(x)$ . As will be shown later, it is the matrix  $S(g)$  that determines the effective Hamiltonian of the system. It may therefore be said that counterterms have to be introduced for the regularization of the Schroedinger equation, and differ from those that are needed for the regularization of the  $S$ -matrix.

In view of all this, we do not adhere to the “renormalization” terminology, but regard the procedure of introducing counterterms as a formal device which guarantees that the results of the calculations are finite.

\*When the nongauge-invariant method of regularization is used similar to the one employed in §27.2.





# Chapter 6

## APPLICATIONS OF THE GENERAL THEORY OF REMOVAL OF DIVERGENCES

### §33. Spinor Electrodynamics. I. General Form of Counter Terms

33.1. *Types of Divergent Diagrams and Furry's Theorem.* As a second example of interacting quantized wave fields we shall examine in detail the important case of spinor electrodynamics, i.e., the system of the vector electromagnetic field interacting with the fermion spinor field with the interaction Lagrangian given by

$$\mathcal{L}(x) = e : \bar{\psi}(x) \gamma^\mu \psi(x) A_\mu(x) : = e : \bar{\psi}(x) \hat{A}(x) \psi(x) :. \quad (23.3)$$

We recall that, in accordance with the structure of (23.3), two fermion lines and one photon line meet at each vertex of the Feynman diagrams, while the causal functions of the fields involved have the form

$$i \overline{A_\mu(x) A_\nu(y)} = -g^{\mu\nu} D_0^c(x-y) = g^{\mu\nu} \frac{1}{(2\pi)^4} \int \frac{dk e^{ik \cdot (x-y)}}{k^2 + i\epsilon}, \quad (24.3)$$

$$i \overline{\psi(x) \bar{\psi}(y)} = S^c(x-y) = \frac{1}{(2\pi)^4} \int \frac{m + \hat{p}}{m^2 - p^2 - i\epsilon} e^{-ip \cdot (x-y)} dp. \quad (24.4)$$

The degree of the polynomial  $P$  in the numerator of the causal function is therefore equal to zero for the photon line, and to unity for the fermion line. The maximum vertex index

$$\omega_i^{\max} = \frac{1}{2} \sum_l (r_l + 2) - 4, \quad (32.7)$$

turns out to be equal to zero, so that the Lagrangian (23.3) belongs to the renormalizable type.

We now carry out the classification of divergent diagrams based on (32.5). We note first that, since the maximum vertex index  $\omega_l^{\max}$  is zero, the index of the diagram  $\omega(G)$  does not depend on the number of vertices and turns out to depend only on the number and on the nature of the external lines:

$$\omega(G) = 4 - \frac{1}{2} \sum_{l_{\text{ext}}} (r_l + 2). \quad (33.1)$$

As was pointed out in §32, the maximum number of external lines in divergent diagrams cannot exceed four. It follows from (33.1) that the only diagram of this nature in spinor electrodynamics is the diagram with four external photon lines. The index  $\omega(G)$  of this diagram turns out to be zero.

We now turn to diagrams with three external lines. Because the spinor lines are continuous (see §23), the number of external spinor lines is always even. It is therefore sufficient to examine diagrams with three external photon lines, and also those with two external fermion lines and one photon line. The total contributions to the matrix elements due to diagrams with an *odd number of external photon* lines in the absence of external spinor lines will be *equal to zero* on the basis of Furry's theorem, given later in this section.

The diagram with two external spinor lines and one photon line has an index  $\omega(G)$  equal to zero, the diagram with two external spinor lines has an index equal to unity, and that with two external photon lines has an index equal to two. Diagrams with one external fermion line do not occur because the fermion line must be continuous, while diagrams with one photon external line are forbidden by Furry's theorem.

This completes the enumeration of the divergent diagrams. Before turning to the investigation of the corresponding quasilocal operators, we shall give a proof of Furry's theorem which was mentioned before. We note first of all that the general classification of divergent diagrams which we have made does not take into account the symmetry properties of the system and its invariance under different transformations. These properties lead to considerable restrictions on the possible types of diagram and, as we shall see later, establish interrelationships between the structure of the regularizing quasilocal operators corresponding to different divergent diagrams. For example, the continuity of spinor lines noted above is essentially a manifestation of the property of conservation of electric charge for the fermions (refer to §8 for the corresponding transformation).

An important restriction on the possible types of diagram is imposed by the property of *charge invariance*, i.e., invariance under a change in the sign of electric charges in processes whose initial and final states contain no electrically charged fermions. Such processes are described by diagrams whose external lines are all photon lines. Charge invariance obviously leads, in this case, to a change in the sign of the charges of virtual fermions in intermediate states. The motion of these virtual particles is described by closed spinor cycles in Feynman diagrams.

*Furry's theorem* (1937) consists of the assertion that the matrix elements corresponding to diagrams containing at least one odd closed spinor cycle mutually cancel out. Let us consider such a diagram  $G$  where  $L$  is an odd closed cycle (Fig. 34). It is evident that the matrix element corresponding to this diagram will consist of the sum of two terms, one of which corresponds to motion along  $L$  in clockwise direction, while the other corresponds to the motion of the charge in the opposite direction. We shall now show (the proof is due to Feynman (1949a)) that the foregoing terms differ only in their signs so that their sum is therefore zero.

The factor in the matrix element which corresponds to the closed cycle  $L$  containing  $n$  vertices has the following form\* according to Feynman's rules:

$$\begin{aligned} \text{Tr } [\gamma S^c(1-2) \gamma S^c(2-3) \dots \gamma S^c(n-1)] = \\ = \sum_{\alpha, \dots, \nu} \{ \gamma_{\alpha\beta} S_{\beta\gamma}^c(1-2) \gamma_{\gamma\delta} S_{\delta\epsilon}^c(2-3) \dots \gamma_{\mu\nu} S_{\nu\alpha}^c(n-1) \}. \end{aligned} \quad (33.2)$$

We now make use of the fact that the whole spinor-field formalism and, in particular, the relation defining the Dirac matrices

$$\gamma^m \gamma^n + \gamma^n \gamma^m = 2g^{mn},$$

as well as the values of the traces of the products of any arbitrary number of matrices, are invariant under the following replacement:

$$\gamma \rightarrow -\overset{\text{T}}{\gamma}, \quad \text{i.e.,} \quad \gamma_{\alpha\beta} \rightarrow -\gamma_{\beta\alpha}. \quad (33.3)$$

As a result of the transformation (33.3), the causal function

$$S_{\beta\gamma}^c(1-2) = \frac{1}{(2\pi)^4} \int \frac{(m + \hat{p})_{\beta\gamma}}{m^2 - p^2 - i\epsilon} e^{ip(2-1)} dp$$

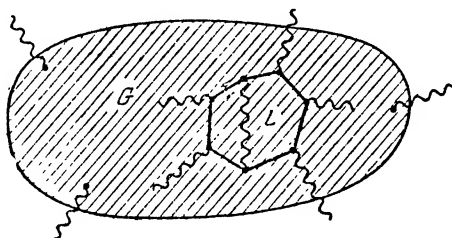


Fig. 34

\*Here, for example,  $S(n-1)$  is an abbreviated notation for  $S(x_n - x_1)$ , and so on.

takes on the form

$$\frac{1}{(2\pi)^4} \int \frac{(m - \hat{p})_{\gamma\beta}}{m^2 - p^2 - i\varepsilon} e^{ip(2-1)} dp = S_{\gamma\beta}^c(2-1).$$

Therefore by subjecting the expression (33.2) to the transformation (33.3) we obtain:

$$\begin{aligned} \text{Tr} [\gamma S^c(1-2) \gamma S^c(2-3) \dots \gamma S^c(n-1)] &= \\ &= (-1)^n \sum_{\alpha, \dots, \nu} \{\gamma_{\beta\alpha} S_{\gamma\beta}^c(2-1) \gamma_{\delta\gamma} S_{\varepsilon\delta}^c(3-2) \dots \gamma_{\nu\mu} S_{\alpha\nu}^c(1-n)\} = \\ &= (-1)^n \text{Sp} \{\gamma S^c(1-n) \dots \gamma S^c(3-2) \gamma S^c(2-1)\}. \end{aligned}$$

From this we see that due to the invariance of the trace under the transformation (33.3), the factor (33.2) for even  $n$  coincides with the expression

$$\text{Tr} \{\gamma S^c(1-n) \dots \gamma S^c(3-2) \gamma S^c(2-1)\}, \quad (33.4)$$

which corresponds to the contour  $L$  being described in the opposite direction, while for odd  $n$  the two differ in sign. In the latter case, the sum of the matrix elements corresponding to different directions around the odd cycle is zero, and the proof of Furry's theorem is complete.

We note that the above proof of Furry's theorem is of a formal nature, since we are dealing with singular products of nonregularized causal functions. However, it is readily seen that if  $S^c$  is replaced by  $\text{reg } S^c$ , the proof remains valid and the sum of the regularized matrix elements is zero. Therefore, by making such diagrams correspond to a quasilocal operator which is equal to zero

$$\Lambda_n(x_1, \dots, x_n) = 0, \quad (33.5)$$

we find that Furry's theorem holds for the complete coefficient functions  $S_n(x_1, \dots, x_n)$  after the divergences have been removed.

We emphasize that the condition given by (33.5), which guarantees the charge invariance of the theory after the removal of divergences is not, generally speaking, necessary.

By replacing it with some other condition we could have obtained some kind of a charge-invariant theory.

We also note that we have proved Furry's theorem for spinor electrodynamics by using the algebraic properties of Dirac matrices. As already noted, the physical basis of Furry's theorem is connected with the symmetry and the change of sign of electrical charges or, more precisely, with the operation of charge conjugation. It is therefore clear that Furry's theorem can be generalized to a broader class of interactions.

We now return to the diagrams under consideration. Since the diagrams containing an odd number of external photon lines have an odd number of vertices, they necessarily contain at least one odd spinor cycle and, therefore, the matrix elements of the diagrams

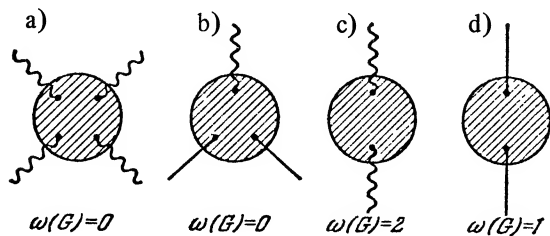


Fig. 35. (a),  $\omega(G) = 0$ ; (b),  $\omega(G) = 0$ ; (c),  $\omega(G) = 2$ ; (d),  $\omega(G) = 1$ .

with an odd number of external photon lines are always equal to zero. We have made use of this fact earlier by excluding diagrams with three and with one external photon lines.

Thus, by taking Furry's theorem into account, we obtain the four types of divergent diagram in spinor electrodynamics shown in Fig. 35. The shaded circles in these diagrams represent the internal parts of the diagrams containing an arbitrary number of vertices (even in a, c and d, and odd in b). We recall that in spinor electrodynamics, the degree of divergence does not depend on the number of vertices because the maximum vertex index is zero.

We now investigate the form of quasilocal operators corresponding to each of these divergent diagrams. For each even  $n$ , starting with  $n = 4$ , diagram (a) corresponds to the following term in  $\Lambda_n(x_1, \dots, x_n)$ :

$$A_n : A_k(x_i) A^k(x_j) A_l(x_m) A^l(x_p) : \delta(x_1 - x_2) \delta(x_1 - x_3) \dots \delta(x_1 - x_n). \quad (33.6)$$

In accordance with the value of the index of diagram (a), the degree of the differential polynomial in this expression is zero. For each odd  $n$ , starting with  $n = 3$ , diagram (b) corresponds to the quasilocal operator

$$B_n : \bar{\psi}(x_i) \hat{A}(x_k) \psi(x_j) : \delta(x_1 - x_2) \delta(x_1 - x_3) \dots \delta(x_1 - x_n) \quad (i \neq k, i \neq j, k \neq j). \quad (33.7)$$

For each even  $n$ , starting with  $n = 2$ , diagram (c) corresponds to the following term in  $\Lambda_n(x_1, \dots, x_n)$ :

$$: A^m(x_i) \left\{ C_n g^{mk} + D_n \frac{\partial}{\partial x_i^m} \frac{\partial}{\partial x_j^k} + E_n g^{mk} \frac{\partial}{\partial x_i^l} \frac{\partial}{\partial x_{jl}} \right\} \times \\ \times \delta(x_1 - x_2) \delta(x_1 - x_3) \dots \delta(x_1 - x_n) \} A^k(x_j) : \quad (i \neq j). \quad (33.8)$$

In accordance with the index of the corresponding diagram, the differential polynomial in the foregoing is of the second degree. The first-degree term is absent from the polynomial, since it is impossible to construct an invariant combination involving it. Finally, for each even  $n$ , starting with  $n = 2$ , diagram (d) corresponds to the operator

$$:\bar{\psi}(x_i) \left\{ \left[ F_n + G_n \frac{i}{2} \left( \frac{\hat{\partial}}{\partial x_i} - \frac{\hat{\partial}}{\partial x_j} \right) \right] \delta(x_1 - x_2) \delta(x_1 - x_3) \dots \delta(x_1 - x_n) \right\} \psi(x_j) :. \quad (33.9)$$

It may be seen from the preceding expressions that the quasilocal operators contain a relatively large number of arbitrary constants.

**33.2. Gauge Invariance of the Scattering Matrix.** However, so far, we have not taken into account the requirement of gauge invariance. In order to formulate this requirement, let us first investigate the infinitesimal gauge transformation of the electromagnetic-field potentials

$$A_m(x) \rightarrow A_m(x) + \frac{\partial f}{\partial x^m}, \quad (33.10)$$

where  $f$  is an arbitrary infinitesimal function. Under this transformation, the  $n$ th-order term of the scattering matrix  $S(1)$

$$\int S_n(x_1, \dots, x_n) dx_1 \dots dx_n, \quad (33.11)$$

acquires, if the linear dependence of  $S_n$  on the potentials  $A(x_i)$  with different arguments is taken into account, the increment

$$\sum_i \int \frac{\partial S_n}{\partial A_m(x_i)} \frac{\partial f}{\partial x_i^m} dx_1 \dots dx_n.$$

On integrating this expression by parts we find that it vanishes if the following identity holds:

$$\text{div}_i^{(4)} \frac{\partial S_n(x_1, \dots, x_n)}{\partial A(x_i)} \equiv \frac{\partial}{\partial x_i^m} \frac{\partial S_n(x_1, \dots, x_n)}{\partial A_m(x_i)} = 0. \quad (33.12)$$

It is readily seen that if this condition holds, then (33.11) also will not be changed by a finite gauge transformation. This is due to the fact that the coefficients of the leading powers of  $\partial f / \partial x$  in the increment of the integrand may be expressed in terms of the derivatives of (33.12) with respect to  $A(x_j)$ .

In view of the foregoing, we shall adopt (33.12) as the *condition for the gauge invariance of the theory*. We shall see later (Chapter 7) that, in addition to the invariance of  $S(1)$ , this condition also guarantees the validity of the differential conservation law for the electric current.

Let us now analyze the degree of arbitrariness in the choice of the coefficients  $A_n, B_n, \dots, F_n, G_n$  in (33.6)–(33.9), which remains after the condition of gauge invariance has been imposed in (33.12). For example, let us consider the vertex part of the operator function

$$\sum_{(i \neq j \neq k)} : \bar{\psi}(x_i) \Gamma^m(x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_n | x_k) \psi(x_j) A_m(x_k) :. \quad (33.13)$$

In accordance with the condition of gauge invariance, the vertex part must satisfy the condition

$$\text{div}_k^{(4)} : \bar{\psi}(x_i) \Gamma^m(x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_n | x_k) \psi(x_j) : = 0.$$

Comparing this with (33.7), we see that the function  $\Gamma^m$  is determined to within the term

$$b_n \gamma^m \delta(x_1 - x_2) \delta(x_1 - x_3) \dots \delta(x_1 - x_n). \quad (33.14)$$

It is evident that the introduction of  $b_n$  corresponds to a change in the coefficient  $B_n$ . However, it is readily seen that if the vertex part (33.13) satisfies condition (33.12), then the addition of the term (33.14) to  $\Gamma^m$  makes it lose this property since

$$\text{div}_k^{(4)} : \bar{\psi}(x_i) \gamma \psi(x_j) : \delta(x_1 - x_2) \dots \delta(x_1 - x_n) \neq 0 \quad \text{when } k \neq i, j.$$

Therefore, if the quasilocal operators (33.7) are chosen in such a way that the result of the subtraction procedure is gauge invariant, then this choice is quite unique, i.e., the ambiguity has been removed by the requirement of gauge invariance.

Similarly, one can establish the uniqueness of the coefficients  $A_n$  in (33.6) and of  $C_n$  in expressions (33.8). It is also evident that (33.9) does not contain the electromagnetic-field potentials and allows an arbitrariness in the choice of the coefficients  $F_n$  and  $G_n$ .

Let us also examine the ambiguity in the operator expressions

$$\sum_{(i \neq j)} : A_m(x_i) \prod^{mk}(x_1, \dots, x_n) A_k(x_j) :$$

with respect to terms of the same structure as the terms containing the coefficients  $D_n$  and  $E_n$  in (33.8). On substituting the term

$$\sum_{(i \neq j)} : A^m(x_i) \left\{ \left( d_n \frac{\partial}{\partial x_i^m} \frac{\partial}{\partial x_j^k} + e_n g^{mk} \frac{\partial}{\partial x_i^l} \frac{\partial}{\partial x_{jl}} \right) \delta(x_1 - x_2) \delta(x_1 - x_3) \dots \delta(x_1 - x_n) \right\} A^k(x_j) :$$

into condition (33.12), and on taking into account the properties of the derivative of the  $\delta$ -function

$$\frac{\partial}{\partial x} \delta(x - y) = - \frac{\partial}{\partial y} \delta(x - y),$$

we find the relation between  $d_n$  and  $e_n$ :



$$d_n + c_n = 0, \quad (33.15)$$

which leaves one degree of freedom in the choice of these coefficients.

Thus for each  $n$  there are only three nonunique constants  $F_n$ ,  $G_n$ , and, for example,  $D_n$ . From the point of view of the counter terms of the Lagrangian only the following three numbers (compare (21.44)) turn out to be nonunique:

$$D = \sum_{n=2}^{\infty} \frac{1}{n!} D_n, \quad F = \sum_{n=2}^{\infty} \frac{1}{n!} F_n, \quad G = \sum_{n=2}^{\infty} \frac{1}{n!} G_n.$$

We shall see later that these three numbers appear in the results in only two combinations and that the ambiguity is completely eliminated by the choice of the mass and of the charge of the spinor particle.

We must now show that, by choosing appropriate counterterms, we may actually obtain a gauge invariant theory. It is well known that the counterterms of the Lagrangian serve to describe the subtraction procedure used in our theory. We recall that this procedure consists of subtracting from the divergent expression a sufficient number of terms of its expansion into a Maclaurin's series about the point  $p = 0$ , and of adding to the result an arbitrary finite polynomial of a definite degree. On the basis of the results just obtained, this arbitrariness is reduced to two constants in terms such as (33.9) and to one constant in terms such as (33.8). We shall show that a gauge invariant theory is obtained as a result of this method of removing the divergences.

In the course of our proof, we use the following variant of regularization by auxiliary masses. We regularize the photon causal functions (24.3) in the usual way (see (27.9)), while the spinor causal functions are regularized not individually, but by replacing their products corresponding to the closed cycles

$$\text{Tr} [\gamma S^c(x_1 - x_2) \gamma S^c(x_2 - x_3) \dots \gamma S^c(x_{n-1} - x_n) \gamma S^c(x_n - x_1)], \quad (33.16)$$

by the expressions

$$\sum_M c_M \text{Tr} [\gamma S_M^c(x_1 - x_2) \gamma S_M^c(x_2 - x_3) \dots \gamma S_M^c(x_n - x_1)], \quad (33.17)$$

where  $S_M^c(x)$  is the fermion causal function with mass  $M$ :

$$S_M^c(x) = \frac{1}{(2\pi)^4} \int \frac{M + \hat{p}}{M^2 - p^2 - i\varepsilon} e^{-ipx} dp. \quad (33.18)$$

The preceding method of regularizing spinor causal functions represents one of the variants of the Pauli-Villars (1949) regularization.

We note in this connection that in accordance with the general properties of the above expressions undergoing regularization, which were established in Chapter 5, a change in the

method of auxiliary regularization does not affect the general structure of the final regularized expressions that are coefficient functions of the operators  $S'_n$ , which determine the scattering matrix containing no infinities.

We shall now show that Pauli-Villars regularization removes divergences from the matrix elements, which correspond to closed cycles. To do this, we consider the result of integrating (33.16) written in the momentum representation:

$$\int dp \frac{\text{Sp} [\gamma (m + \hat{p}) \gamma (m + \hat{p} + \hat{k}_1) \dots \gamma (m + \hat{p} + \hat{k}_{n-1})]}{(m^2 - p^2 - i\varepsilon) (m^2 - (p + k_1)^2 - i\varepsilon) \dots (m^2 - (p + k_{n-1})^2 - i\varepsilon)}. \quad (33.19)$$

For large  $p$ , the integrand behaves like  $p^{-n}$  and for  $n < 5$  the integral diverges like

$$\int \frac{p^3 dp}{p^n} \sim \int \frac{dp}{p^{n-3}}.$$

We note that, for large  $p$ , the integrand (33.19)

$$\frac{P_n(p) + m^2 P_{n-2}(p) + \dots + m^n}{P_{2n}(p) + m^2 P_{2n-2}(p) + \dots + m^{2n}} \simeq \left\{ \frac{P_n(p)}{P_{2n}(p)} + \frac{P_n(p)}{P_{2n}(p)} \left[ \frac{P_{n-2}(p)}{P_n(p)} - \frac{P_{2n-2}(p)}{P_{2n}(p)} \right] m^2 + \dots \right\}$$

(where  $P_i(p)$  is a polynomial of the  $i$ th degree in the components of  $p$ ) may be written as a power series in  $m^2$ , where the rate at which the coefficient of the  $k$ th power of  $m^2$  increases with  $p$  as  $p \rightarrow \infty$  is equal to  $p^{-n-2k}$ . It is therefore clear that if the coefficients  $C_M$  in (33.17) satisfy  $k$  relations of the form

$$\sum_M C_M = 0, \quad \sum_M C_M M^2 = 0, \quad \dots, \quad \sum_M C_M M^{2k-2} = 0,$$

then, as  $p \rightarrow \infty$ , the integrand corresponding to (33.17) behaves like  $p^{-2k-n}$ , and the Pauli-Villars procedure regularizes the above expressions together with their  $2k + n - 5$  first derivatives with respect to  $x_j$ . This property of the Pauli-Villars procedure could also have been established directly by going over to the "α-representation" of causal functions.

Thus, we shall introduce auxiliary masses only for photon lines and for closed spinor cycles. We shall, in general, not subject open spinor cycles to regularization. It is readily seen that it is sufficient to regularize the photon functions by one auxiliary mass, and to regularize closed spinor cycles by two auxiliary masses. Indeed, the maximum degree of divergence of diagrams with closed spinor cycles is equal to two (diagrams such as Fig. 35c) ( $n = 2$ ), while the introduction of two auxiliary masses lowers the degree by four. As a result of this procedure, the integral now converges as  $p \rightarrow \infty$  like

$$\int dp p^{-3}.$$

On the other hand, the maximum degree of divergence of diagrams with internal photon

lines is equal to unity (diagrams of the type of Fig. 35d). The introduction of one auxiliary mass into the photon function lowers the degree of divergence by two, and the integral now converges like

$$\int^{\infty} dp p^{-2}.$$

Therefore, there is no need to regularize open spinor cycles. We see from the foregoing that all the matrix elements turn out to be convergent for finite values of the auxiliary masses  $M_i$ . However, we recall that the regularization by means of auxiliary masses is only a technical auxiliary device employed at an intermediate stage of the argument, and that the actual removal of divergences is accomplished by the subtraction procedure. We shall therefore now apply to the regularized expression for the scattering matrix obtained above, the usual procedure of subtracting a Maclaurin's series in the momentum representation, at the same time adding three finite arbitrary constants. In view of the invariance of the results of this subtraction with respect to the method of introducing the auxiliary masses, which was established earlier, it only remains to establish the gauge invariance of the Pauli-Villars regularization and of the subtraction procedure that follows it.

We shall now demonstrate the gauge invariance of the regularized matrix  $S(1)$  before the subtraction procedure has been applied to it. In doing so, we shall use the fact that the set of  $n$ th-order diagrams with  $k(k \geq 1)$  external photon lines may be obtained from the  $(n-1)$ th-order diagram with  $(k-1)$  external photon lines by *inserting* an additional  $n$ th vertex  $\xi$  into any arbitrary external or internal spinor line. This process of insertion (which for the sake of brevity we shall refer to as the  $\xi$  process) at the same time establishes a graphical correspondence between the expressions

$$\sum_i \frac{\partial S_n(x_1, \dots, x_i)}{\partial A(\xi)} \bigg|_{\xi=x_i} \quad \text{and} \quad S_{n-1}(x_1, \dots, x_{n-1}).$$

We first examine the process of insertion into an internal spinor line. Inserting the  $\xi$ -vertex into the line  $x_1, x_2$ , we obtain Fig. 36. On evaluating the divergence (33.12) with respect to  $\xi$  of the factor

$$S^c(x_2 - \xi) \gamma S^c(\xi - x_1),$$

we find, after making use of the equations for  $S^c$ ,

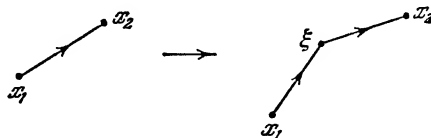
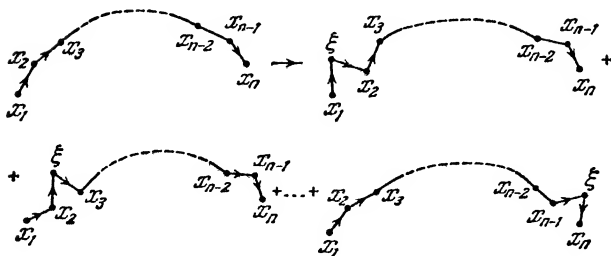


Fig. 36. The process of the insertion of a  $\xi$ -vertex into an internal fermion line.

Fig. 37. The  $\xi$ -process for internal lines of a fermion cycle.

$$\begin{aligned} \operatorname{div}_{\xi}^4 [S^c(x_2 - \xi) \gamma S^c(\xi - x_1)] &\equiv \\ &\equiv \frac{\partial}{\partial \xi_i} S^c(x_2 - \xi) \gamma^i S^c(\xi - x_1) = \frac{1}{i} (\delta(x_2 - \xi) - \delta(\xi - x_1)) S^c(x_2 - x_1). \end{aligned} \quad (33.20)$$

The  $\xi$ -process for the internal lines of a complicated fermion cycle may be represented by the diagram of Fig. 37. By evaluating the divergence of the sum of the terms corresponding to the right-hand side of Fig. 37 we see without difficulty with the aid of (33.20) that it is proportional to the term which corresponds to the diagram on the left-hand side of Fig. 37, with the proportionality constant equal to

$$\frac{1}{i} (\delta(x_n - \xi) - \delta(x_1 - \xi)). \quad (33.21)$$

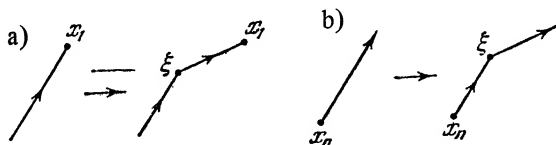
From this it follows that the preceding divergence vanishes for closed cycles (for  $x_1 = x_n$ ).

By applying this argument to more complicated diagrams consisting of internal photon lines and of an arbitrary number of closed and open fermion cycles, we arrive at the relation

$$\operatorname{div}_{\xi}^4 K_{n+1}(x_1, \dots, x_n | \xi) = i \left( \sum_a \delta(\xi - x_a) - \sum_b \delta(\xi - x_b) \right) K_n(x_1, \dots, x_n), \quad (33.22)$$

where the sums  $\Sigma_a$  and  $\Sigma_b$  are evaluated over all the vertices of the diagram which are entered by  $(x_a)$ , and from which there emerge  $(x_b)$  external fermion lines; here  $K_n(x_1, \dots, x_n)$  is the  $n$ th-order coefficient function, while  $K_{n+1}(x_1, \dots, x_n | \xi)$  is the  $(n+1)$ th-order coefficient function obtained from the preceding one by the  $\xi$ -process.

To complete the demonstration of gauge invariance of the regularized matrix  $S(1)$ , we also consider the  $\xi$ -process for the external (entering and emerging) fermion lines (Fig. 38). By evaluating the appropriate divergences we find that

Fig. 38. The  $\xi$ -process for external fermion lines.

$$\operatorname{div}_{\xi}^4 S^c(x_1 - \xi) \gamma \psi(\xi) = \frac{1}{i} \delta(x_1 - \xi) \psi(x_1), \quad (33.23)$$

$$\operatorname{div}_{\xi}^4 \bar{\psi}(\xi) \gamma S^c(\xi - x_n) = -\frac{1}{i} \delta(\xi - x_n) \bar{\psi}(x_n). \quad (33.24)$$

Comparison of these expressions with (33.22) shows that  $S_n(x_1, \dots, x_n)$  do in fact satisfy the condition of gauge invariance given by (33.12). Indeed, as we have just shown, the divergence of the factors describing closed cycles is equal to zero. The divergence of the operator corresponding to an open cycle is made up of the divergence of the sum of the  $\xi$ -insertions into the coefficient function and of insertions into the external lines, and also vanishes because (33.24) compensates the first term in (33.21), while (33.23) compensates the second term.

This completes the formal demonstration of gauge invariance of the matrix  $S(1)$  prior to regularization. However, we note that only the closed fermion cycles are subjected to the Pauli-Villars regularization with the mass having the same value in each of the additional terms in (33.17) for all the causal functions. Because of this, the foregoing additional terms also have zero divergence after the  $\xi$ -process, and we conclude that the regularized matrix  $S(1)$  satisfies the condition of gauge invariance after the application of the Pauli-Villars procedure. We note here that the above property of retaining gauge invariance is an important advantage of the Pauli-Villars regularization method, which we had in mind in departing from the method of introducing auxiliary masses adopted by us earlier.

We emphasize that (33.22) may be looked upon as the condition of gauge invariance for the coefficient functions. In actual fact, we have only verified that (33.22) is sufficient to ensure the gauge invariance of the scattering matrix (32.12). The formula given by (33.22) can usefully be rewritten in the momentum representation, and this can be done by taking translational invariance explicitly into account:

$$K_n(x_1, \dots, x_n) = \frac{1}{(2\pi)^{4(n-1)}} \int e^{-i \sum p_j x_j} \tilde{K}_n(p_1, \dots, p_n) \delta\left(\sum_j p_j\right) dp_1 \dots dp_n, \quad (33.25)$$

$$K_{n+1}(x_1, \dots, x_n | \xi) = \frac{1}{(2\pi)^{4n}} \int e^{-i \sum p_j x_j - i q \xi} \tilde{K}_{n+1}(p_1, \dots, p_n | q) \delta\left(\sum p_j + q\right) \times \\ \times dp_1 \dots dp_n dq. \quad (33.26)$$

Instead of (33.22) we have

$$q \tilde{K}_{n+1}(p_1, \dots, p_n | q) = \sum_b \tilde{K}_n(p_1, \dots, p_a + q, \dots, p_n) - \\ - \sum_a K_n(p_1, \dots, p_b + q, \dots, p_n). \quad (33.27)$$

This expression is the condition for the gauge invariance of the coefficient functions in the momentum representation. As in the case of the arguments  $x_1, \dots, x_n$  in (33.22), the only

essential arguments among the  $p_1, \dots, p_n$  in (33.27) are those that correspond to external lines. To verify this, it is sufficient to substitute  $K_n$  in  $S_n$  and integrate over all the  $x_i$  corresponding to internal vertices in the diagram.

**33.3. Ward's Identities.** Let us consider the simplest case when the diagram corresponding to  $K_{n+1}$  there is one external photon line, one entering electron line, and one emerging electron line (Fig. 35b). In this case,  $K_{n+1}$  is the vertex function

$$\tilde{K}_{n+1}(p_1, \dots, p_n | q) \rightarrow \Gamma_{n+1}(p_b, p_a | q),$$

and  $K_n$  is the electron self-energy

$$\tilde{K}_n(p_1, \dots, p_n) \rightarrow \Sigma_n(p_b, p_a).$$

Instead of (33.27), we now have

$$q\Gamma_{n+1}(p_b, p_a | q) = \Sigma_n(p_b, q + p_a) - \Sigma_n(p_b + q, p_a). \quad (33.28)$$

This formula establishes the connection between the vertex function of order  $n + 1$  and the electron self-energy in order  $n$ . It is called the *generalized Ward identity*. Differentiating in (33.28) with respect to  $q_m$ , and substituting  $Q = 0$ , we obtain the usual Ward identity

$$\Gamma_{n+1}^m(p, 0) = -\frac{\partial \Sigma_n(p)}{\partial p_m}, \quad (33.29)$$

where we have transformed to the notation of (27.4) and (28.3):

$$\Gamma_{n+1}(p, k) = \Gamma_{n+1}(p + k, -p | -k), \quad \Sigma_n(p) = \Sigma_n(p, -p).$$

Ward's identities for the corresponding complete vertex and self-energy functions can be obtained by summing over  $n$  in (33.28) and (33.29). We postpone this operation until §34.3. For the diagrams of Figs. 35a and c, which do not contain external electron lines on the right-hand side of (33.27), we obtain zero, so that the coefficient functions for the corresponding expressions

$$:A_p(x_1) A_q(x_2) A_r(x_3) A_s(x_4): \square^{pqrs}(x_1, \dots, x_4), \quad :A_p(x_1) A_q(x_2): \Pi^{pq}(x_1, x_2)$$

satisfy the conditions

$$(k_1)_p \tilde{\square}^{pqrs}(k_1, \dots, k_4) = 0, \quad (33.30)$$

$$k_p \Pi^{pq}(k, -k) = 0. \quad (33.31)$$

We have considered the application of the gauge invariance condition (33.27) to diagrams with nonnegative index  $\omega(G)$ , shown in Fig. 35. The application of this condition has led us

to the restrictions given by (33.30) and (33.31) and the coefficient functions for diagrams with two and four photon ends, and also the generalized Ward identity relating the vertex function and the electron self-energy. The resulting relationships couple the arbitrary constants that arise when the ultraviolet divergences are removed.

Other useful relationships can be obtained by applying gauge invariance (33.27) to diagrams with a large number of external ends (and negative indices  $\omega(G)$ ). For example, if we interpret  $K_{n+1}$  on the left-hand side of (33.27) as the coefficient function for Compton scattering, we obtain a connection of the type of generalized Ward's identity (33.28) between Compton scattering and the vertex function, and so on. The analog of the usual Ward identity (33.29), which is subsequently obtained by differentiating with respect to the photon momentum, leads to the so-called threshold theorems.

We now turn to the proof of the gauge invariance of the subtraction procedure. Since condition (33.12) is satisfied prior to the subtraction process, we must evidently demonstrate the gauge invariance of the quasilocal operators  $\Lambda_n$  that are being subtracted. If it should turn out that the operators being subtracted which correspond to the diagrams of Figs. 33a, c are individually gauge invariant, while the operators which correspond to the diagrams of Figs. 33b, d satisfy (33.27), then the gauge invariance of the subtraction procedure will have been established. We shall show that this is in fact the case.

Since we have already established (see §28) the condition for the gauge invariance of the subtraction procedure in the second and third orders in  $e$ , we shall carry out the proof by induction. Let us suppose that the quasilocal operators  $\Lambda_n$  satisfy the above requirements up to a certain odd

$$n - 1 = 2v - 1.$$

We consider an element of the scattering matrix  $S'_n = S_n - i\Lambda_n$  before the quasilocal operator  $\Lambda_n(x_1, \dots, x_n)$  has been subtracted from it. Since we have demonstrated the gauge invariance of the  $S$ -matrix before the subtraction process, and since we have assumed the gauge invariance of the quasilocal operators of lower orders,

$$\Lambda_v(x_1, \dots, x_v) \quad (v = 1, \dots, n - 1)$$

the expression for  $S'_n$  will also be gauge invariant. The specific  $n$ th-order divergences for even  $n$  correspond to the diagram shown in Figs. 33a, c, d.

By virtue of the gauge invariance of  $S'_n$ , its parts corresponding to these diagrams will also be invariant. The gauge invariance of the diagram of Fig. 35d is obvious, and the conditions for the invariance of the diagrams of Figs. 35a and c have the form given by (33.30) and (33.31).

Differentiating (33.30) with respect to  $(k_1)_p$  and assuming that  $k_1 = 0$ , we obtain

$$\tilde{\square}^{pqrs}(0, k_2, k_3, k_4) = 0. \quad (33.32)$$

This formula is *exceptionally important*. Since the divergence of Fig. 35a is only

logarithmic, the removal of the divergence reduces to the subtraction from  $\tilde{\square}$  of its value at zero external-photon momenta, i.e., the quantity

$$\tilde{\square}^{pqrs}(0, 0, 0, 0),$$

which by virtue of (33.22) is equal to zero. The quantity  $\square$  is thus found to be *convergent*, and it is in general unnecessary to remove the divergences from Fig. 35a.

Gauge invariance has thus led us to the conclusion that diagrams describing scattering of light by light must converge and, consequently, quasilocal operators of the form of (33.6) are absent. This is in accordance with the absence of gauge invariance in the case of expressions which are similar to (33.6) in that they directly contain the electromagnetic potential  $A_m$  (and not its derivatives that reduce to the electric and magnetic fields).

We now consider the two-photon diagram. Condition (33.31) together with relativistic covariance lead us to the conclusion that the polarization operator

$$\Pi^{pq}(k, -k) \equiv \Pi^{pq}(k)$$

has the transverse form

$$\Pi^{pq}(k) = (g^{pq}k^2 - k^p k^q) \pi(k^2). \quad (33.33)$$

Since the operator  $\Pi^{pq}$  is defined to within an arbitrary finite polynomial of the second degree in  $k$ , it is clear that this polynomial has the form

$$d_n(g^{pq}k^2 - k^p k^q), \quad (33.34)$$

and this is in accordance with (33.15).

Gauge invariance imposes no restrictions on the quasilocal operator  $\Lambda_n$  corresponding to the electron self-energy diagram (Fig. 35d). This operator is defined to within the polynomial

$$\hat{f}_n m - g_n \hat{p}. \quad (33.35)$$

This concludes our analysis of the terms  $\Lambda_n$ . In the next odd order  $n + 1$ , the only diagram from which the divergence has to be removed is the vertex diagram (see Fig. 35b). When the divergence is removed from it, we must ensure that condition (33.29) is satisfied. When this is written down for the subtracted terms, i.e., for the coefficient functions of quasilocal operators

$$\Gamma_{n+1}^{\text{subtr}}(p, 0) = - \frac{\partial \Sigma_n^{\text{subtr}}(p)}{\partial p}, \quad (33.36)$$

we must remember the subtracted parts  $\Gamma$  and  $\Sigma$  are equal to their Maclaurin expansions to within finite polynomials:



$$\Gamma_{n+1}^{\text{subtr}}(p, k) = \Gamma_{n+1}(0, 0) + \gamma b_{n+1},$$

$$\Sigma_n^{\text{subtr}}(p) = \Sigma_n(0) + p \left[ \frac{\partial \Sigma_n(p)}{\partial p} \right]_{p=0} + f_n m - g_n \hat{p}.$$

Substituting these expressions in (33.36), and using (33.29), we obtain

$$b_{n+1} = g_n. \quad (33.37)$$

Since we have previously established the uniqueness of the constant  $b_{n+1}$ , it is therefore determined by the above relation. The subtraction procedure in order  $n+1$  also turns out to be gauge invariant if (33.37) holds. This completes the proof.

We also note that the two relationships (33.36) and (33.37) establish for each order in  $n$  the equality of the constants  $G_n$  and  $B_{n+1}$  in the operators (33.7) and (33.9), i.e., it is equivalent to the *Ward identity for the renormalization constants*.

**33.4. Counterterms.** We now write down the counterterms of the Lagrangian that regularize the matrix  $S(1)$ . On substituting expressions (33.7)–(33.9) into (21.24) we obtain, after integrating by parts and summing over  $\nu$ , the complete interaction Lagrangian in the form

$$\begin{aligned} \mathcal{L}(x) = \mathcal{L}(x; 1) = & eZ_1 : \bar{\psi}(x) \hat{A}(x) \psi(x) : - \delta m : \bar{\psi}(x) \psi(x) : + \\ & + (Z_2 - 1) \left\{ \frac{i}{2} : \left( \bar{\psi}(x) \gamma^k \frac{\partial \psi}{\partial x^k} - \frac{\partial \bar{\psi}}{\partial x^k} \gamma^k \psi(x) \right) : - m : \bar{\psi}(x) \psi(x) : \right\} - \\ & - (Z_3 - 1) \left\{ \frac{1}{2} : \frac{\partial A_n(x)}{\partial x^m} \frac{\partial A_n(x)}{\partial x_m} : - \frac{1}{2} : \left( \frac{\partial A_n}{\partial x_n} \right)^2 : \right\}, \end{aligned} \quad (33.38)$$

where the following notation has been used:

$$\begin{aligned} Z_1 = 1 + \sum_{\nu=1}^{\infty} \frac{e^{2\nu}}{(2\nu)!} B_{2\nu+1}, \quad Z_2 = 1 + \sum_{\nu=1}^{\infty} \frac{e^{2\nu}}{(2\nu)!} G_{2\nu}, \\ Z_3 = 1 + \sum_{\nu=1}^{\infty} \frac{e^{2\nu}}{(2\nu)!} D_{2\nu}, \quad \delta m = \sum_{\nu=1}^{\infty} \frac{e^{2\nu}}{(2\nu)!} (F_{2\nu} + G_{2\nu}). \end{aligned} \quad (33.39)$$

In view of the relation

$$B_{2\nu+1} = G_{2\nu},$$

which was established earlier, Ward's identity in the present notation takes the form

$$Z_1 = Z_2. \quad (33.40)$$

The constants  $Z_1$ ,  $Z_2$ ,  $Z_3$ , and  $\delta m$ , determined by (33.39), depend on the auxiliary masses

$M_i$  with the coefficients in the expansions (33.39) diverging logarithmically as these masses tend to infinity. However, the matrix elements of the matrix

$$S(1) = T \left( \exp i \int \mathcal{L}(x; 1) dx \right)$$

approach constant values in the limit  $M_i \rightarrow \infty$ .

Since the Pauli-Villars procedure is gauge invariant, the Lagrangian (33.38), in contrast to the Lagrangian (27.45) which was obtained by means of a nongauge-invariant regularization, does not contain the term involving the photon mass

$$\delta m_\Phi^2 : A_n(x) A^n(x) : \quad (33.41)$$

and is therefore manifestly gauge invariant.

We see from the above that the form and the properties of the counterterms *depend in an essential way on the method of auxiliary regularization*. Thus, by using the gauge-noninvariant regularization of the photon and fermion pairings which we employed in Chapter V for the removal of infinities from  $S_2$  and  $S_3$ , we have arrived at the necessity for the introduction of *gauge-noninvariant counterterms* such as (33.41). On the other hand, as we have just shown, by using the Pauli-Villars auxiliary regularization, which does have the property of gauge invariance, we have arrived at (33.38), which does not contain any gauge-noninvariant terms. Moreover, it may turn out that the counterterms will not have even the property of Lorentz invariance. As Stepanov (1956) has shown, such a situation in fact arises when Lorentz-noninvariant auxiliary regularization is used. Of course, in such a case, after the auxiliary regularization has been removed, the scattering matrix turns out to be both gauge-invariant and Lorentz-invariant.

### §34. Spinor Electrodynamics. II. Mass and Charge Renormalization

**34.1. Gauge Transformation of the Pairing  $\overline{AA}$ .** We now turn to a more detailed examination of the scattering matrix  $S(1)$ . Its gauge invariance was established in the preceding section. We first note that the property of gauge invariance allows one to add to the pairing of electromagnetic potentials

$$\overline{A_n(k) A_n(k')},$$

which is used in the process of reducing the terms of the matrix  $S(1)$  to the normal form, an expression of the type

$$k_m k_n f(k^2),$$

where  $f(k^2)$  is an arbitrary function of  $k^2$ . In other words, the elements of the scattering matrix are not altered when

$$\overline{A_m(k) A_n(k')} \equiv \langle T(A_m(k) A_n(k')) \rangle_0 = -\frac{ig^{mn}}{k^2 + i\epsilon} \delta(k + k') \quad (34.1)$$

is replaced by the expression

$$\frac{1}{i} \left( g^{mn} \frac{1}{k^2} + k_m k_n f(k^2) \right) \delta(k + k').$$

To prove this, we consider the gauge transformation

$$A_n \rightarrow A'_n = A_n + k_n F(k^2) (k \cdot A(k)) \quad (34.2)$$

and determine the chronological pairing of the new operators  $A'$ :

$$\overline{A'_n(k) A'_m(k')} = \frac{1}{i} \left\{ \frac{g^{mn}}{k^2} + \frac{k_m k_n}{k^2} (2F(k^2) + k^2 F(k^2) F(k^2)) \right\} \delta(k + k'). \quad (34.3)$$

Since it was shown in the preceding section that  $S(1)$  is gauge invariant, its matrix elements will not depend on the function

$$f(k^2) = \frac{2F(k^2)}{k^2} + F(k^2) F(k^2),$$

which is what we wanted to show. We also note that by setting

$$f(k^2) = \frac{d^l - 1}{k^2 k^2},$$

we may write expression (34.3) in the form

$$i \overline{A'_n(k) A'_m(k')} = \frac{1}{k^2} \left( g^{mn} - \frac{k_m k_n}{k^2} \right) \delta(k + k') + \frac{d^l}{k^2} \frac{k_m k_n}{k^2} \delta(k + k') = D_{nm}(k) \delta(k + k'), \quad (34.4)$$

$$D_{nm}(k) = \frac{1}{k^2} \left( g_{nm} - \frac{k_n k_m}{k^2} \right) + \frac{d^l}{k^2} \cdot \frac{k_n k_m}{k^2}. \quad (34.5)$$

The formula given by (34.5) is the most convenient one. In it we have carried out an explicit decomposition of the pairing  $\overline{AA}$  into its *transverse and longitudinal (in the four-dimensional sense) parts* in which the arbitrariness of the gauge has been completely associated with the longitudinal part (the coefficient  $d_l$ ). In the foregoing in accordance with (34.4)  $d_l$  may, generally speaking, depend on  $k^2$ . However, for purposes of our future discussion it will be sufficient to take  $d_l$  constant.

By setting  $d_l = 0$  we obtain from (34.5) the expression

$$i A_n^{\text{tr}}(k) \overline{A_m^{\text{tr}}(k')} = \frac{1}{k^2} \left( g^{mn} - \frac{k_m k_n}{k^2} \right) \delta(k' + k), \quad (34.6)$$

which has the property of transversality

$$k^n \overline{A_n^{\text{tr}}(k)} A_m^{\text{tr}}(k') = 0. \quad (34.7)$$

On the other hand, for  $d_l = 1$  we obtain the usual diagonal pairing (34.1). We note that the diagonal gauge (34.1) is sometimes called the *Feynman gauge*, and the transverse gauge (34.6) is frequently referred to as the *Landau gauge*.

**34.2. Nonuniqueness of the Process of Removal of Infinities.** In §33 it was shown that in the process of obtaining a gauge invariant matrix  $S(1)$  which contains no infinities there is a certain arbitrariness connected with three finite constants which may be represented in the form of coefficients of operator expressions of the same type as the counterterms in the Lagrangian (33.44).

Thus, after we have completed the removal of infinities which may be accomplished either by introducing divergent counterterms (33.44), or by an appropriate redefinition of chronological products, the resulting expression for the  $S$ -matrix contains a certain degree of arbitrariness which is most conveniently recorded in the form of finite additive terms to the initial interaction Lagrangian

$$\begin{aligned} \delta \mathcal{L} = & e(z_1 - 1) : \bar{\psi}(x) \hat{A}(x) \psi(x) : - \delta m : \bar{\psi}(x) \psi(x) : + \\ & + (z_2 - 1) \left\{ \frac{i}{2} : \left( \bar{\psi}(x) \gamma^n \frac{\partial \psi}{\partial x^n} - \frac{\partial \bar{\psi}}{\partial x^n} \gamma^n \psi(x) \right) : - m : \bar{\psi}(x) \psi(x) : \right\} - \\ & - (z_3 - 1) \left\{ \frac{1}{2} : \frac{\partial A_n(x)}{\partial x^m} \frac{\partial A^n(x)}{\partial x_m} : - \frac{1}{2} \left( \frac{\partial A}{\partial x} \right)^2 \right\}. \end{aligned} \quad (34.8)$$

Here  $z_1, z_2, z_3$  and  $\delta m$  are *finite* constants which, by virtue of Ward's identity, satisfy

$$z_1 = z_2. \quad (34.9)$$

We shall now examine the influence of the finite counterterms (34.8) on the scattering matrix and we shall show that it is equivalent to a certain finite renormalization of the mass and the charge of the fermion. With this end in view we shall write (34.8) in the momentum representation:\*

\*Here we have used the symbolic notation

$$\bar{\psi} \hat{A} \psi = \frac{1}{(2\pi)^4} \int \bar{\psi}(p) \hat{A}(q) \psi(p') \delta(p + q + p') dq dp'.$$

$$\delta \mathcal{L}(p) = (z_1 - 1) e : \bar{\psi} \hat{A} \psi : + (z_2 - 1) : \bar{\psi}(p) (\hat{p} - m) \psi(-p) : - \delta m : \bar{\psi}(p) \psi(-p) : - \\ - (z_3 - 1) \frac{1}{2} : A_m(p) (g^{mn} p^2 - p^m p^n) A_n(-p) :. \quad (34.10)$$

In contrast to the case of the usual "initial" interaction Lagrangian

$$\mathcal{L} = e \bar{\psi} \hat{A} \psi$$

vertices of three types will correspond to the Lagrangian  $\mathcal{L} + \delta \mathcal{L}$  in Feynman diagrams. The term  $(z_2 - 1)$  will correspond to vertices in which two fermion lines meet and to which we shall refer for the sake of brevity as  $z_2$ -vertices. The term  $(z_3 - 1)$  will correspond to the  $z_3$ -vertices which join two photon lines. Finally, the term  $z_1$  will correspond to a vertex of the usual type ( $z_1$ -vertex).

We now consider the structure of the "propagation factors" which correspond to the internal lines of the new, more complicated, diagrams. We begin our discussion with the internal fermion lines. By associating the factor  $(z_2 - 1)(\hat{p} - m)$  with the  $z_2$ -vertex, and the factor  $e z_1$  with the  $z_1$ -vertex, we shall find that the pairing\*

$$\overbrace{\psi(-p) \bar{\psi}(p)} = \frac{i}{p - m}$$

will correspond to internal fermion lines independently of the type of vertices joined by the given internal line. We now consider the "total propagation factor" corresponding to the motion of a particle between two  $z_1$ -vertices. It is clear that this will be the sum of factors corresponding to open fermion cycles that begin and end in  $z_1$  type vertices and contain an arbitrary number of type  $z_2$  vertices. By evaluating these factors in succession, and taking into account the presence of the factor  $i^n$  in the  $n$ -th order of the  $S$ -matrix, and the compensation of the factorial  $n!$  in the denominator on transforming to the momentum representation, we obtain

$$\frac{i}{\hat{p} - m}; \quad \frac{i}{\hat{p} - m} i (z_2 - 1) (\hat{p} - m) \frac{i}{\hat{p} - m} = - (z_2 - 1) \frac{i}{\hat{p} - m},$$

i.e., the introduction of each new  $z_2$ -vertex corresponds to the appearance of the factor  $y_2 = 1 - z_2$ .

Hence, by summing factors corresponding to different numbers of  $z_2$ -vertices between zero and infinity,

$$\frac{i}{\hat{p} - m} (1 + y_2 + y_2^2 + \dots) = \frac{i}{\hat{p} - m} \frac{1}{1 - y_2} = \frac{1}{z_2} \frac{i}{\hat{p} - m},$$

\*For the sake of brevity, we have omitted here the delta function of the total momentum  $\delta(p + p')$  and have set  $p' = p$ .

we conclude that the appearance of the term  $(z_2 - 1)$  in the interaction Lagrangian is equivalent from the point of view of internal fermion lines to the following renormalization of the fermion causal function:

$$S^c(p) \rightarrow S'^c(p) = z_2^{-1} S^c(p). \quad (34.11)$$

In precisely the same way, we can verify that by taking into account the term  $(z_2 - 1)$  in the external fermion lines, we introduce the factor  $z_2^{-1}$  into the operators  $\psi$  and  $\bar{\psi}$  corresponding to free ends of the lines.

However, we note that when the corresponding calculation is performed for external electron lines, we encounter expressions of the form

$$\frac{1}{\hat{p} - m} (\hat{p} - m) \psi(p),$$

which contain an obvious indeterminacy. A result of the form of (34.11) can be obtained by assuming that  $\hat{p} \neq m$ , i.e., by leaving the mass surface and first performing the multiplication  $(\hat{p} - m)(\hat{p} - m) = 1$ . The reverse order of operation yields zero.

Medvedev and Polivanov (1967) have established that a more rigorous analysis will show that renormalization "multipliers" such as  $z_i$  for the external lines are, in general, integrodifferential operators outside the mass surface. They become numbers only on the mass surface. In that case, we have

$$\psi(p) \rightarrow z_2^{-1/2} \psi(p), \quad \bar{\psi}(p) \rightarrow z_2^{-1/2} \bar{\psi}(p). \quad (34.12)$$

We now turn to internal photon lines. We associate a propagator of the form of (34.5) with the internal lines in arbitrary gauge

$$\overline{A_m(-k)} A_n(k) = \frac{1}{ik^2} (P_{mn}^{\text{tr}} + d^l P_{mn}^l), \quad (34.13)$$

where

$$P_{mn}^{\text{tr}} = g_{mn} - \frac{k_m k_n}{k^2}, \quad P_{mn}^l = \frac{k_m k_n}{k^2} \quad (34.14)$$

are the projection operators with the following properties:

$$P^{\text{tr}} P^{\text{tr}} = P^{\text{tr}}, \quad P^l P^l = P^l, \quad P^{\text{tr}} P^l = 0.$$

With the vertex  $z_3$  we associate the factor

$$\frac{1 - z_3}{2} (g^{mn} k^2 - k^m k^n) = \frac{y_3}{2} (P^{\text{tr}})^{mn} k^2.$$

If we investigate the structure of the expressions corresponding to the inclusion of the  $z_3$ -vertices in the internal photon lines we obtain, successively,

$$\begin{aligned} & \frac{1}{ik^2} (P^{\text{tr}} + d_l P^l); \\ & \frac{1}{ik^2} (P^{\text{tr}} + d_l P^l) 2i \frac{y_3}{2} P^{\text{tr}} k^2 \frac{1}{ik^2} (P^{\text{tr}} + d_l P^l) = \frac{y_3}{ik^2} P^{\text{tr}}, \\ & \dots = \frac{y_3^2}{ik^2} P^{\text{tr}}, \end{aligned}$$

and so on.

The factor 2 in the  $z_3$ -vertex is due to the two possible orders of pairing of the operators  $A$  in the term  $z_3$  in (34.10). Summation over all the possible numbers of  $z_3$ -vertices yields

$$1 + y_3 + y_3^2 + \dots = \frac{1}{1 - y_3} = z_3^{-1},$$

so that, finally, we have

$$\frac{1}{ik^2} \left( \frac{1}{z_3} P_{mn}^{\text{tr}} + d_l P_{mn}^l \right).$$

Thus the term  $(z_3 - 1)$  leads to the renormalization of only the transverse part of the photon Green's function while its longitudinal term does not undergo any changes. This term is often neglected in the usual discussions of the scattering matrix, by making use of the gauge invariance of the scattering matrix noted above. In the theory of the  $S$ -matrix, the "incomplete renormalization" of the photon function therefore does not turn out to be essential.

The situation is altered when we go over to the general theory of Green's functions of interacting fields, which are sums of diagrams made up of internal lines only. In this case, the basic apparatus of the theory is formulated without any reference to matrix elements or the Lorentz condition. The longitudinal term may then no longer be neglected. However, the difficulty that arises may be avoided if one chooses for the zero-order approximation to the Green's function the purely transverse expression

$$D_{mn}^{c, \text{tr}}(p) = -\frac{1}{p^2} \left( g^{mn} - \frac{p_m p_n}{p^2} \right).$$

In this case, taking into account the counter term  $(z_3 - 1)$  leads to a purely multiplicative renormalization:

$$D_{mn}^{c, \text{tr}}(p) \rightarrow D_{mn}^{c, \text{tr}}(p) = z_3^{-1} D_{mn}^{c, \text{tr}}(p). \quad (34.15)$$

By considering further the insertion of  $z_3$ -vertices into the external photon lines of the

diagram, we find, on taking into account the weakened Lorentz condition imposed on the allowed states, that the terms quadratic in  $A$  lead to the transformation

$$A_m(p) \rightarrow z_3^{-1/2} A_m(p). \quad (34.16)$$

Strictly speaking, this yields an expression of the form

$$\left\{ \frac{1}{z_3} P^{\text{tr}} + d_l P^l \right\}_{mn} A^n(k),$$

which contains indeterminacies of the form  $(kA)/k^2 \sim 0/0$ . However, since the field  $A$  is transverse and outside the mass surface (for  $k^2 \neq 0$ ), it follows that if we chose the appropriate order of passing to the limit, and take account of the remark before (34.12), we obtain (34.16). It is also clear that the effect of the term  $(z_1 - 1)$  amounts to a change in the value of the charge

$$e \rightarrow e' = z_1 e. \quad (34.17)$$

In summarizing the results of the preceding discussion we conclude on the basis of (34.11), (34.12), (34.15)–(34.17) that the introduction into the interaction Lagrangian of the finite terms  $(z_1 - 1)$ ,  $(z_2 - 1)$ , and  $(z_3 - 1)$  is equivalent, from the point of view of the structure of the  $S$ -matrix, to the following transformation of propagation factors, of the field operators, and of the electron charge [this is the *Dyson transformation* (1949b)]:

$$S^c(p) \rightarrow z_2^{-1} S^c(p), \quad D^{c, \text{tr}}(p) \rightarrow z_3^{-1} D^{c, \text{tr}}(p), \quad e \rightarrow z_1 e, \quad (34.18)$$

$$\bar{\psi}(p) \rightarrow z_2^{-1/2} \bar{\psi}(p), \quad \psi(p) \rightarrow z_3^{-1/2} \psi(p), \quad A(p) \rightarrow z_3^{-1/2} A(p). \quad (34.19)$$

The formulas (34.18) may be generalized to the case  $d_l \neq 0$ . To do this, we note that the transformation

$$D^c(k) = -\frac{1}{k^2} (P^{\text{tr}} + d_l P^l) \rightarrow -\frac{1}{k^2} \left\{ \frac{1}{z_3} P^{\text{tr}} + d_l P^l \right\}$$

can be written in the form

$$D^c(k | d_l) \rightarrow z_3^{-1} D^c(k | z_3 d_l),$$

so that, instead of (34.18), we obtain

$$S^c \rightarrow z_2^{-1} S^c, \quad D^c \rightarrow z_3^{-1} D^c, \quad e \rightarrow z_1 e, \quad d_l \rightarrow z_3 d_l. \quad (34.20)$$

We note here that, in contrast to all the other quadratic counterterms, the effect of the expression  $\delta m \bar{\psi} \psi$  cannot be reduced to any sort of renormalization or to a replacement of



the causal functions and the potentials, and must therefore necessarily be retained in the interaction Lagrangian. Indeed, it is readily shown that the inclusion of the vertices such as  $\delta m$  into the internal fermion lines leads us to a new fermion mass:

$$\frac{1}{\hat{p} - m} \rightarrow \frac{1}{\hat{p} - m'}, \quad m' = m + \delta m. \quad (34.21)$$

The introduction of the  $\delta m$ -vertices into the external fermion lines leads to the expression

$$\psi(p) \rightarrow \frac{\hat{p} - m}{p - m'} \psi(p),$$

which gives zero when the matrix element is evaluated. Thus, from the point of view of the  $S$ -matrix the introduction of the term  $\delta m \bar{\psi} \psi$  cannot be described in a consistent way by changing the fermion mass.

**34.3. Complete Green's Functions  $G$ ,  $D$ , and the Vertex Part  $\Gamma$ .** It may also be shown that the set of transformations (34.19) and (34.20), which is equivalent to the terms (34.10) in respect of the internal lines of Feynman diagrams, may be completely reduced to a change in the mass  $m$  and the charge  $e$  of the electron. To do this, we must examine in more detail the structure of the factors corresponding to the internal elements of the diagram, taking radiation corrections into account.

We begin this discussion with an internal fermion line. The radiative corrections for such a line are determined by diagrams of the self-energy type. We denote the sum of all such possible strongly connected diagrams of all orders by a circle with two fermion lines entering it (Fig. 39). and we denote the corresponding factor by  $\Sigma/i$ .\*

It is clear that  $\Sigma$  is a function of the momenta  $p_1$  and  $p_2$  which correspond to particles entering and leaving the diagram. But since  $p_1 + p_2 = 0$ , we have

$$\Sigma(p_1, p_2) = \Sigma(p),$$

where, in accordance with (27.4),

$$-p_1 = p_2 = p.$$

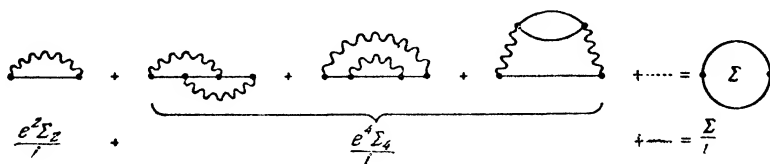


Fig. 39.

\*The  $\Sigma$  introduced in this way is the generalization of the fermion self-energy operator of order  $e^2$ , considered in § 27.

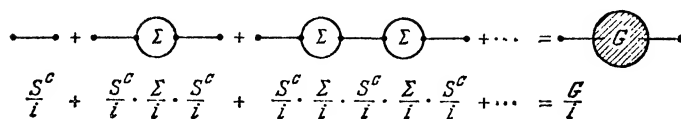


Fig. 40. The set of diagrams corresponding to the total electron Green's function.

The complete fermion propagation factor  $G$  which contains all possible radiation insertions of the self-energy type corresponds to the set of diagrams shown in Fig. 40 and may be represented in the form

$$\begin{aligned} & \frac{1}{i(m-\hat{p})} + \frac{1}{i(m-\hat{p})} \frac{\Sigma}{i} \frac{1}{i(m-\hat{p})} + \frac{1}{i(m-\hat{p})} \frac{\Sigma}{i} \frac{1}{i(m-\hat{p})} \frac{\Sigma}{i} \frac{1}{i(m-\hat{p})} + \dots = \\ & = \frac{i}{\hat{p}-m} \left( 1 + \Sigma \frac{1}{\hat{p}-m} + \Sigma \frac{1}{\hat{p}-m} \Sigma \frac{1}{\hat{p}-m} + \dots \right) = \frac{i}{\hat{p}-m} \left( 1 - \frac{\Sigma}{\hat{p}-m} \right)^{-1} = \frac{i}{\hat{p}-m-\Sigma(p)}, \end{aligned}$$

i.e.,

$$G(p) = \frac{1}{m + \Sigma(p) - \hat{p}}. \quad (34.22)$$

Similarly, it is possible to construct the complete photon Green's function and the generalized vertex part. Thus, by summing the contributions from the self-energy parts of the photon  $\Pi(k)$ :

$$\Pi^{mn}(k) = \pi(k) \left( g^{mn} - \frac{k^m k^n}{k^2} \right), \quad (34.23)$$

we obtain the complete photon propagation factor

$$D_{mn}(k) = \frac{1}{\pi(k) - k^2} \left( g^{mn} - \frac{k_m k_n}{k^2} \right) - \frac{d_I}{k^2} \frac{k_m k_n}{k^2}. \quad (34.24)$$

An important feature of (34.24) is the fact that, due to the transverse nature of the operator  $\Pi$ , the radiative corrections to the longitudinal term are zero. In this connection we recall that the transverse form of  $\Pi$  is determined by the requirement of gauge invariance of the scattering matrix (see (24.39) and (30.35)). It may also be shown that  $\Pi(k)$  is likewise a gauge-invariant quantity and does not depend on  $d_I$ .

Finally, in considering vertex-type diagrams, i.e., connected diagrams with one fermion entering, one fermion emerging, and one external photon entering, we may introduce their sum  $\Gamma$  which may sometimes be represented conveniently in the form

$$\Gamma^n(p, q; k) = \gamma^n + \Lambda^n(p, q; k), \quad (34.25)$$

where  $\Lambda^n$  is the sum of all possible strongly connected radiative connections. The weakly connected diagrams of this type will always be referred to radiative corrections in internal lines, i.e., corrections to electron and photon Green's functions.

According to §33.3, the strongly connected vertex part  $\Lambda$  and the electron self-energy operator  $\Sigma$  are related by the generalized Ward identity which, in the notation of the present section, has the form

$$k\Lambda(p+k, p|k) = \Sigma(p) - \Sigma(p+k). \quad (34.26)$$

This formula is obtained from (33.28) by summing over all orders of perturbation theory. Differentiating it with respect to the components of the four-vector  $k$ , and substituting  $k=0$ , we obtain the usual Ward identity

$$\Lambda_m(p, p|0) = -\frac{\partial \Sigma(p)}{\partial p^m}. \quad (34.27)$$

In (34.26) we can transform to the complete vertex part  $\Gamma$  and complete Green's function  $G$  by using (34.22) and (34.25):

$$k\Gamma(p+k, p|k) = G^{-1}(p) - G^{-1}(p+k). \quad (34.28)$$

This is occasionally referred to as the *Ward-Takahashi relation*.

We now note that although the relations given by (34.22)–(34.28) may formally hold even before the removal of infinities, we shall be principally interested in relations of this type between quantities not containing any divergences. We shall therefore consider that the foregoing relations have been formulated after the process of removing infinities has been completed.

The finite quantities  $G$ ,  $D$ , and  $\Gamma$  introduced into the above relations then contain an ambiguity related to the possibility of introducing the finite terms (34.10) into the interaction Lagrangian. Let us therefore establish the transformation properties of the quantities  $G$ ,  $D$ , and  $\Gamma$  when (34.10) is added to the Lagrangian. We examine first of all the transformation of  $\Sigma$  and  $\Pi$ . It was shown that the functions  $S^c$ ,  $D^{c, \text{tr}}$ , and the charge  $e$  transform according to formulas (34.18) when the aforementioned terms are introduced. We now note that since two fermion and one photon lines meet at each vertex, we may consider that the transformation (34.18) is equivalent to one in which the factors  $S^c$  and  $D^{c, \text{tr}}$  undergo no

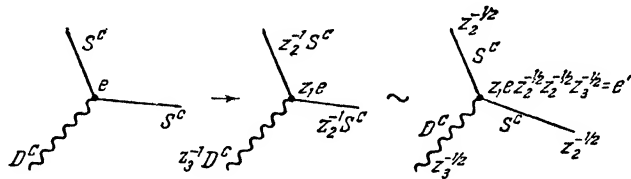
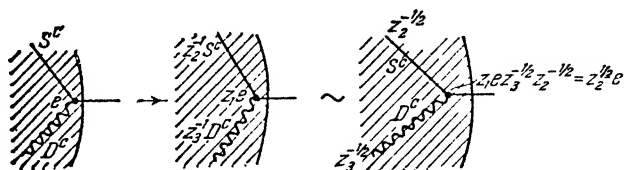


Fig. 41. Transformation of a vertex internal with respect to  $\Sigma$  and  $\Pi$ .

Fig. 42. Transformation of an internal vertex in  $\Sigma$ .

change while the charge  $e$  corresponding to each vertex internal with respect to  $\Sigma$  and  $\Pi$  undergoes a transformation of the type

$$e \rightarrow e' = z_1 z_2^{-1} z_3^{-1/2} e \quad (34.29)$$

(Fig. 41). With respect to the two external vertices contained in  $\Sigma$  and in  $\Pi$ , i.e., those vertices which connect  $\Sigma$  and  $\Pi$  to the other parts of the diagrams, we note that for a complete renormalization of (34.29), each of these vertices will lack the square root of the corresponding  $z$  (see Fig. 42). Therefore, the transformation law for  $\Sigma$  and  $\Pi$  may be represented in the form

$$\Sigma(p, e) \rightarrow z_2 \Sigma(p, e'), \quad (34.30)$$

$$\Pi(k, e) \rightarrow z_3 \Pi(k, e'), \quad (34.31)$$

where  $e'$  is given by (34.29).

Turning to the transformation of the complete Green's functions  $G$  and  $D$ , we note that in the process of summation leading to (34.22) and (34.24), we need only make the replacements

$$\Sigma(p, e) \rightarrow z_2 \Sigma(p, e'), \quad \pi(k, e) \rightarrow z_3 \pi(k, e'), \quad (34.32)$$

$$\frac{1}{\hat{p} - m} \rightarrow \frac{1}{z_2} \frac{1}{(\hat{p} - m)}, \quad \frac{1}{k^2} \rightarrow \frac{1}{z_3} \frac{1}{k^2}, \quad (34.33)$$

which yield

$$G(p, e) \rightarrow z_2^{-1} G(p, e'), \quad (34.34)$$

$$D^{\text{tr}}(k, e) \rightarrow z_3^{-1} D^{\text{tr}}(k, e'). \quad (34.35)$$

Finally, on taking into account the counterterm  $\delta m$  in all the internal fermion lines, we are led to the following mass renormalization:

$$m \rightarrow m' = m + \delta m. \quad (34.36)$$

It is readily shown by an analogous argument that by taking into account the terms (34.10) in diagrams of the vertex type, we obtain:

$$e\Gamma^n(p, k, q|e, m) \rightarrow z_2 z_3^{1/2} e' \Gamma^n(p, k, q|e', m') = z_1 e \Gamma^n(p, q, k|e', m'). \quad (34.37)$$

In summary, we have shown that the introduction of the terms (34.10) is equivalent to the following transformation of the factors  $G$ ,  $D$ ,  $\Gamma$ ,  $e$ , and  $m$ :

$$\left. \begin{aligned} G(p, e, m) &\rightarrow z_2^{-1} G(p, e', m'), & D^{\text{tr}}(k, e) &\rightarrow z_3^{-1} D^{\text{tr}}(k, e', m'), \\ \Gamma(p, q, k|e, m) &\rightarrow z_1 \Gamma(p, q, k|e', m'), \\ \text{where} \quad e' &= z_1 z_2^{-1} z_3^{-1/2} e, & m' &= m + \delta m. \end{aligned} \right\} \quad (34.38)$$

We also note that with the aid of the factors  $G$ ,  $D$ , and  $\Gamma$ , calculations of arbitrarily high order may be carried out for any process on the basis of the so-called “*skeleton*” *Feynman diagrams*. The set of all the skeleton diagrams is obtained from the set of all the connected diagrams by excluding from them all those diagrams that contain elements of the type shown in Fig. 35. Therefore, skeleton diagrams contain no self-energy parts and no vertex parts. However, for the calculation of the corresponding coefficient functions, we must employ not the ordinary propagation factors  $S^c$  and  $D^c$  but the complete propagation factors  $G$  and  $D$ , and we must make the vertices of the skeleton diagram correspond to  $\Gamma^n$  and not to  $\gamma^n$ . We emphasize that in this method of calculation, utilizing as it does the renormalized functions  $G$ ,  $D$ , and  $\Gamma$ , we need no longer use the subtraction procedure, since, as we have shown earlier, subtraction has to be applied only to the elements of diagrams of the type shown in Fig. 35.

Moreover, it follows from the above that the effect of the terms (34.10) on the coefficient functions of the  $S$ -matrix which correspond to more complicated irreducible diagrams is described completely by the renormalization (34.33) of the quantities  $G$ ,  $D^{\text{tr}}$ ,  $\Gamma$  which correspond to elements of the appropriate skeleton diagram, and does not lead to any additional effects.

It is also clear that the factors  $z_2^{-1}$ ,  $z_3^{-1}$  and  $z_1$  which correspond to the factors  $G$ ,  $D$ , and  $\Gamma$  of the skeleton diagram lead in the final result to the charge renormalization (34.26) at the internal vertices of the skeleton diagram.

Thus from the point of view of the internal parts of arbitrarily complicated diagrams of any arbitrary order, the introduction of the terms  $(z_1 - 1)$ ,  $(z_2 - 1)$ ,  $(z_3 - 1)$  is equivalent to the renormalization of the quantities  $G$ ,  $D$ , and  $\Gamma$ :

$$\Gamma \rightarrow z_1 \Gamma, \quad G \rightarrow z_2^{-1} G, \quad D^{\text{tr}} \rightarrow z_3^{-1} D^{\text{tr}}, \quad (34.39)$$

which in turn is equivalent to the charge renormalization (34.29) or, on taking Ward's identity into account, to

$$e \rightarrow e' = z_3^{-1/2} e. \quad (34.40)$$

Taken together with the nature of the effect produced by the term  $\delta m$  established above

this means that the effect of the four terms (34.10) amounts to a renormalization of the two quantities—the mass  $m$  and the charge  $e$  as described by formulas (34.36) and (34.40).

Therefore, if simultaneously with the introduction of the terms (34.10) the mass  $m$  and the charge  $e$  in the initial equation are replaced by the quantities

$$\left. \begin{aligned} m'' &= m - \delta m, \\ e'' &= z_3^{1/2} e, \end{aligned} \right\} \quad (34.41)$$

then as a result of the transformations (34.36) and (34.40) we obtain the initial values of the mass  $m$  and the charge  $e$ . Conversely, instead of introducing the counterterms (34.10) it would have been sufficient to go over at the outset to the new mass and charge  $m'$  and  $e'$ .

Thus, the simultaneous introduction of the renormalizations (34.38) and (34.41) leads us to a theory which is equivalent to the initial one. These transformations evidently have the group property, and lead us to a *group of transformations* which leave invariant the observable values of  $m$  and  $e$ . We shall call this group the *renormalization group*. It will be discussed in greater detail in Chapter IX. Equations (34.39) and (34.40) may be directly generalized to the case  $d_l \neq 0$ :

$$\left. \begin{aligned} \Gamma_1 &\rightarrow z_1 \Gamma, & G &\rightarrow z_2^{-1} G, & D &\rightarrow z_3^{-1} D, \\ m &\rightarrow m + \delta m, & e &\rightarrow z_3^{-1/2} e, & d_l &\rightarrow z_3 d_l. \end{aligned} \right\} \quad (34.42)$$

It is important to note that in order to preserve the physical meaning of the renormalized mass and charge, the arbitrary constants must satisfy certain conditions. Thus the requirements that the mass should be finite and positive, and that the charge should be finite and real, lead to the following restrictions on  $\delta m$  and  $z_3$ :

$$0 < m + \delta m < \infty, \quad 0 < z_3 < \infty. \quad (34.43)$$

**34.4. Radiative Corrections to External Lines and the Choice of Finite Constants.** We shall now make the subtraction procedure more specific and shall remove entirely all arbitrariness of the polynomials being subtracted. In our general definition of the subtraction procedure (§29), we agreed to subtract from the divergent expressions (in the present case  $\Sigma$ ,  $\Pi$ , and  $\Lambda$ ) the first terms of their expansions into Maclaurin series. However, for practical purposes, it turns out to be more convenient to expand the self-energy part of the fermion about the point  $\hat{p} = m$ . We therefore define the regular function  $\Sigma_{\text{reg}}$  as follows:

$$\Sigma_{\text{per}}(\hat{p}) = \Sigma(\hat{p}) - \Sigma(\hat{p}) \Big|_{\hat{p}=m} - \frac{\partial \Sigma(\hat{p})}{\partial \hat{p}} \Big|_{\hat{p}=m} (\hat{p} - m). \quad (34.44)$$

In the foregoing, we have made use of the fact that  $\Sigma$  depends on  $p$  only through  $\hat{p}$  and  $p^2 = (\hat{p})^2$ . In making the subtraction in this manner, we obtain

$$\Sigma_{\text{reg}}(\hat{p}) \Big|_{\hat{p}=m} = 0 \quad (34.45)$$

and

$$\left. \frac{\partial \Sigma_{\text{reg}}(\hat{p})}{\partial \hat{p}} \right|_{\hat{p}=m} = 0, \quad (31.46)$$

from which it also follows that

$$\left. \frac{1}{\hat{p}-m} \Sigma_{\text{reg}}(\hat{p}) \right|_{\hat{p}=m} = \Sigma_{\text{reg}}(\hat{p}) \left. \frac{1}{\hat{p}-m} \right|_{\hat{p}=m} = 0. \quad (34.47)$$

It follows from (34.45) that the mass  $m$  turns out to be equal to the experimental mass of the electron. This may be seen directly from the fact that the pole of the complete Green's function

$$G(p) = \frac{1}{m - \hat{p} + \Sigma_{\text{reg}}(\hat{p})}$$

coincides with the pole of the function

$$S^c(p) = \frac{1}{m - \hat{p}}.$$

Condition (34.46) leads to the result that the radiative corrections to the external fermion lines turn out to be equal to zero.

In connection with the expansion of the function  $\Sigma$  about the point  $\hat{p} = m$  ( $p^2 = m^2$ ) it is also necessary to note that as we have shown in Chapter V (§29.2, §31.1), the region in which the regularized expressions are analytic in the limit as  $\epsilon \rightarrow 0$  is restricted by the condition (31.1) which, in our case, assumes the form

$$p^0 < \min m'_i.$$

Since in the present case, the minimum mass is equal to zero (photon mass), the function  $\Sigma(p)$  cannot be analytic at the point  $p^2 = m^2$ , which is in fact the case. The derivative  $\partial \Sigma / \partial \hat{p}$  diverges at this point. However, this divergence is a manifestation of the well-known "infrared catastrophe" (see §35), and for its removal it turns out to be sufficient to introduce an auxiliary quantity in the intermediate stages of the argument—the fictitious infinitesimal photon mass  $\lambda_0$ . Therefore, in future, we shall always assume that, wherever necessary, the photon has the small mass  $\lambda_0$ .

We now consider the subtraction process for the vertex part  $\Gamma$ . As was shown in §33, it follows from the requirement of gauge invariance that the subtraction procedure for  $\Lambda$  is determined completely by the subtraction procedure used for  $\Sigma$ . In fact, from (33.25) we have for the complete vertex function

$$\Gamma^n(p, 0) = -\frac{\partial \Sigma(p)}{\partial p_n} + \gamma^n. \quad (34.48)$$

If we demand that this should also hold for the regularized functions, i.e., that

$$\Gamma_{\text{reg}}^n(p, 0) = -\frac{\partial \Sigma_{\text{reg}}(p)}{\partial p_n} + \gamma^n, \quad (34.49)$$

we find that in accordance with (34.44), the subtracted constant  $\Lambda$  in

$$\Gamma_{\text{reg}}^n(p, q; k) = \Gamma^n(p, q; k) - \gamma^n \Lambda$$

is given by

$$\Lambda = -\left(\frac{\partial \Sigma(p)}{\partial \hat{p}}\right)_{\hat{p}=m}.$$

All that remains is to define the subtraction operation for the photon self-energy part. If, as usual, we place the center of expansion at  $k = 0$ , we have

$$\Pi_{\text{reg}}^{mn}(k) = \Pi^{mn}(k) - \Pi^{mn}(0) - \frac{\partial \Pi^{mn}(k)}{\partial k^l} \Big|_{k=0} k^l - \frac{1}{2} \frac{\partial^2 \Pi^{mn}(k)}{\partial k^l \partial k^s} \Big|_{k=0} k^l k^s. \quad (34.50)$$

This expression, which satisfies the conditions

$$\Pi_{\text{reg}}^{mn}(0) = 0 \quad (34.51)$$

and

$$\frac{1}{k^2} \Pi_{\text{reg}}^{mn}(k) \Big|_{k=0} = 0, \quad (34.52)$$

by virtue of (34.51), does not change the pole of the photon function and, in accordance with (34.52), leads to the absence of radiative corrections to external photon lines.

Moreover, it may also be shown that (34.50) guarantees that the constant  $e$  in the interaction Lagrangian coincides with the observable value of the electron charge. Indeed, let us add to the right-hand side of equation (34.50) the expression

$$c(g^{mn}k^2 - k^m k^n), \quad (34.53)$$

which violates condition (34.52). This expression is equivalent to adding to the Lagrangian a finite term of the type  $z_3 - 1 = c$ , so that the constant  $c$  leads to the charge being altered by the factor  $(1 + c)^{-1/2}$ . We thus arrive at the possibility of determining the arbitrary finite



constant  $c$  from the condition that  $e$  should agree with the experimental value of the fermion charge. Assuming that the experimental value of the charge is determined by the scattering of the zero-energy photon by the fermion, we obtain the result that such a process is described by the expression

$$e\bar{\psi}(p)\Gamma_{\text{reg}}^n(p, -p, 0)\psi(-p). \quad (34.54)$$

We note the connection with (34.54) that, according to (34.49) and (34.44),

$$\Gamma_{\text{per}}^n(p, -p; 0) = \gamma^n - \frac{\partial \Sigma(p)}{\partial p_n} + \gamma^n \left( \frac{\partial \Sigma}{\partial \hat{p}} \right)_{\hat{p}=m}. \quad (34.55)$$

We shall now show that

$$\bar{\psi}(p) \left\{ \gamma^n \left( \frac{\partial \Sigma}{\partial \hat{p}} \right)_{\hat{p}=m} - \frac{\partial \Sigma}{\partial p_n} \right\} \psi(-p) = 0. \quad (34.56)$$

To do this, we write

$$\Sigma(\hat{p}) = (\hat{p} - m)a(p^2) + mb(p^2)$$

and, evaluating the difference in (34.55), we find that

$$\gamma^n \left( \frac{\partial \Sigma}{\partial \hat{p}} \right)_{\hat{p}=m} - \frac{\partial \Sigma}{\partial p_n} = 2m(\gamma^n m - p^n) b'(m^2).$$

Taking into account the field equations

$$\bar{\psi}(p)(\hat{p} - m) = 0, \quad (\hat{p} - m)\psi(-p) = 0$$

and making use of the relation

$$2(\gamma^n m - p^n) = (m - \hat{p})\gamma^n + \gamma^n(m - \hat{p})$$

we verify the validity of (34.56).

Taking renormalization into account, we obtain finally

$$e\bar{\psi}\Gamma_{\text{reg}}^n\psi = \frac{e}{\sqrt{1+c}}\bar{\psi}\gamma^n\psi,$$

from which it follows that the condition that  $e$  should agree with the experimental value of the charge has the form

$$c = 0.$$

Thus, the formula given by (34.50) does in fact guarantee both the absence of radiative corrections to external photon lines and also the fact that  $e$  agrees with the experimental value of the electron charge.

### §35. Spinor Electrodynamics. III. Radiative Corrections of the Second Order

We now turn to the application of the preceding results to the calculation of the radiative corrections of lowest order to the different effects of spinor electrodynamics. With this end in view, we shall first obtain expressions for the corrections to the propagation factors for the photon  $D$ , and for the electron  $G$ , and for the vertex function  $\Gamma$ .

*35.1. Correction to the Photon Function.* The photon Green's function  $D$  in zero-order approximation corresponding to the absence of interaction has, in the case of arbitrary  $d_I$ , the form

$$D_{mn}^c(k) = -\frac{1}{k^2} \left( g^{mn} - \frac{k_m k_n}{k^2} \right) - \frac{d_I}{k^2} \frac{k_m k_n}{k^2}. \quad (35.1)$$

To calculate the radiative correction to (35.1), we must add a term corresponding to the photon line and containing the insertion of the second-order photon self-energy (Fig. 43). This term has the form

$$i^2 D_{mk}^c \Pi^{kl}(k) D_{in}^c(k). \quad (35.2)$$

Using the gauge invariant expression (27.40) for the polarization operator  $\Pi$ , and determining the constant in it from the condition that the second-order partial derivatives at the point  $k = 0$  should be equal to zero, we find that

$$\Pi^{kl}(k) = \frac{e^2}{4\pi^2} (g^{kl} k^2 - k^k k^l) I(k^2), \quad (35.3)$$

where

$$I(k^2) = 2 \int_0^1 dx (1-x) x \ln \left[ \frac{m^2 - x(1-x)k^2}{m^2} \right]. \quad (35.4)$$

Substituting (35.3) into (35.2), and adding to (35.1), we obtain

$$D_{mn}(k) = -\frac{d(k^2)}{k^2} \left( g^{mn} - \frac{k_m k_n}{k^2} \right) - \frac{d_I}{k^2} \frac{k_m k_n}{k^2}, \quad (35.5)$$



Fig. 43. Diagrams corresponding to the principal term of  $D_{mn}$  and to the first radiative correction.

where

$$d(k^2) = 1 + \frac{\alpha}{\pi} I(k^2); \quad \alpha = \frac{e^2}{4\pi}. \quad (35.6)$$

From (35.4)–(35.6) it follows that as was to be expected, on the one hand, the longitudinal function  $d_l$  has no influence on the radiative corrections to the transverse function  $d(k^2)$ , while, on the other hand, the radiative corrections make no contribution to  $d_l$ . Therefore, we shall restrict ourselves to the examination of the purely transverse part of the photon Green's function, writing it in the form

$$D_{mn}^{\text{tr}}(k) = -\frac{1}{k^2} \left( g^{mn} - \frac{k_m k_n}{k^2} \right) d(k^2). \quad (35.7)$$

The integral  $I$  which appears in the definition of  $d(k^2)$  may be put, by means of the change of variable  $z = 1 - 2x$  and by integration by parts, in the form

$$I(k^2) = \frac{1}{2} \int_0^1 dz (1 - z^2) \ln \left[ 1 - \frac{1 - z^2}{4} \frac{k^2}{m^2} \right] = -k^2 \int_0^1 \frac{z^2 dz (1 - z^2/3)}{4m^2 - k^2 (1 - z^2)}. \quad (35.8)$$

For  $k^2 > 4m^2$ , the denominator in the integral (35.8) has zeros, which at first glance seems to lead to a nonintegrable singularity. However, we recall that, in accordance with the general prescription, we should in such cases regard the electron mass as having an infinitesimal imaginary additional part ( $m^2 \rightarrow m^2 - i\epsilon$ ), so that the above singularities turn out to be integrable.

By introducing into (35.8) another change of variables  $4m^2/(1 - z^2) = M^2$ , and by taking into account the addition of the above imaginary term, we obtain

$$I(k^2) = -\frac{k^2}{3} \int_{4m^2}^{\infty} \frac{dM^2 (1 + 2m^2/M^2) \sqrt{1 - 4m^2/M^2}}{M^2 (M^2 - k^2 - i\epsilon)}. \quad (35.9)$$

Substituting (35.9) in (35.6) and (35.7), we obtain the well-known parametric representation of the photon Green's function:

$$D_{mn}^{\text{tr}}(k) = \left( g^{mn} - \frac{k_m k_n}{k^2} \right) \left\{ \frac{1}{-k^2 - i\epsilon} + \int_{4m^2}^{\infty} \frac{\rho(M^2) 4M^2}{M^2 - k^2 - i\epsilon} \right\} \quad (35.10)$$

This is the so-called *Källén-Lehmann spectral representation* [Källén (1952), Lehmann (1954)]. Further details are given in §53.

The kernel of the spectral representation, determined in second-order perturbation theory in our case, has the form

$$\rho(x) \rightarrow \rho_2(x) = \frac{\alpha}{3\pi} \frac{x+2m^2}{x^2} \sqrt{1 - \frac{4m^2}{x}}.$$

The integral in (35.9) can be evaluated. To do this, it is convenient to write it in the form

$$I(k^2) = -\frac{5}{9} - \frac{4}{3} \frac{m^2}{k^2} + \frac{1}{3} \left(1 + \frac{2m^2}{k^2}\right) J\left(\frac{k^2}{4m^2}\right),$$

where

$$J(x) = (1-x) \int_1^{\infty} \frac{dz}{\sqrt{z(z-1)}(z-x+i\epsilon)}. \quad (35.11)$$

For  $x < 1$ , this has no singularities in the integrand:

$$\begin{aligned} J(x) &= 2 \sqrt{\frac{1-x}{x}} \arctan \left( \frac{x}{1-x} \right)^{1/2} \quad \text{for } 0 \leq x \leq 1, \\ &= 2 \left( \frac{x-1}{x} \right)^{1/2} \tan^{-1} \left( \frac{x}{x-1} \right)^{1/2} = \\ &= 2 \sqrt{\frac{x-1}{x}} \ln \left( \sqrt{1-x} + \sqrt{-x} \right) \quad \text{for } x \leq 0. \end{aligned} \quad (35.12)$$

When  $x > 1$ , the integrand in (35.11) contains a pole which can be bypassed by adding the infinitesimal quantity  $i\epsilon$ . Using the symbolic formula given by (A2A.7) in Appendix 2, we have

$$J(x) = -i\pi \sqrt{\frac{x-1}{x}} + 2 \sqrt{\frac{x-1}{x}} \ln(\sqrt{x} + \sqrt{x-1}) \quad \text{for } x > 1. \quad (35.13)$$

Combining (35.12) and (35.13), and using the relation

$$ix \tan^{-1} \frac{1}{ix} = x \arctan \frac{1}{x},$$

we obtain

$$\begin{aligned} I(k^2) &= -\left(\frac{5}{9} + \frac{4}{3} \frac{m^2}{k^2}\right) + \frac{2}{3} \left(1 + \frac{2m^2}{k^2}\right) \sqrt{1 - \frac{4m^2}{k^2}} \tan^{-1} \frac{1}{\sqrt{1 - 4m^2/k^2}} - \\ &\quad - \theta(k^2 - 4m^2) \frac{i\pi}{3} \left(1 + \frac{2m^2}{k^2}\right) \sqrt{1 - \frac{4m^2}{k^2}}. \end{aligned} \quad (35.14)$$

For small  $k^2$  we obtain from the above

$$I(k^2) = -\frac{1}{15} \frac{k^2}{m^2} + O\left(\frac{k^2}{m^2}\right)^2 \quad (k^2 \ll m^2).$$

Comparing this result with (35.6) we obtain an important property\* of the radiative correction to the function  $d(k^2)$ :

$$d(k^2) = 1 - \frac{e^2}{60\pi^2} \frac{k^2}{m^2} + \dots \quad (k^2 \ll m^2)$$

it vanishes in the limit as  $k^2 \rightarrow 0$ .

For large  $|k^2|$ , we correspondingly obtain:

$$I(k^2) = \frac{1}{3} \ln \frac{|k^2|}{m^2} - \frac{5}{9} + O\left(\frac{m^2}{k^2}\right) \quad (|k^2| \gg m^2).$$

This yields:

$$d(k^2) = 1 + \frac{e^2}{12\pi^2} \ln \frac{|k^2|}{m^2} + \dots \quad (35.15)$$

**35.2. Corrections to the Electron Green's Function.** The second-order correction to the electron Green's function has the form

$$\frac{i}{\hat{p} - m} \frac{\Sigma^{(2)}(p)}{i} \frac{i}{\hat{p} - m}.$$

For the determination of  $\Sigma^{(2)}(p)$  we shall start with the photon causal function for arbitrary  $d_I$ . We shall determine the arbitrary constants in the expression for  $\Sigma^{(2)}(p)$  (compare (27.30)) from the conditions

$$\Sigma^{(2)}(m) = 0, \quad \left. \frac{\partial \Sigma^{(2)}(\hat{p})}{\partial \hat{p}} \right|_{\hat{p}=m} = 0. \quad (35.16)$$

It turns out that the derivative  $\partial \Sigma^{(2)}(\hat{p})/\partial \hat{p}$  with respect to  $\hat{p}$  contains a term of the form  $\ln(1 - p^2/m^2)$  which diverges logarithmically as  $\hat{p} \rightarrow m$ . This divergence is a manifestation of the infrared catastrophe and is related to the fact that it is not permissible to make expansions in terms of the number of emitted photons in the case of processes involving photons with small momenta. The nature of infrared divergences was investigated in its time by Bloch and Nordsieck (1937). For a more detailed analysis of this difficulty we refer the reader to §35.4 (see also §50.3, Chapter IX), and at present restrict ourselves to

\*Actually, this property is a consequence of the vanishing of the second derivatives of the polarization operator for  $k = 0$  (see 31.52).

pointing out that the total transition probabilities in each order in  $e^2$  are free of infrared divergences, while the actual regularization of the intermediate expressions is usually achieved by introducing the small fictitious photon mass  $\lambda_0$ .

By modifying the corresponding calculations of §27 we obtain in place of (27.30)

$$\Sigma^{(2)}(p) = \frac{e^2}{16\pi^2} \left\{ \frac{p^2 - m^2}{m^2} \left( d_l \frac{p^2 + m^2}{p^2} \hat{p} - (d_l + 3)m \right) A(p^2) + \right. \\ \left. + d_l \hat{p} \frac{p^2 - m^2}{p^2} + c_1(\hat{p} - m) + c_2 m \right\}, \quad (35.17)$$

where (omitting the imaginary part equal to  $-\pi i(m^2/p^2)\theta(p^2 - m^2)$ )

$$A(p^2) = \int_0^1 \frac{dx (x-1) m^2}{(1-x)(m^2 - xp^2) + x\lambda_0^2} = \begin{cases} \frac{m^2}{p^2} \ln \left[ \frac{m^2 - p^2}{m^2} \right] & \text{for } |p^2 - m^2| \gg \lambda_0^2, \\ \frac{1}{2} \ln \frac{\lambda_0^2}{m^2} & \text{for } p^2 = m^2. \end{cases} \quad (35.18)$$

By determining the constants  $c_1$  and  $c_2$  from the conditions (35.16) we obtain in place of (35.17)

$$\Sigma^{(2)}(p) = \frac{e^2}{16\pi^2} \left\{ \frac{p^2 - m^2}{m^2} \left( d_l \frac{p^2 + m^2}{p^2} \hat{p} - (d_l + 3)m \right) A(p^2) + d_l \hat{p} \frac{p^2 - m^2}{p^2} - \right. \\ \left. - 2d_l(\hat{p} - m) + 2(3 - d_l) A(m^2)(\hat{p} - m) \right\} = \frac{e^2}{16\pi^2} (\hat{p} - m) \left\{ A(p^2) \left( d_l \frac{p^2}{m^2} - 3 \right) + \right. \\ \left. + A(m^2)(6 - 2d_l) - d_l + \frac{\hat{p}}{m} \left[ A(p^2) \left( d_l \frac{m^2}{p^2} - 3 \right) + d_l \frac{m^2}{p^2} \right] \right\}. \quad (35.19)$$

Substituting (35.19) into the sum

$$G(p) = \frac{1}{m - \hat{p}} \left( 1 + \Sigma(p) \frac{1}{\hat{p} - m} \right),$$

we obtain

$$G(p) = \frac{1}{m - \hat{p}} \left\{ 1 + \frac{e^2}{16\pi^2} \left[ A(p^2) \left( d_l \frac{p^2}{m^2} - 3 \right) + A(m^2)(6 - 2d_l) - d_l \right] + \right. \\ \left. + \frac{e^2}{16\pi^2} \frac{\hat{p}}{m} \left[ A(p^2) \left( d_l \frac{m^2}{p^2} - 3 \right) + d_l \frac{m^2}{p^2} \right] \right\}. \quad (35.20)$$

Sometimes it is also convenient to represent  $G(p)$  in the form

$$G(p) = \frac{\hat{p}a(p^2) + mb(p^2)}{m^2 - p^2}, \quad (35.21)$$

where in accordance with (35.20)

$$a(p^2) = 1 + \frac{e^2}{16\pi^2} \left\{ \frac{(p^2 - m^2)^2}{p^2 m^2} d_l A(p^2) + [A(p^2) - A(m^2)] (2d_l - 6) - d_l + d_l \frac{m^2}{p^2} \right\}, \quad (35.22)$$

$$b(p^2) = 1 + \frac{e^2}{16\pi^2} \left\{ \frac{p^2 - m^2}{m^2} (d_l - 3) A(p^2) + [A(p^2) - A(m^2)] (2d_l - 6) \right\}. \quad (35.23)$$

For  $p^2 \gg m^2$  we obtain:

$$G(p) = -\frac{1}{\hat{p}} \left\{ 1 + \frac{e^2 d_l}{16\pi^2} \ln \frac{p^2}{m^2} \right\}. \quad (35.24)$$

35.3. *Corrections to the Vertex Part.* We now turn to the vertex part

$$\Gamma^n(p, q, k) = \gamma^n + \Lambda^n(p, q, k),$$

where, as usual,  $p$  and  $q$  are respectively the four-moment of the emerging and the incident electron, while  $k = p - q$  is the momentum of the incident photon (Fig. 44). It was shown in §28 that the most general expression for  $\Lambda_3^n$  contains the term  $C\gamma^n$  where the coefficient  $C$  is an arbitrary constant. In accordance with §33, this coefficient must be determined by the condition of gauge invariance of the scattering matrix which has the form

$$\Lambda_3^n(p, p, 0) = -\frac{\partial \Sigma_2(p)}{\partial p_n}. \quad (35.25)$$

By generalizing the corresponding calculations of §25 to the case  $d_l \neq 0$ ,  $\lambda_0 \neq 0$ , we obtain

$$\begin{aligned} \Gamma_{\text{reg}}^n(p, q, k) = & \gamma^n + \frac{e^2}{16\pi^2} \left\{ \gamma^n \left[ (1 + d_l) \int_0^1 dx \ln \left[ \frac{m^2}{m^2 - x(1-x)k^2} \right] - \frac{3}{2} - 2d_l - C \right] + \right. \\ & + \int_0^1 dx \int_0^{1-x} dy \frac{2A^n + A^n(1-x-y)(d_l-1) - 4\hat{Q}Q^n - 2R^n - (d_l-1)D^n}{Z} + \\ & \left. + (d_l-1) \int_0^1 dx \int_0^{1-x} dy \frac{(1-x-y)B^n}{Z^2} - 2\gamma^n \int_0^1 dx \int_0^{1-x} dy \ln \left[ \frac{(x+y)m^2}{Z} \right] \right\}, \quad (35.26) \end{aligned}$$

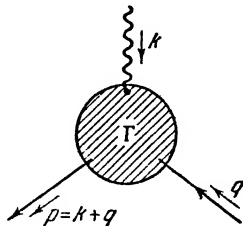


Fig. 44. Momentum variables for the vertex part.

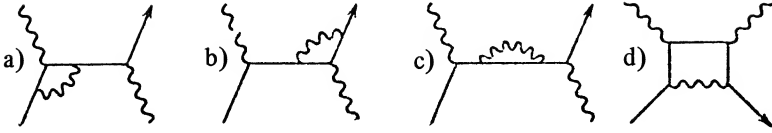


Fig. 45. Radiative corrections to the Compton effect which give matrix elements different from zero.

where we have used the notation

$$\left. \begin{aligned} A^n &= \gamma^n (\hat{q} + m) \hat{p} - (\hat{p} - m) (\hat{q} + m) \gamma^n - (\hat{p} - m) \gamma^n (\hat{q} - 2m), \\ B^n &= \hat{Q} (\hat{p} + m) \gamma^n (\hat{q} + m) \hat{Q}, \\ D^n &= \gamma^n (\hat{q} + m) \hat{Q} + \hat{Q} (\hat{p} + m) \gamma^n, \\ Q &= xq + yp, \\ R^n &= \hat{Q} \hat{k} \gamma^n - \gamma^n \hat{k} \hat{Q} - (\hat{p} - 2m) \gamma^n \hat{Q} - \hat{Q} \gamma^n (\hat{q} - 2m), \\ Z &= (1 - x - y) yp^2 + (1 - x - y) xq^2 + xyk^2 - m^2(x + y) - (1 - x - y) \lambda_0^2. \end{aligned} \right\} \quad (35.27)$$

By determining the constant  $C$  from condition (35.25), we find for it the value

$$C = 2 + (3 - d_l) \ln \frac{\lambda_0^2}{m^2}. \quad (35.28)$$

**35.4. Evaluation of Corrections to the Klein-Nishina Formula.** Formulas (35.5), (35.6), (35.20), and (35.26) for  $D$ ,  $G$ , and  $\Gamma$ , which contain radiative corrections of order  $e^2$  to the fundamental expressions, allow us to calculate directly radiative corrections of order  $e^2$  to the different effects in spinor electrodynamics. As an illustration of this, we shall discuss *schematically* the procedure of calculating radiative corrections to the Klein-Nishina formula, obtained in §26. For this it is evidently necessary to take into account fourth-order diagrams containing two external photon lines and two external electron lines. Altogether there are eight such diagrams. Four of them differ from the basic Compton-effect diagram by self-energy insertions into the external lines and, in accordance with conditions (34.47) and (34.52), give no contribution to the matrix elements. Two other diagrams contain vertex parts of the second order (Figs. 45a, b), one diagram contains a self-energy insertion into the internal fermion line (Fig. 45c), and, finally, the last diagram corresponds to the emission of an additional virtual photon (Fig. 45d).

The structure of the matrix elements corresponding to the diagrams of Figs. 45a, b is analogous to the structure of the basic second-order matrix elements (26.2) with the exception of the fact that the vertex parts contained in these diagrams correspond not to the matrix  $\gamma^n$ , but to the expression  $\Gamma^n(p, q, k)$  determined by (35.26).

We note in this connection that the vertex factors  $\Gamma^n(p, q, k)$  appearing in the above matrix elements correspond to one of the two groups of conditions:

$$\begin{aligned} 1) \quad & \hat{p} = m, \quad k^2 = 0, \quad e_n k^n = 0, \\ 2) \quad & \hat{q} = m, \quad k^2 = 0, \quad e_n k^n = 0. \end{aligned}$$



In these two cases, the integrals in (35.26) may be evaluated almost completely and the vertex factor  $\Gamma^n$  may be expressed in terms of algebraic combinations of the Dirac matrices, momenta, logarithms, and the function

$$w(z) = \frac{1}{z-1} \int_1^z \frac{dt}{t} (1-t) \ln(1-t). \quad (35.29)$$

The matrix elements corresponding to the diagram of Fig. 45c may be constructed by using (35.20) for the fermion propagation function in the intermediate state (this includes the radiative correction due to the self-energy loop contained in this diagram).

Thus, the matrix elements corresponding to all the fourth-order Compton scattering diagrams with the exception of the diagram of Fig. 45d may be evaluated very simply with the aid of the explicit finite expressions for  $G$  and  $\Gamma$  obtained above. This calculation no longer contains any integrations over the four-momenta, and actually reduces to the evaluation of the trace of the square of the sum of matrix elements (cf. (26.3), (26.4)).

Finally, the evaluation of the matrix elements corresponding to the last diagram (Fig. 43d) requires the evaluation of integrals of the type

$$\int \frac{dq}{q^2 + ie} \gamma_n \frac{1}{\hat{p}_2 - \hat{q} - m} \gamma^m \frac{1}{\hat{p}_1 + \hat{k}_1 - \hat{q} - m} \gamma^l \frac{1}{\hat{p}_1 - \hat{q} - m} \gamma^n, \quad (35.30)$$

where  $p_1 + k_1 = p_2 + k_2$ .

The integrals (35.30) do not contain ultraviolet divergences in an obvious manner. However, they diverge logarithmically for small  $q$  when  $\hat{p}_1 = m$ ,  $\hat{p}_2 = m$ . Indeed, in this case we have a singularity of the form

$$\int \frac{dq}{q^2} \frac{1}{(p_2 q)(p_1 q)},$$

and this may be removed, as usual, by introducing the photon mass  $\lambda_0$ . The actual evaluation of the integrals (35.30) is readily carried out by going over to the  $\alpha$ -representation of the causal functions (such as (27.12)). Because it is rather laborious, we shall not perform the evaluation of integrals such as (35.30) or of traces, and instead we refer the reader who is interested in the final expression for the Klein-Nishina formula, including radiative corrections of order  $e^4$ , to the article by Brown and Feynman (1952).

We shall here add only a few additional words on the subject of compensation of the infrared divergences in the final expressions. As we have already seen, terms proportional to  $\ln(\lambda_0^2/m^2)$  are contained in several fourth-order Compton scattering matrix elements. Such terms will also occur in the expressions for the transition probabilities. Thus, the expression for the probability of Compton scattering corresponding to the preceding eight fourth-order diagrams turns out to be *meaningless*.

The reason for this is that the problem has been *incorrectly formulated physically*. As we have mentioned, the infrared catastrophe is a consequence of the fact that it is not permissible to make expansions in the number of emitted photons at low photon energies.

From the experimental point of view, this situation corresponds to the physical indistinguishability of processes that differ from each other by the emission of a certain number of low-energy photons. Indeed, the fact that a photon of very low energy has been emitted may be established only by indirect methods by measuring the energy of all the other particles taking part in the process, and this may be carried out only with a certain finite precision. Therefore, there always remains the possibility of emission of photons with total energy smaller than a certain value  $E_{\max}$ , determined by the possible experimental error.

In the present case, the first radiative corrections to the Klein-Nishina formula of order  $e^6$ , whose origin lies in the interference between second- and fourth-order terms, turn out to be physically indistinguishable from the so-called *double* Compton scattering in which, in addition to the usually observed Compton photon, a second photon of very low energy is emitted, the probability of this being also of order  $e^6$ . If we take into account the probability of the double Compton effect, integrated over the energies of the second photon from zero to  $E_{\max}$ , we obtain an expression which is free of the infrared catastrophe, but which explicitly involves the quantity  $E_{\max}$ .

### §36. Some Models of Strong Interactions

In this section, we shall consider three relatively simple quantum-field models that are sometimes used in qualitative descriptions of the interaction of mesons and nucleons.

**36.1. The  $\varphi^3$  Model.** We begin with the simplest model of a real scalar field with a non-linear cubic term in Lagrangian:

$$\mathcal{L}_{\text{tot}} = \frac{1}{2} \varphi; n \varphi; n - \frac{m^2}{2} \varphi^2 + \frac{g^3}{3!} \varphi^3. \quad (36.1)$$

This term may be looked upon as the interaction of the field  $\varphi$  with itself. Transforming to the interaction representation, we obtain the interaction Lagrangian

$$\mathcal{L}(x) = \frac{g}{3!} : \varphi^3(x) :. \quad (36.2)$$

We note at once that the wave field defined by (36.1) is essentially *fictitious*. It is, however, of *methodologic interest* because the Lagrangian (36.2) is analytically the simplest, and the description of this field within the scheme provided by interaction theory is based on the single and simplest scalar-field causal function

$$D^c(x) = \frac{1}{(2\pi)^4} \int \frac{e^{-ikx}}{m^2 - k^2 - i\varepsilon} dk. \quad (36.3)$$

Following the general prescription of §32, we shall now classify the diagrams corresponding to divergent matrix elements. The structure of the “interaction” Lagrangian (36.2) shows that three identical lines are encountered at each vertex of the Feynman diagram.

Since the degree of the polynomial in the numerator of the causal function (36.3) is zero, and if we use (32.1), we find that the maximum vertex index is negative:

$$\omega_i^{\max} = -1.$$

The Lagrangian (36.2) is therefore of the renormalized type. The formula given by (32.5) now enables us to perform the complete classification of divergent diagrams.

We begin by considering the vacuum diagrams ( $l_{\text{ext}} = 0$ ) which, in the present case, will always contain an even number of vertices. It follows from (32.5) that the index of the vacuum diagrams is nonnegative only for  $i = 2$  and  $i = 4$ . In the former case, the positron corresponds to the diagram shown in Fig. 46a where  $\omega(G) = 2$ , and we have a quadratic divergence. In the second case (Fig. 46b), we have  $\omega(G) = 0$  and the divergence is logarithmic in character.

For diagrams with one external line ( $l_{\text{ext}} = 1$ ), the number of vertices corresponding to (36.2) is always odd. There are no diagrams with one vertex corresponding to one external line, and the diagram with three vertices ( $i = 3$ ), shown in Fig. 47a, leads to a logarithmic divergence  $\omega(G) = 0$ . There are no other divergences when  $l_{\text{ext}} = 1$ .

Finally, the only divergent diagram corresponding to two external lines is the diagram with two vertices (Fig. 47b). Its index is zero and the corresponding divergence is logarithmic.

The diagrams shown in Figs. 46–47 exhaust all the divergent diagrams in the  $\varphi^3$  model. We emphasize that the existence of a *finite number of divergent diagrams* is an important distinguishing feature of the Lagrangian given by (36.2). We shall see later that other theories usually involve an infinite number of divergent diagrams in all arbitrarily large orders in the coupling constant.

In the above case, the divergences are present only in the second, third, and fourth orders. The corresponding regularizing quasilocal operators are therefore of the form

$$\begin{aligned}\Lambda_2(x_1, x_2) &= (A_1 + A_2 \square_{x_1}) \delta(x_1 - x_2) + A_3 : \varphi^2(x_1) : \delta(x_1 - x_2), \\ \Lambda_3(x_1, x_2, x_3) &= B \varphi(x_1) \delta(x_1 - x_2) \delta(x_1 - x_3), \\ \Lambda_4(x_1, x_2, x_3, x_4) &= C \delta(x_1 - x_2) \delta(x_1 - x_3) \delta(x_1 - x_4)\end{aligned}$$

and, after integration, leads to the following counterterms:

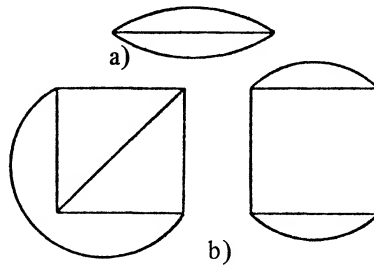


Fig. 46.

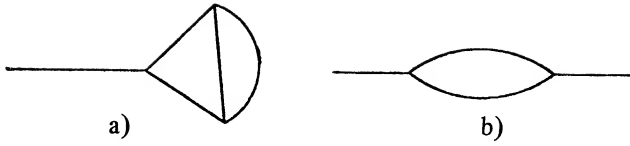


Fig. 47.

$$\Delta\mathcal{L} = \frac{A_3}{2!} : \varphi^2(x) : + \frac{B}{3!} : \varphi(x) : + \frac{C}{4!} + A_{1,2}, \quad (36.4)$$

where the constants  $A_3$ ,  $B$ , and  $C$  diverge logarithmically as  $M \rightarrow \infty$  and  $A_{1,2}$  diverges quadratically.

From the point of view of renormalization, the counterterms  $C$  and  $A$  are unimportant additive constants, the term  $A_3 : \varphi^2(x) :$  leads to mass renormalization, and the term  $B\varphi$  produces an additive renormalization of the potential and an additional mass renormalization:

$$\begin{aligned} \mathcal{L} + \Delta\mathcal{L} &= \frac{1}{2} : \varphi;_n \varphi;^n : - \frac{m^2}{2} : \varphi^2 : + \lambda : \varphi^3(x) : + A : \varphi^2(x) : + B : \varphi(x) : + C + A_{1,2} = \\ &= \frac{1}{2} : \varphi';_n \varphi';^n : - \frac{m'^2}{2} : \varphi'^2(x) : + \lambda : \varphi'^3(x) : + C', \end{aligned} \quad (36.5)$$

where

$$\varphi'(x) = \varphi(x + \varphi_0), \quad (36.6)$$

and  $\varphi_0$ ,  $m'^2$ ,  $C'$  are constants expressed algebraically in terms of  $m$ ,  $\lambda$ ,  $A$ . We also note that because of the absence of differential operators from the quasilocal operators which depend on the field functions  $\varphi(x)$ , the counterterms for the Schroedinger equation have the form

$$\Delta\mathcal{L}(x; g) = \frac{A_3}{2!} : \varphi^2(x) : g(x) + \frac{B}{3!} : \varphi(x) : g^3(x). \quad (36.7)$$

Thus, in the model defined by (36.2) there are no infinite renormalizations of the field function  $\varphi$  and the coupling constant  $g$ . The only physically essential divergence is the divergence corresponding to the second-order self-energy diagram shown in Fig. 47b. This divergence leads to mass renormalization. Let us examine it in greater detail.

The diagram of Fig. 47b gives the following addition to the three Green's function for the field  $\varphi$ :

$$\overline{\varphi(x) \varphi(y)} \rightarrow \overline{\varphi(x) \varphi(y)} + \frac{i^2 g^2}{2} \int \overline{\varphi(x) \varphi(z)} dz [\overline{\varphi(z) \varphi(t)}]^2 dt \overline{\varphi(t) \varphi(y)}, \quad (36.8)$$

or, in the momentum representation,

$$D(k^2) = \frac{1}{m^2 - k^2} \rightarrow D(k^2) - D(k^2) M_2(k^2) D(k^2). \quad (36.9)$$

The second-order mass operator can be written in the form

$$M_2(k^2) = \frac{g^2}{32\pi^2} I(k^2), \quad (36.10)$$

where

$$I(k^2) = 16\pi^2 i \int e^{ik(z-t)} [i\varphi(z)\varphi(t)]^2 dt. \quad (36.11)$$

In the momentum representation, the integral  $I$  can be written in the form

$$I(k^2) = \frac{i}{\pi^2} \int \frac{d^4 p}{(m^2 - p^2 - i\varepsilon) [m^2 - (p+k)^2 - i\varepsilon]} \quad (36.12)$$

and diverges logarithmically. It can be evaluated with the aid of the  $R$ -operation. Performing the subtraction at the point  $k^2 = \lambda$ , and following the standard procedure with the aid of (27.12), (27.13), and (27.23), we obtain

$$R_\lambda I(k^2) = I_\lambda(k^2) = \int_0^1 dx \ln \left[ \frac{k^2 x(1-x) - m^2 + i\varepsilon}{\lambda x(1-x) - m^2 + i\varepsilon} \right]. \quad (36.13)$$

We now use on the right-hand side of (36.13) the transformations employed in §35.1 in the evaluation of the polarization operator. We reduce  $I_\lambda$  to the spectral form

$$I_\lambda(k^2) = (\lambda - k^2) \int_{4m^2}^{\infty} \frac{\sqrt{1 - \frac{4m^2}{M^2}} dM^2}{(M^2 - k^2 - i\varepsilon)(M^2 - \lambda)}, \quad (36.14)$$

and this can then be evaluated by writing it in the form

$$I_\lambda(k^2) = J\left(\frac{k^2}{4m^2}\right) - J\left(\frac{\lambda}{4m^2}\right), \quad (36.15)$$

where  $J(x)$  is defined by (35.11).

The parameter  $\lambda$  must be chosen so that the subtracted constant which, when the intermediate regularization is removed, transforms to the mass counterterm

$$\Delta \mathcal{L} \sim \frac{g^2}{32\pi^2} J\left(\frac{\lambda}{4m^2}\right) \varphi^2(x),$$

is real and the counterterm  $\Delta\mathcal{L}$  is Hermitian. It follows from (36.14) and (36.11) that this will be so provided

$$\lambda \leq 4m^2. \quad (36.16)$$

In particular, we may substitute  $\lambda = m^2$ , i.e., subtract the self-energy diagram on the "mass surface." We then have

$$I_{m^2}(k^2) = (m^2 - k^2) \int_{4m^2}^{\infty} \frac{\sqrt{1 - \frac{4m^2}{M^2}} dM^2}{(M^2 - k^2 - i\varepsilon)(M^2 - m^2)}, \quad (36.17)$$

or

$$I_{m^2}(k^2) = J\left(\frac{k^2}{4m^2}\right) - J\left(\frac{1}{4}\right) = J\left(\frac{k^2}{4m^2}\right) - \frac{\pi}{\sqrt{3}}. \quad (36.18)$$

The subtraction of the radiative corrections to the single-particle Green's function on the mass surface is convenient because the complete Green's function  $D(k^2, g^2)$  is then found to have the same pole structure at  $k^2 = m^2$  as the free Green's function

$$D(k^2, g^2) = \frac{d(k^2, g^2)}{m^2 - k^2 - i\varepsilon}. \quad (36.19)$$

The dimensionless function  $d$  which we have introduced here is finite at  $k^2 = m^2$ . According to (36.9), (36.10), and (36.17), we have in our case

$$d \rightarrow d_2(k^2, g^2) = 1 - \frac{g^2}{32\pi^2} \frac{I_{m^2}(k^2)}{m^2 - k^2}. \quad (36.20)$$

This expression is finite in the neighborhood of the mass surface

$$d_2(m^2, g^2) = 1 - \frac{g^2}{96\pi^2 m^2} \left( \frac{2\pi}{\sqrt{3}} - 1 \right), \quad (36.21)$$

and has a root singularity at the two-particle threshold

$$d_2(k^2 \sim 4m^2, g^2) \simeq 1 - \frac{g^2}{96\pi m^2} \left[ i \sqrt{1 - \frac{4m^2}{k^2}} + \frac{\pi}{\sqrt{3}} \right] \quad (36.22)$$

and the logarithmic singularity in the ultraviolet asymptotic region

$$d_2(k^2, g^2) \simeq 1 + \frac{g^2}{32\pi^2 k^2} \left[ \ln \frac{|k^2|}{m^2} - \frac{\pi}{\sqrt{3}} - i\pi\theta(k^2) \right] \quad \text{for } |k^2| \gg m^2. \quad (36.23)$$

36.2. *Pseudoscalar Field with Nonlinear Interaction.* We now consider a real scalar field described by the nonlinear Lagrangian

$$\mathcal{L}(x) = \frac{1}{2} : \varphi; \pi \varphi; \pi : - \frac{m^2}{2} : \varphi^2 : + \lambda : \varphi^4 :. \quad (36.24)$$

Since  $\mathcal{L}$  depends only on the even powers of  $\varphi$ , the field  $\varphi$  may be looked upon not as a scalar but a pseudoscalar. If we now replace the function  $\varphi$  by the isospin triplet  $\varphi_0$  (or  $\pi(x)$ ; see §3.5)

$$\mathcal{L}(x) = \frac{1}{2} : (\pi; \pi \pi; \pi) : - \frac{m^2}{2} : (\pi \pi) : + \lambda : (\pi \pi)^2 :, \quad (36.25)$$

we obtain a model for the description of real interacting pions.

The Lagrangians given by (36.24) and (36.25) differ only by their isospin structure. For the sake of simplicity, we shall therefore confine our attention to the first of them, but we shall introduce reservations referring to (36.25) whenever necessary.

Consider the structure of the perturbation-theory terms, based on the interaction Lagrangian

$$\mathcal{L}_{\text{int}}(x) = \mathcal{L}(x) = \lambda : \varphi^4(x) :$$

and the propagator given by (36.3). There are four lines at each vertex of the Feynman diagrams. Accordingly, the maximum index (32.1) is

$$\omega_l^{\text{max}} = 0,$$

so that this model is renormalizable and the degree of divergence of the diagram is wholly determined by the number of external lines and is independent of the perturbation-theory order.

Since the field  $\varphi$  is pseudoscalar (there are no odd powers of  $\varphi$  in  $\mathcal{L}_{\text{int}}$ ), there are no diagrams with an odd number of external lines. The most strongly divergent are the vacuum diagrams. According to (32.5), their degree of divergence is 4. Diagrams with two external ends (Fig. 48) diverge quadratically and, finally, diagrams with four external ends (scattering diagrams of Fig. 49) diverge logarithmically.

The quasilocal operators that regularize the diagrams of Figs. 48 and 49 lead to the following counterterms after integration:

$$\Delta \mathcal{L} = \frac{Z_3 - 1}{2} \{ : \varphi; \pi \varphi; \pi : - m^2 : \varphi^2 : \} - \frac{\delta m^2}{2} : \varphi^2 : + \lambda (Z_4 - 1) : \varphi^4(x) :. \quad (36.26)$$

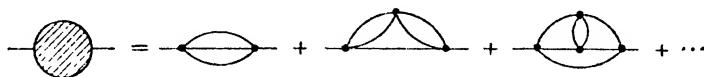
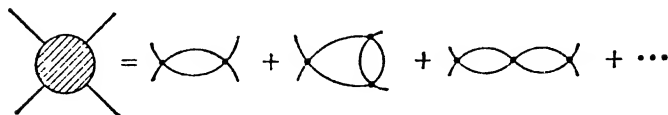


Fig. 48. Proper energy diagrams in the  $\varphi^4$  model.

Fig. 49. Scattering diagrams in  $\varphi^4$  model.

The quantities  $Z_3$ ,  $Z_4$  and  $\delta m^2$  are then given by series in powers of  $\lambda^2$  and the corresponding coefficients in the expansions of  $Z_3$ ,  $Z_4$  diverge logarithmically for  $M \rightarrow \infty$  whereas the coefficients in  $\delta m^2$  diverge quadratically.

We now consider the lowest radiative corrections.

The first radiative correction to the single-particle propagator is of the form given by (36.13) where the second-order mass operator now corresponds to the diagram with three internal lines (first diagram on the right-hand side of Fig. 48) and can be written in the form

$$M_2(k^2) = \frac{96h^2}{(2\pi)^8 i} \int d^4p d^4q D(q) D(p) D(p+q+k) = \frac{3h^2}{4\pi^4 i} \tilde{J}(k^2), \quad (36.27)$$

$$\tilde{J}(k^2) = \frac{1}{\pi^4} \int \frac{d^4q d^4p}{(m^2 - q^2)[m^2 - (k+q+p)^2](m^2 - p^2)}. \quad (36.28)$$

The integral  $\tilde{J}$  diverges quadratically. Two subtractions at  $k^2 = \lambda$  yield

$$R_\lambda \tilde{J}(k^2) = \tilde{J}_\lambda(k^2) = \int_0^1 dx \int_0^{1-x} \frac{dy}{d^2(x, y)} \left\{ (ak^2 - m^2) \ln \left[ \frac{ak^2 - m^2 + i\epsilon}{a\lambda - m^2} \right] + a(\lambda - k^2) \right\}. \quad (36.29)$$

where

$$d(x, y) = xy + (x+y)(1-x-y),$$

$$a = a(x, y) = \frac{xy(1-x-y)}{d(x, y)}.$$

Now consider the vertex function with four ends. The first radiative correction corresponds to the first diagram on the right-hand side of Fig. 49. It is clear that it is identical with the diagram for the self-energy in the  $\varphi^3$  model, shown in Fig. 47b. Accordingly, the first corrections to the four-meson vertex can be written in the form

$$h \rightarrow \Gamma_4(k_1, k_2, k_3, k_4) = h - \frac{3h^2}{4\pi^2} \{I(s) + I(t) + I(u)\}, \quad (36.30)$$

where

$$s = (k_1 + k_2)^2, \quad t = (k_1 + k_3)^2, \quad u = (k_1 + k_4)^2,$$



and the functions  $I(k^2)$  were introduced in §36.1 [see (36.15)–(36.17)].

The question now is how to choose the subtraction points in the divergent contributions to the vertex functions  $\Gamma_4$ . Such subtraction points can, in general, be chosen independently for different diagrams. However, it is often convenient in practice to choose them in the same way for all the contributions to  $\Gamma_4$ . Moreover, we note that Lorentz invariance ensures that the scalar function  $\Gamma_4$  depends on the invariant squares of four-momenta  $k_1, \dots, k_4$  and their paired sums:

$$k_1^2, k_2^2, k_3^2, k_4^2, (k_1 + k_2)^2 = s, (k_1 + k_3)^2 = t, (k_1 + k_4)^2 = u. \quad (36.31)$$

In addition, since the four-momentum is conserved,

$$k_1 + k_2 + k_3 + k_4 = 0$$

we have

$$s + t + u = k_1^2 + k_2^2 + k_3^2 + k_4^2. \quad (36.32)$$

Of the seven invariant arguments (36.31) of the vertex function  $\Gamma_4$ , only six are therefore linearly independent and the subtraction point in the seven-dimensional manifold (36.31) must be chosen in the light of (36.22).

In practice, it is convenient, from the point of view of physical applications and the use of the  $R$ -operation in iterations, to take the symmetric choice of the subtraction point:

$$k_1^2 = k_2^2 = k_3^2 = k_4^2 = m^2, \quad s = u = t = \frac{4m^2}{3}. \quad (36.33)$$

This corresponds to

$$\Gamma_4 \left( (k_1^2 = k_2^2 = k_3^2 = k_4^2 = m^2, \quad s = t = u = \frac{4}{3} m^2) \right) = h, \quad (36.34)$$

because of all the radiative corrections are zero at the point defined by (36.33). If, at the same time, we take the subtraction point of the propagator as being equal to  $m^2$ , we can look upon (36.34) as a definition of the renormalized coupling constant. With this choice of the points of subtraction, we have

$$M_2(k^2) = \frac{3\hbar^2}{4\pi^2} \tilde{J}_{m^2}(k^2), \quad (36.35)$$

$$\tilde{J}_{m^2}(k^2) = \int_0^1 dx \int_0^{1-x} \frac{dy}{d^2} \left\{ (ak^2 - m^2) \ln \left[ \frac{ak^2 - m^2}{(a-1)m^2} \right] + a(m^2 - k^2) \right\} \quad (36.36)$$

and

$$\Gamma_4(\dots) = h - \frac{3h^2}{4\pi^2} \{ \tilde{I}(s) + \tilde{I}(t) + \tilde{I}(u) \}, \quad (36.37)$$

$$\tilde{I}(k^2) = \left( \frac{4m^2}{3} - k^2 \right) \int_{4m^2}^{\infty} \frac{\sqrt{1 - \frac{4m^2}{M^2}} dM^2}{(M^2 - k^2 - i\varepsilon) \left( M^2 - \frac{4}{3}m^2 \right)}. \quad (36.38)$$

**36.3. Pseudoscalar Model of the Meson-Nucleon Interaction.** As the third model of strong interactions, consider the pseudoscalar theory of the meson-nuclear interaction, i.e., the theory of interaction between the pseudoscalar meson and the spinor nucleon fields. The interaction Lagrangian will be taken in the pseudoscalar version which does not contain derivatives [such as (8.4)]. This model of the meson-nucleon interaction is sometimes referred to as the *pseudoscalar Yukawa model*.

We shall consider the so-called *symmetric* version which leads to nuclear forces that are symmetric with respect to the electric charge (for example, identical proton-proton and neutron-neutron scattering matrix elements). The mathematical description of the symmetric pseudoscalar theory is most conveniently performed within the framework of the *isospin formalism* (see Appendix 1).

The interaction Lagrangian invariant under “rotations in isospin space” has the form

$$\mathcal{L}(x) = ig : \bar{\Psi}(x) \gamma^5 \tau \Psi(x) \varphi(x) :. \quad (8.8)$$

The constant  $g$  plays the role of the coupling constant and is called the *mesonic charge*.

We also note that, for the sake of simplicity, one sometimes considers the so-called “neutral model” of the pseudoscalar theory which has the Lagrangian

$$\mathcal{L}(x) = ig : \bar{\psi}(x) \gamma^5 \psi(x) \varphi(x) :. \quad (36.39)$$

Here  $\bar{\psi}$ ,  $\psi$  are four-component spinors and  $\varphi$  is a real one-component function. The neutral model differs from the symmetric model by the absence of the “isospin algebra,” but at the same time retains the principal elements of its structure. In future, we shall sometimes for the sake of brevity combine (8.8) and (36.39) in the form

$$\mathcal{L}(x) = g : \bar{\psi}(x) \Gamma \psi(x) \varphi(x) :, \quad (36.40)$$

without specifying the structure of the vertex matrix  $\Gamma$ .

Turning now to the analysis of divergent diagrams, we note that the elementary pairings have the form

$$\overline{\Psi_\alpha(x)} \bar{\Psi}_\beta(y) = \frac{1}{i} \frac{1}{(2\pi)^4} \int dp e^{-ip(x-y)} \frac{(M + \hat{p})_{\alpha\beta}}{M^2 - p^2 - i\varepsilon}, \quad (36.41)$$

$$\overline{\varphi_i(x)} \varphi_j(y) = \frac{\delta_{ij}}{i(2\pi)^4} \int \frac{dk e^{ik(x-y)}}{\mu^2 - k^2 - i\epsilon}, \quad (36.42)$$

where  $M_{\alpha\beta} = MI_{\alpha\beta}$ ,  $\hat{p}_{\alpha\beta} = \Sigma_n g^{nn} p^n \gamma_{\alpha\beta}^n$ , and  $I_{\alpha\beta}$  is the unit matrix of the eighth rank, while  $\gamma_{\alpha\beta}^n$  are matrices of the eighth rank, related to the ordinary fourth-rank Dirac matrices by expressions of the form

$$\gamma_{\alpha\beta}^n = \begin{pmatrix} \gamma^n & 0 \\ 0 & \gamma^n_{\alpha\beta} \end{pmatrix}.$$

Finally,  $M$  and  $\mu$  are the nucleon and the pion masses, respectively.

It is essential for us in this case that, in accordance with (36.41) and (36.42), the degree of the polynomial  $P$  in the numerator of the causal function is zero for the meson line and unity for the nucleon line. Therefore, the maximum vertex index

$$\omega_i^{\max} = \frac{1}{2} \sum_i (r_i + 2) - 4$$

turns out to be zero, and the Lagrangian (36.40) belongs to the renormalizable type.

Turning to the classification of divergent diagrams, we note that the situation in this case is very reminiscent of spinor electrodynamics and differs from it only in the following respects: (a) the meson mass is different from zero, which results in the absence of gauge invariance, (b) the meson functions form an isospin three-vector and not a four-vector as in the case of the electromagnetic field, and (c) the structure of the vertex matrix  $\Gamma$  differs from the four-vector of the Dirac matrices  $\gamma^k$ .

In view of this, some of the results of the discussion given for the case of spinor electrodynamics in §33 and §34 may be directly carried over to the pseudoscalar meson theory.

Thus, taking Furry's theorem into account, we obtain the types of divergent diagrams shown in Fig. 50. In the foregoing, due to the absence of considerations of gauge invariance, the diagram with four external boson lines (Fig. 50a) leads, in contrast to spinor electrodynamics, to a divergence. To compensate this, we must introduce a special "four-meson" counterterm of the form  $Z:\varphi^4:$ , while for the subtraction of divergences connected with the meson self-energy diagram (Fig. 50c) we must introduce in addition to a counterterm of

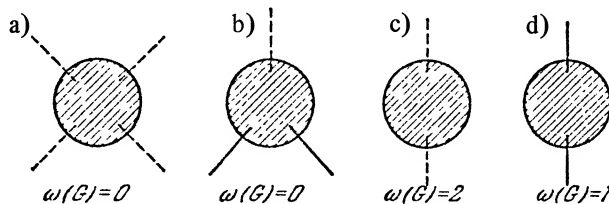


Fig. 50. Types of divergent diagrams in the model defined by (36.40).

the type  $(Z_3 - 1)\varphi(p^2 - \mu^2)\varphi$  the counterterm for the proper mass of the meson  $-\delta\mu\varphi^2$ . As a result, we obtain the complete interaction Lagrangian in the form

$$\mathcal{L}_{\text{tot}} = \mathcal{L} + \Delta\mathcal{L} = g: \bar{\Psi}\Gamma\Psi\varphi: + (Z_1 - 1)g: \bar{\Psi}\Gamma\Psi\varphi: + (Z_2 - 1): \bar{\Psi}(\not{p} - M)\Psi: - \delta M: \Psi\Psi: + (Z_3 - 1): \varphi(p^2 - \mu^2)\varphi: - \delta\mu^2: \varphi\varphi: + Z: \varphi^4: \quad (36.43)$$

*36.4. Second Charge, Multiplicative Renormalizations, and External Lines.* It follows from the general theory that the introduction of the counterterms (36.43) into the interaction Lagrangian may be replaced by a redefinition of the corresponding chronological products. However, it is important to note that this introduces a degree of arbitrariness into the theory, which is related to the possibility of introducing into the interaction Lagrangian finite terms with the six arbitrary coefficients

$$z_1, z_2, z_3, \delta M, \delta\mu^2 \text{ and } z,$$

which have the same operator structure as (36.43).

By analyzing the influence of these terms on nucleon and meson Green's functions  $G$  and  $\Delta$ , the vertex part  $\Gamma$ , and the masses and the charge, we conclude that the introduction of the constants  $z_1$ ,  $z_2$ , and  $z_3$  is equivalent to a renormalization of  $G$ ,  $D$  and  $\Gamma$ , and that, consequently, it corresponds to a change in the charge  $g$ , while  $\delta M$  and  $\delta\mu^2$  lead to a change in the meson and nucleon masses in the internal lines of the diagrams.

However, the term  $z\varphi^4$  does not amount to a change in  $g$ ,  $M$ , and  $\mu^2$ . This means that, in addition to the masses  $M$ ,  $\mu$ , and the mesonic charge  $g$ , the theory actually contains one other constant  $h$  which may be introduced into the interaction Lagrangian from the outset as the coefficient of  $\varphi^4$ . Thus,  $\mathcal{L}(x)$ , takes the form

$$\mathcal{L}(x) = g: \Psi(x)\Gamma\Psi(x)\varphi(x): + h: \varphi^2\varphi^2: \quad (36.44)$$

(in the case of the symmetric theory  $\varphi^2 \rightarrow \varphi^2$ ).

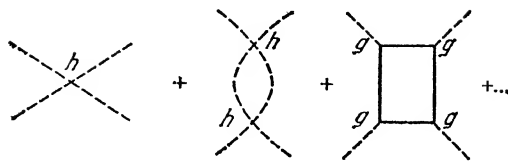
The constant introduced above in fact plays a role of a *second charge*. The magnitude of this second charge like the quantity  $g$  must be determined by experiment. The introduction of the term containing  $z$ , which may be conveniently represented in the form

$$z = (z_4 - 1)h,$$

now amounts to a renormalization of the charge  $h$ . Arguments analogous to these given in §34.2 lead us to conclude that the introduction into the interaction Lagrangian of the finite terms

$$\delta\mathcal{L} = (z_1 - 1)g: \bar{\Psi}\Gamma\Psi\varphi: + (z_2 - 1): \bar{\Psi}(\not{p} - M)\Psi: - \delta M: \bar{\Psi}\Psi: + \\ + (z_3 - 1): \varphi(p^2 - \mu^2)\varphi: - \delta\mu^2: \varphi^2: + (z_4 - 1)h: \varphi^2\varphi^2: \quad (36.45)$$

is equivalent to the following renormalization of the nucleon and meson Green's functions  $G$ ,  $\Delta$  and of the vertex functions  $\Gamma$  and  $\square$ :

Fig. 51. Diagrams corresponding to the four-vertex function  $\square$ .

$$\left. \begin{aligned} G(\mu, M, g, h) &\rightarrow z_2^{-1} G(\mu', M', g', h'), \\ \Delta(\mu, M, g, h) &\rightarrow z_3^{-1} \Delta(\mu', M', g', h'), \\ \Gamma(\mu, M, g, h) &\rightarrow z_1 \Gamma(\mu', M', g', h'), \\ \square(\mu, M, g, h) &\rightarrow z_4 \square(\mu', M', g', h'). \end{aligned} \right\} \quad (36.46)$$

Here

$$m' = m + \delta m, \quad M' = M + \delta M, \quad g' = z_1 z_2^{-1} z_3^{-1/2} g, \quad h' = z_4 z_3^{-2} h, \quad (36.47)$$

while  $\square$  is a four-vertex function which may be represented by the sum of diagrams with four external meson lines (Fig. 51). By continuing the argument of §34.2, we see that from the point of view of the internal parts of the diagram, the introduction of the six terms (36.45) is equivalent to the following transformation of the two masses and the two charges:

$$\mu^2 \rightarrow \mu'^2 = \mu^2 + \delta\mu^2, \quad M \rightarrow M' = M + \delta M, \quad g \rightarrow g' = z_1 z_2^{-1} z_3^{-1/2} g, \quad h \rightarrow h' = z_4 z_3^{-2} h. \quad (36.48)$$

We now turn to the external lines. By arguments similar to those given in §34.5, we find that in order to eliminate the arbitrariness in the external lines, it is particularly convenient to expand the self-energy parts of the meson and of the nucleon about the points  $k^2 = \mu^2$  and  $\bar{p} = M$ , respectively. These conditions fix the constants  $\delta\mu$ ,  $\delta M$ ,  $z_2$ , and  $z_3$ .

The constants  $z_1$  and  $z_4$  may be uniquely determined by specifying an experimental method for the determination of the charges  $g$  and  $h$ , for example, by assuming that the experimental value of the charge  $g$  is determined from the scattering by a nucleon of a meson of zero velocity, while the value of  $h$  is determined from the mutual scattering of two mesons of zero velocity.

### §37. Complete Green's Functions and Vertex Functions

*37.1. Higher Green's Functions.* Chronological pairings of the free-field operators  $\Delta^c(x-y)$ , which are Green's functions for the corresponding free-field equations

$$(\square - m^2) \Delta^c(x-y) = -\delta(x-y)$$

play a basic role in quantum-field perturbation theory.

From the physical point of view,  $\Delta^c$  are the propagation factors for particles moving between points  $x$  and  $y$  without interacting with other fields. Such interactions result in radiative corrections to the elementary propagators  $\Delta^c$ , the inclusion of which leads to the so-called complete Green's functions  $\Delta(x; g)$  which include interaction effects and depend on the coupling constants  $g_i$ .

The complete Green's functions were introduced above in a semi-intuitive fashion by summing terms corresponding to the Feynman diagrams. Such complete Green's functions are the propagation factors for particles moving between points  $x$  and  $y$  *with allowance for the interaction between these particles*. They can, therefore, be looked upon as generalizations of Green's functions for free fields to the case of interactions.

We shall now give precise definitions of complete single-particle Green's functions  $\Delta(x; g)$ , and also the higher Green's functions  $\Gamma(x_1, \dots, x_n; g)$  which are generalizations of the vertex factors (vertices), similar to the three-vertex function  $\Gamma_\mu$  of quantum electrodynamics, and are represented by sums of connected diagrams with a given number (equal to  $k$ ) of external lines.

The higher and single-particle Green's functions will be constructed on the basis of the vacuum expectations

$$\langle T \{u_1(x_1) u_2(x_2) \dots u_k(x_k) S\} \rangle_0, \quad (37.1)$$

containing the scattering matrix

$$S = T(\exp i \int \mathcal{L}(x) dx) \quad (37.2)$$

within the chronological product sign.

Expressions such as (37.1) are natural generalizations for  $k = 2$  of the single-particle free-field Green functions to the case where interaction is present,  $\mathcal{L} \neq 0$ . By expanding the operator  $S$  in powers of the interaction  $\mathcal{L}$ , we obtain the series

$$\begin{aligned} \langle T \{u_1(x) u_2(y) S\} \rangle_0 &= \langle T \{u_1(x) u_2(y)\} \rangle_0 + i \langle T \{u_1(x) u_2(y) A\} \rangle_0 + \\ &+ \frac{i^2}{2} \langle T \{u_1(x) u_2(y) A^2\} \rangle_0 + \dots; \quad A \equiv \int \mathcal{L}(x) dx, \end{aligned} \quad (37.3)$$

the last terms of which contain the radiative corrections to the first term  $\langle T \{u_1 u_2\} \rangle_0$  which represents elementary (free) pairing.

It is convenient at this point to introduce an important technical remark relating to expressions which contain within the operation

$$\langle T(\dots) \rangle_0$$

operator factors that do not explicitly depend on time and are similar to the argument of the exponential in (37.2) or, for example, field operators in momentum representation.

One might be tempted to take a "time independent" factor of this kind outside the  $T$ -product sign. However, the result of this operation after the evaluation of the vacuum

expectation is an expression in which chronological pairing may turn out to be replaced by ordinary pairing, i.e., we may be led to an incorrect result.

The correct procedure in expressions of this kind is always to begin with the operation  $\langle T \dots \rangle_0$  and only after this has been done integrate over configuration space. For example, the third term on the right-hand side of (37.3) is, by definition,

$$\begin{aligned} \langle T \{u_1(x) u_2(y) A^2\} \rangle_0 &\equiv \int dz_1 dz_2 G(x, y; z_1, z_2), \\ G(x, y; z_1, z_2) &= \langle T \{u_1(x) u_2(y) \mathcal{L}(z_1) \mathcal{L}(z_2)\} \rangle_0. \end{aligned}$$

The operations  $\langle T \dots \rangle_0$  and  $\int dx$  do not therefore commute, and (37.2) is a purely symbolic expression.

The terms on the right-hand side of (37.3) contain terms corresponding to Feynman diagrams with two external lines which "enter" the points  $x$  and  $y$ . Figure 52 shows Feynman diagrams corresponding to some of the lower-order terms in the theory of the scalar field with the interaction  $\mathcal{L} \sim \varphi^4$ . Diagram (a) corresponds to the first term on the right-hand side of (37.3), diagrams (b) and (c) correspond to the third term, and diagram (d) to the fourth term.

Diagram (c) is interesting. As can be seen, it is disconnected. Such disconnected diagrams appear in all the higher-order terms of the expansion (37.3). All such contributions correspond to a particular method of pairing the operators in which  $u_1(x)$  and  $u_2(x)$  are paired through a "part" of the intermediate vertices  $z_1, \dots, z_m$ . The remaining vertices  $z_{m+1}, \dots, z_n$  in diagrams of a given order  $n$ , are paired with each other only. The corresponding contributions can be factorized:

$$\langle T \{u_1 u_2 (iA)^m\} \rangle_0^{\text{con}} \langle T (iA)^{n-m} \rangle_0.$$

The superscript in this expression corresponds to connected contributions. The vertices  $z_1, \dots, z_m$  can be chosen out of the  $n$  vertices in  $n!/[m!(n-m)!]^{-1}$  ways. Therefore

$$\langle T \{u_1 u_2 S\} \rangle_0 = \sum_{0 \leq n \leq \infty} \frac{i^n}{n!} \sum_{1 \leq m \leq n} \frac{n!}{m!(n-m)!} \langle T \{u_1 u_2 A^m\} \rangle_0^{\text{con}} \langle T A^{n-m} \rangle_0.$$

Rearranging the order of summation, we obtain

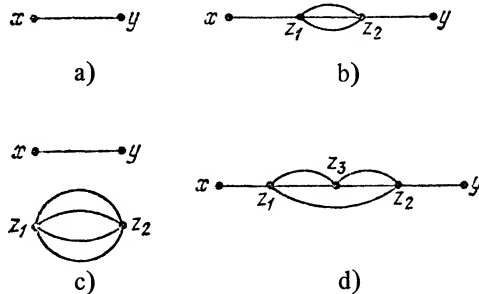


Fig. 52. Lower-order diagrams with two external lines in  $\varphi^4$  theory.

$$\begin{aligned}\langle T \{u_1 u_2 S\} \rangle_0 &= \sum_{0 \leq m \leq \infty} \frac{i^m}{m!} \langle T \{u_1 u_2 A^m\} \rangle^{cb} \sum_l \frac{i^l}{l!} \langle T A^l \rangle_0 = \\ &= \langle T \{u_1 u_2 S\} \rangle_0^{\text{con}} S_0; \quad S_0 = \langle S \rangle_0. \quad (37.4)\end{aligned}$$

The vacuum contributions  $S_0$  are thus seen to be factorized out in the complete expression as well.

The complete single-particle Green's functions will therefore be defined as follows:

$$\Delta_{12}(x, y) = i \frac{\langle T \{u_1(x) u_2(y)\} S \rangle_0}{S_0}. \quad (37.5)$$

This expression takes into account all the radiative corrections corresponding to connected Feynman diagrams, and becomes identical with the free-field Green's function when the interaction is turned off.

It is readily verified that the above discussion, which leads to the factorization (37.5) of the vacuum contributions, can be repeated for the vacuum expectations (37.1) for  $k > 2$ . However, in general, this may give rise to contributions in which some of the operators  $u_1 \dots u_l$  appear in one of the components of a disconnected diagram and the others,  $u_{l+1}, \dots, u_k$ , in another (or others). Expressions of the form

$$\frac{1}{S_0} \langle T \{u_1 \dots u_k S\} \rangle_0 \quad (37.6)$$

will therefore contain disconnected contributions for  $k \geq 4$ . The only exception is provided by the three-vertex

$$\frac{1}{S_0} \langle T \{u_1(x) u_2(y) u_3(z) S\} \rangle_0, \quad (37.7)$$

which is the sum of connected contributions. We shall return later to the question of connected higher Green's functions.

We note, further, that we have used throughout the value of the scattering matrix  $S = S(1)$  which is obtained from  $S(g)$  by allowing  $g(x)$ , which characterizes the interaction region, to unity throughout space-time. It must be remembered, however, that by definition of the  $S$ -matrix, even in this limiting case the function  $g(x)$  will adiabatically tend to zero for positive and negative infinite times.

**37.2. Sources and Generating Functionals.** To obtain the relationship between different Green's functions, and also to introduce strongly connected Green's functions, we shall follow Schwinger (1949) and introduce the general device of the *generating functional*. To do this, we add auxiliary terms that are linear in the field operators, to the Lagrangian for the system under consideration:

$$\mathcal{L}(x) \rightarrow \mathcal{L}(x) + \sum_i \varphi_i(x) J^i(x). \quad (37.8)$$



The nonoperator coefficients  $J^i$  are called “classical currents” and are the sources of the field  $\varphi_i$ . This terminology is historically connected with the fact that, in the case of the electromagnetic field  $\varphi \rightarrow A$ , the term in the Lagrangian

$$A_n(x) J^n(x)$$

corresponds exactly to the interaction with the given external electromagnetic current  $J^n$ . In the general case, this terminology is somewhat arbitrary.

The operation (37.8) ensures that the scattering matrix  $S$ , its vacuum expectation  $S_0$ , and also certain definite higher Green's functions begin to depend functionally on  $J^i$ . They become *current functionals*. We emphasize that the introduction of the classical currents  $J$  is an auxiliary mathematical device and does not necessarily imply the actual existence of external currents and charges. The functions  $J^i(x)$  are essentially auxiliary objects (to some extent analogous to the function  $g(x)$  in Chapter IV), which enable us to transform to the functional formulation of the equations for the different Green's functions and the connections between them. When actual external currents are not in fact present, the function  $J^i$  will be set equal to zero at the concluding stage of our analysis.

In statistical mechanics, the generating functional (and also its exponential representation) was first defined by Bogolyubov (1946, 1947) by introducing the auxiliary classical functions  $f$  into the partition function  $Z$ . The multiparticle distribution functions were then expressed in terms of variational derivatives of  $\ln Z(f)$  with respect to  $f$ .

Let us consider, to begin with, the vacuum expectation of the scattering matrix in the presence of the currents  $J$ :

$$S_0(J) = \langle S(J) \rangle_0, \quad (37.9)$$

$$S(J) = T \left\{ \exp i \left( \int \mathcal{L}(x) dx + \varphi J \right) \right\}. \quad (37.10)$$

where we have used the abbreviated notation

$$\varphi J \equiv \int dx \sum_i \varphi_i(x) J^i(x).$$

The expression given by (37.9) can be represented by a functional series in powers of  $J$ :

$$S_0(J) = S_0(0) + \sum_{n \geq 1} \frac{1}{n!} \int dx_1 \dots dx_n \sum_{(v_1, \dots, v_n)} C_{v_1 \dots v_n}^{(n)}(x_1, \dots, x_n) J^{v_1}(x_1) \dots J^{v_n}(x_n). \quad (37.11)$$

The coefficients  $C^{(n)}$  can be expressed in terms of the functional derivatives

$$\frac{\delta^n S_0(J)}{\delta J_\alpha(x_1) \dots \delta J_\omega(x_n)} = i^n \langle T \{ \varphi_\alpha(x_1) \dots \varphi_\omega(x_n) S(J) \} \rangle_0 \quad (37.12)$$

for  $J = 0$ , i.e.,

$$C_{\alpha \dots \omega}^{(n)}(x_1, \dots, x_n) = \frac{\delta^n S_0(J)}{\delta J_{\alpha}(x_1) \dots \delta J_{\omega}(x_n)} \Big|_{J=0}. \quad (37.13)$$

The quantity  $S'_0(J)$  is therefore the generating functional for the vacuum expectations (37.1).

We now consider separately the procedure for introducing sources for fields with half-integer spin:

$$\mathcal{L}(\psi, \bar{\psi}) \rightarrow \mathcal{L}(\psi, \bar{\psi}) + \sum_i (\bar{\psi}_i(x) \eta_i(x) + \bar{\eta}_i(x) \psi_i(x)). \quad (37.14)$$

Because of the anticommutation property of the Fermi field operators  $\psi, \bar{\psi}$ , the corresponding source functions  $\eta, \bar{\eta}$  must also be looked upon as anticommuting objects satisfying the commutation relations

$$\left. \begin{aligned} \eta(x) \eta(y) + \eta(y) \eta(x) &= 0, & \eta(x) \eta(x) &= 0, \\ \bar{\eta}(x) \bar{\eta}(y) + \bar{\eta}(y) \bar{\eta}(x) &= 0, & \bar{\eta}(x) \bar{\eta}(x) &= 0. \\ \eta(x) \bar{\eta}(y) + \bar{\eta}(y) \eta(x) &= 0, \end{aligned} \right\} \quad (37.15)$$

With the exception of the last, the commutation relations differ from the Fermi commutation relations. The classical anticommuting fields  $\eta, \bar{\eta}$  can be regarded as the generators of a *Grassman algebra*. [A more detailed account of the mathematical aspects of these questions is given by Berezin (1965)]. The commutation relations given by (37.15) will be called the Grassman commutation relations and, for the sake of brevity, the fields  $\eta, \bar{\eta}$  will be referred to as the Fermi sources.

It is also convenient to assume about the sources  $\eta, \bar{\eta}$  completely anticommute with the quantized Fermi fields  $\psi, \bar{\psi}$ :

$$\{\eta, \psi\}_+ = \{\eta, \bar{\psi}\}_+ = \{\bar{\eta}, \psi\}_+ = \{\bar{\eta}, \bar{\psi}\}_+ = 0.$$

When the variational derivatives with respect to the Fermi sources are introduced, it is important to remember that they anticommute. It is also important to distinguish between the *left* and the *right* variational derivatives. The left (right) variational derivative of a functional  $F(\eta)$  with respect to the anticommuting Fermi source  $\eta(x)$  is defined as the coefficient in the principal part of the increment  $\delta F$  when the argument is incremented by  $\delta\eta(x)$  and is taken to the left (to the right). Denoting the left derivative by  $'\delta F/\delta\eta$  and the right by  $\delta'F/\delta\eta$ , we have by definition

$$\delta F = \int \delta\eta(x) \frac{'\delta F(\eta)}{\delta\eta(x)} dx = \int \frac{\delta'F(\eta)}{\delta\eta(x)} \delta\eta(x) dx. \quad (37.16)$$

Thus, for example, the left variational derivative of a homogeneous polynomial functional

$F$  of "degree"  $n$  will be equal to the right derivative for odd  $n$  and will differ in sign from the right derivative for even  $n$ .

For the sake of convenience, we shall always use the left derivative. We shall therefore use the notation

$$\frac{\delta}{\delta\eta} = \frac{'\delta}{\delta\eta}, \quad \frac{\delta}{\delta\bar{\eta}} = \frac{'\delta}{\delta\bar{\eta}}.$$

The anticommutation property of the fields  $\eta, \bar{\eta}$  leads to the anticommutation of the operations of variational differentiation with each other in the case of multiple differentiation, for example,

$$\frac{\delta^2 F}{\delta\eta_1 \delta\eta_2} = - \frac{\delta^2 F}{\delta\eta_2 \delta\eta_1}, \quad (37.17)$$

and also to the anticommutation with the Fermi fields and Fermi sources. Products are differentiated in accordance with the formula

$$\frac{\delta}{\delta\eta} (AB) = \frac{\delta A}{\delta\eta} B + (-1)^a A \frac{\delta B}{\delta\eta}, \quad (37.18)$$

where  $a$  is the total degree of linearity of the functional  $A$  with respect to all the Fermi fields  $(\eta, \bar{\eta}, \psi, \bar{\psi})$ .

The above definitions and properties enable us to obtain formulas analogous to (37.12). For example,

$$\frac{\delta^2 S_0(\eta, \bar{\eta})}{\delta\eta(x) \delta\bar{\eta}(y)} = i^2 \langle \{T\psi(x) \bar{\psi}(y) S\} \rangle_0, \quad (37.19)$$

and so on. We shall use (37.12) in our general analysis, and assume that it describes both Bose and Fermi fields and sources.

**37.3. Generating Functional for the Higher Green's Function.** We now establish the connection between the right-hand sides of (37.12) and the complete Green's functions.

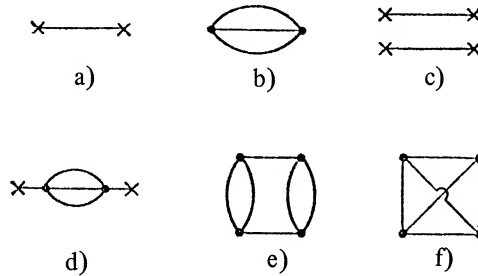


Fig. 53. Second-and fourth-order vacuum diagrams in  $\varphi^3$  theory in the presence of external current.

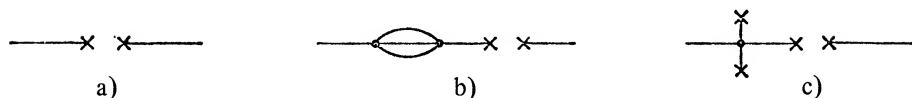


Fig. 54. Disconnected diagrams with two external ends for the model given by (37.25) with zero vacuum contribution.

The expressions

$$\langle T \{ u(\xi_1) \dots u(\xi_n) S(J) \} \rangle_0 \quad (37.20)$$

are the sums of all contributions with  $n$  external ends in the presence of external sources. Let us first segregate the disconnected contributions. The disconnected diagrams can be resolved in two classes.

Thus, first, we have diagrams containing the vacuum subdiagrams. In contrast to §37.1, the class of vacuum diagrams is now much more extensive.

As an illustration, Fig. 53 shows the lower-order vacuum diagram in the model

$$\mathcal{L} = \lambda \varphi^3 + \varphi J. \quad (37.21)$$

Crosses represent "current" vertices corresponding to  $\varphi J$ . It is clear that, roughly speaking, the number of vacuum diagrams has doubled. Repeating the discussion given in the last section, we can readily show that all the disconnected diagrams of this class are removed by dividing by  $S_0(J)$ .

The second class of disconnected contributions to (37.20) corresponds to diagrams for which the disconnectedness is, so to speak, organically related to the presence of the currents  $J$ . Examples of such diagrams with two external ends are shown in Fig. 54 for the model defined by (37.21).

It is clear that the diagrams corresponding to the situation where the sequence of pairings that begins on the operator  $\varphi(x)$  corresponding to an external line will not "penetrate" the diagram but ends in its interior on a certain number of "current" vertices. It is readily verified that such possible pairings within the expressions given by (37.20) correspond to partial or complete "factorization," for example,

$$\begin{aligned} \langle T \{ \varphi_1(x) \varphi_2(y) S(J) \} \rangle_0 &\rightarrow \langle T \{ \varphi_1(x) S(J) \} \rangle_0 \langle T \{ \varphi_2(y) S(J) \} \rangle_0, \\ \langle T \{ \varphi_1 \varphi_2 \varphi_3 S \} \rangle_0 &\rightarrow \langle T \{ \varphi_1 S \} \rangle_0 \langle T \{ \varphi_2 S \} \rangle_0 \langle T \{ \varphi_3 S \} \rangle_0 + \langle T \{ \varphi_1 \varphi_2 S \} \rangle_0 \langle T \{ \varphi_3 S \} \rangle_0 + \\ &\quad + \langle T \{ \varphi_2 \varphi_3 S \} \rangle_0 \langle T \{ \varphi_1 S \} \rangle_0 + \langle T \{ \varphi_1 \varphi_3 S \} \rangle_0 \langle T \{ \varphi_2 S \} \rangle_0. \end{aligned}$$

Let

$$\Phi(x, J) = \frac{1}{S_0(J)} \langle T \{ \varphi(x) S(J) \} \rangle_0 = \frac{-i}{S_0(J)} \frac{\delta S_0(J)}{\delta J(x)}, \quad (37.22)$$

This represents the sum of connected diagrams with one external end. It can be interpreted

as the expectation value of the operator  $\varphi(x)$  in the presence of the external current  $J$ .

The two-end and three-end Green's functions that are free from disconnected contributions can now be written in the form

$$S_0^{-1}(J) \langle T \{ \varphi_1(x) \varphi_2(y) S \} \rangle_0 - \Phi_1(x|J) \Phi_2(y|J) = \frac{1}{i} \Delta_{12}(x, y|J), \quad (37.23)$$

$$S_0^{-1}(J) \langle T \{ \varphi_1 \varphi_2 \varphi_3 S \} \rangle_0 - \Phi_1 \Phi_2 \Phi_3 - \Phi_1 S_0^{-1} \langle T \{ \varphi_2 \varphi_3 S \} \rangle_0 - \Phi_2 S_0^{-1} \langle T \{ \varphi_1 \varphi_3 S \} \rangle_0 - \\ - \Phi_3 S_0^{-1} \langle T \{ \varphi_1 \varphi_2 S \} \rangle_0 = \frac{1}{i^2} \Delta_{123}(x_1, x_2, x_3|J). \quad (37.24)$$

In (37.23) we have introduced the symbol  $\Delta_{12}$  to represent the sum of connected two-end diagrams. The function  $\Delta_{12}(x, y|J)$  will be referred to as the complete two-end (single-particle) Green's function. It is a direct generalization of the function  $\Delta_{12}(x, y)$  introduced in (37.5). The function  $\Delta_{123}$  introduced in (37.24) is the complete (connected) three-end Green's function. The higher-connected Green's functions can be introduced in an analogous fashion.

We now note that the functions (37.22)–(37.24) can be very simply expressed in terms of the variational derivatives of the function  $Z(J)$  which is related to  $S_0(J)$  by

$$Z(J) = -i \ln S_0(J), \quad S_0(J) = \exp iZ(J). \quad (37.25)$$

We have

$$\frac{\delta Z(J)}{\delta J(x)} = \Phi(x|J), \quad (37.26.1)$$

$$\frac{\delta^2 Z(J)}{\delta J_1(x) \delta J_2(y)} = \frac{\delta \Phi_1(x|J)}{\delta J_2(y)} = \Delta_{12}(x, y|J), \quad (37.26.2)$$

$$\frac{\delta^3 Z(J)}{\delta J_1(x) \delta J_2(y) \delta J_3(z)} = \frac{\delta \Delta_{12}(x, y)}{\delta J_3(z)} = i \Delta_{123}(x, y, z|J). \quad (37.26.3)$$

It is thus clear that the functional  $Z(J)$  introduced in (37.25) is the generating functional for the connected Green's functions

$$\frac{\delta^n Z(J)}{\delta J_1(x_1) \dots \delta J_n(x_n)} = \Delta_{1, \dots, n}^{\text{con}}(x_1, \dots, x_n|J). \quad (37.27)$$

**37.4. Vertex Functions.** We note that the higher Green's functions introduced in (37.27) include all the weakly connected (i.e., single-particle reducible) contributions. In Fig. 55, which illustrates this remark, the strongly connected diagrams are shown cross hatched.

To introduce the quantities corresponding to strongly-connected diagrams, let us consider the functional Legendre transformation

$$Z(J) \rightarrow W(\Phi) = Z - \int d^4x J(x) \Phi(x), \quad (37.28)$$

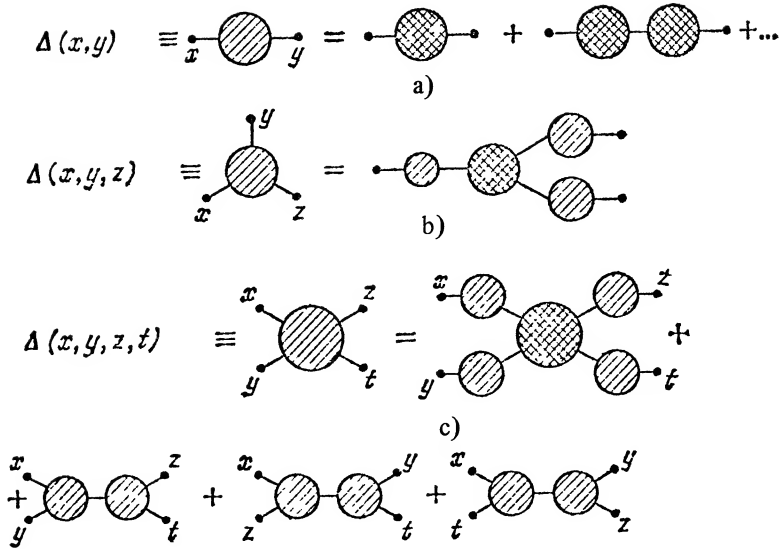


Fig. 55.

in which instead of the functional argument  $J$  we have introduced a new independent functional argument  $\Phi$  (i.e., we have inverted the functional relation  $\Phi = \Phi(J)$ ).

It will be seen later that  $W(\Phi)$  is the generating functional for strongly connected Green's functions. We shall call them the vertex functions (or simply vertices, for the sake of brevity).

We begin by introducing a useful auxiliary relation. Differentiating (37.26.1) with respect to  $\Phi(y)$ , we obtain

$$\frac{\delta^2 Z(J)}{\delta \Phi(y) \delta J(x)} = \int \frac{\delta J(z)}{\delta \Phi(y)} dz \frac{\delta^2 Z(J)}{\delta J(z) \delta J(x)} = \int \frac{\delta J(z)}{\delta \Phi(y)} dz \Delta(z, x) = \delta(x - y).$$

Using (37.26.2), we therefore have

$$\frac{\delta J(z)}{\delta \Phi(y)} = \Delta^{-1}(z, y). \quad (37.29)$$

Here we have introduced the function  $\Delta^{-1}$  which is the inverse of  $\Delta$  in the sense of the following integral operation in the  $x$ -representation:

$$\int \Delta(x, z) dz \Delta^{-1}(z, y) = \int \Delta^{-1}(x, z) dz \Delta(z, y) = \delta(x - y). \quad (37.30)$$

We note that in the momentum representation (in the translationally invariant limit  $J \rightarrow 0, \Phi \rightarrow 0$ ), the relation given by (37.30) assumes the algebraic form

$$\tilde{\Delta} \tilde{\Delta}^{-1} = 1. \quad (37.31)$$

$$\begin{aligned}
 \text{a) } \Gamma(x, y) &= x \text{ --- } \text{---} y \\
 \text{b) } \Gamma(x, y, z) &= \text{---} \\
 \text{c) } \Gamma(x, y, z, t) &= \text{---}
 \end{aligned}$$

Fig. 56.

We now consider the successive variational derivatives of  $W$ . Using (37.26.1), we have

$$\begin{aligned}
 \frac{\delta W(\Phi)}{\delta \Phi(x)} &= \frac{\delta Z}{\delta \Phi(x)} - \int \frac{\delta J(z)}{\delta \Phi(x)} dz \Phi(z) - J(x) = \\
 &= \int \left[ \frac{\delta Z}{\delta J(z)} - \Phi(z) \right] dz \frac{\delta J(z)}{\delta \Phi(x)} - J(x) = -J(x). \quad (37.32)
 \end{aligned}$$

The functional  $W(\Phi)$  is thus seen to have the important *stationary* property. In the limit of zero external currents ( $J \rightarrow 0$ ), its first derivative will vanish.

Differentiating again with respect to  $\Phi(y)$ , and using (37.29), we have

$$\frac{\delta^2 W(\Phi)}{\delta \Phi(x) \delta \Phi(y)} = - \frac{\delta J(x)}{\delta \Phi(y)} = -\Delta^{-1}(x, y). \quad (37.33)$$

This is the Fourier transform of the reciprocal (in the algebraic sense) propagator

$$\tilde{\Delta}^{-1}(p) = m^2 - p^2 - \Gamma(p),$$

and is therefore the sum of the strongly connected diagrams. Differentiating (37.33) once again with respect to  $\Phi(z)$ , and rearranging, we finally obtain

$$\begin{aligned}
 \frac{\delta^3 W(\Phi)}{\delta \Phi(x) \delta \Phi(y) \delta \Phi(z)} &= \\
 &= \int \Delta^{-1}(x, \xi) d\xi \frac{\delta^3 Z}{\delta J(\xi) \delta J(\eta) \delta J(\tau)} d\eta \Delta^{-1}(\eta, y) d\tau \Delta^{-1}(\tau, z). \quad (37.34)
 \end{aligned}$$

This shows that the vertex function

$$\Gamma(x, y, z) = \frac{\delta^3 W(\Phi)}{\delta \Phi(x) \delta \Phi(y) \delta \Phi(z)} \quad (37.35)$$

differs from the three-end Green's function  $\Delta(x, y, z)$  [see (37.26.3)] by the absence of factors corresponding to external lines [transition from  $\Delta(x, y, z)$  to  $\Gamma(x, y, z)$  corresponds to the "amputation" of the external factors (see Fig. 55c and Fig. 56c)].

We note that (37.33) also leads to

$$\Gamma(x, y, z) = -\frac{\delta \Delta^{-1}(x, y)}{\delta \Phi(z)}. \quad (37.36)$$

We leave it as an exercise for the reader to derive the formula relating  $\Delta(x, y, z, t)$  and the four-end vertex function  $\Gamma(x, y, z, t)$  illustrated graphically in Fig. 56c.

The functional  $W(\Phi)$  is thus the generating functional

$$\frac{\delta^n W(\Phi)}{\delta \Phi_1(x_1) \dots \delta \Phi_n(x_n)} = \Gamma_{1, \dots, n}(x_1, \dots, x_n) \quad (37.37)$$

for the vertex (i.e., strongly connected) functions.

### §38. Schwinger and Dyson Equations

We now turn to the derivation of the equations for the complete Green's functions introduced in the last section. It will be shown in §38.2 that these Green's functions can be relatively simply (generally speaking, linearly) related to the elements of the  $S$ -matrix, and completely characterize the behavior of a dynamic system. On the other hand, by considering the homogeneous equations for the "generalized wave functions" corresponding to the inhomogeneous equations for the complete Green's functions, we can determine the radiative corrections to the energy eigenvalues of closed systems, and also solve other similar problems (see Chapter VII).

One might try to obtain such equations for the complete Green's functions by analyzing in detail the radiative corrections. For example, the photon polarization operator can be expressed in spinor electrodynamics in terms of an integral of the product of the electron Green functions and the vertex parts. The sum of the radiative corrections to the vertex part can, in turn, be expressed in terms of the single-particle Green's functions and the more complicated vertex parts, and so on.

This procedure would yield a sequence of increasingly complicated equations (in fact, an infinite sequence of equations) with structure not amenable to simple analysis. More elegant relationships can be obtained with the aid of the generating functional introduced in §37.2. This approach yields a more manageable system of equations in functional derivatives—the so-called Schwinger equations—which, in turn, can be used to obtain the equations relating the different Green's functions (these are the Dyson equations).

**38.1. Generalized Wick's Theorem.** Before we derive the equations for Green's functions, let us consider one preliminary auxiliary proposition which may be called "generalized Wick's theorem" and which asserts that the vacuum expectation value of the chronological product of  $n + 1$  linear operators  $A, B_1, \dots, B_n$  is equal to the sum of  $n$  vacuum expectation values of the same chronological products with all possible pairings of one of these operators (for example,  $A$ ) with all the others, i.e.,

$$\langle T(AB_1 \dots B_n) \rangle_0 = \sum_{1 \leq i \leq n} \langle T(\overline{AB_1 \dots B_i \dots B_n}) \rangle_0. \quad (38.1)$$



It should be noted that, in contrast to the usual Wick theorem (compare, for example, (17.17)), there are no expressions involving a number of pairings greater than one on the right-hand side of (38.1).

Nevertheless, the validity of (38.1) follows directly from the usual Wick theorem. Since the vacuum expectation value of the normal product of any number other than zero of unpaired operators vanishes, the left-hand side of (38.1) is equal to the sum over all the possible variants of the complete mutual pairings within the product of the operators

$$AB_1 \dots B_n, \quad (38.2)$$

i.e., over pairings in which all the operators have been paired with one another. In a completely analogous manner, any term of the sum appearing on the right-hand side of (38.1), for example, the first, may be represented in the form

$$\overline{AB_1} \langle T(B_2 \dots B_n) \rangle_0$$

and is equal to the product of the pairing  $\overline{AB_1}$  and the sum of all possible complete pairings of the operators  $B_2, \dots, B_n$ .

On carrying out the summation over  $i$  in the right-hand side of (38.1), we obtain the sum over all possible pairings of the operators (38.2). This completes the proof of the generalized Wick theorem.

Since the  $T$ -products in the vacuum expectation values defining the complete Green's functions also contain the  $S$ -matrix which is not a linear operator, we shall generalize theorem (38.1) to that case also. We shall therefore consider the pairing of the linear operator  $A$  with the  $n$ th term in the expansion of the scattering matrix

$$\frac{i^n}{n!} \int T \mathcal{L}(x_1) \dots \mathcal{L}(x_n) dx_1 \dots dx_n. \quad (38.3)$$

We introduce the operation of pairing the linear operator  $A$  with the Lagrangian  $\mathcal{L}(x)$

which is also a nonlinear operator. It is natural to define  $\overline{A\mathcal{L}}(x)$  as the sum of products  $A\mathcal{L}$  with all possible pairings of the operator  $A$  with the operators appearing in  $\mathcal{L}$ . For example, in the case of spinor electrodynamics, if we take into account the external current term

$$\mathcal{L}(x) = e : \bar{\psi}(x) \gamma^\mu \psi(x) A_\mu(x) : + J^\mu(x) A_\mu(x), \quad (38.4)$$

we obtain, by definition,

$$\overline{\psi(x) \mathcal{L}(y)} = e \overline{\psi(x) \bar{\psi}(y) \gamma^\mu : \psi(y) A_\mu(y) :}, \quad (38.5)$$

$$\overline{\bar{\psi}(x) \mathcal{L}(y)} = e : A_\mu(y) \bar{\psi}(y) : \overline{\gamma^\mu \psi(y) \bar{\psi}(x)}, \quad (38.6)$$

$$\overline{A_l(x) \mathcal{L}(y)} = e : \bar{\psi}(y) \gamma^k \psi(y) : \overline{A_l(x) A_k(y)} + J^k(y) \overline{A_l(x) A_k(y)}. \quad (38.7)$$

In this connection, we note that the complete Green's functions defined above, which are sums of contributions corresponding to the internal lines of Feynman diagrams, naturally contain divergences of the same type as in the scattering matrix. For the removal of these divergences, one would from the outset introduce the usual counterterms into the interaction Lagrangian. However, for the sake of simplifying the presentation, we shall first derive Schwinger's equations for Green's functions from the Lagrangian (38.4), and shall then also take the counterterms into account.

We now return to the pairing of (38.3) with the linear operator  $A$ . We obtain the sum of terms

$$\frac{i^n}{n!} \sum_j \int T(\overline{A \mathcal{L}(x_1) \dots \mathcal{L}(x_j)} \dots \mathcal{L}(x_n)) dx_1 \dots dx_n,$$

which, because of the symmetry of (38.3) with respect to integration variables  $x_1, \dots, x_n$ , can be written in the form

$$\frac{i^n}{(n-1)!} \int d\tau \int dx_1 \dots dx_{n-1} T(\overline{A \mathcal{L}(\tau)} \mathcal{L}(x_1) \dots \mathcal{L}(x_{n-1})).$$

If we sum over  $n$ , we obtain the result of the pairing of the operator  $A$  with the matrix  $S$  in the form

$$\overline{AS} = i \int d\tau T(A \mathcal{L}(\tau) \exp i \int \mathcal{L}(x) dx) = i \int d\tau T(\overline{A \mathcal{L}(\tau)} S). \quad (38.8)$$

This can also be written in a different form if we use the idea of the *variational derivative* of the  $S$ -matrix with respect to the *operator field function*.

These derivatives can be introduced by defining them as the limits of the corresponding functional derivatives with respect to the classical addition  $\eta(x)$  to the quantum function  $u(x)$ :

$$u(x) \rightarrow u(x) + \eta(x)$$

for  $\eta(x) = 0$ , i.e.,

$$\frac{\delta S}{\delta u_i(x)} = \lim_{\eta=0} \frac{\delta S(\eta)}{\delta \eta_i(x)}. \quad (38.9)$$

To ensure that the variational derivatives with respect to the Fermi fields, obtained in this way, have the required anticommutation properties, we must assume that the classical additions to the Fermi fields are algebraically similar to the Fermi field  $\eta, \bar{\eta}$  introduced in §37.3, and satisfy the commutation relations (37.15) of Grassman algebra. As in §37.3,

we shall agree to consider all the variational derivatives with respect to the Fermi fields as being left derivatives.

Since (38.5)–(38.7) can be written in the form

$$\overline{u(x)} \mathcal{L}(y) = \overline{u(x)} u(y) \frac{\delta \mathcal{L}(y)}{\delta u(y)},$$

we can rewrite (38.8) in the form

$$\overline{AS} = \int \overline{Au}(\eta) d\eta \frac{\delta S}{\delta u(\eta)}. \quad (38.10)$$

Finally, we introduce the commutation relations for the  $S$ -matrix and its functional derivatives for the Bose operator  $\varphi$ :

$$[\varphi(x), S]_- = \int [\varphi(x), \varphi(\xi)]_- d\xi \frac{\delta S}{\delta \varphi(\xi)} \quad (38.11)$$

and for the Fermi operator  $\psi$ :

$$[\psi(x), S]_- = \int [\psi(x), \bar{\psi}(\xi)]_+ d\xi \frac{\delta S}{\delta \bar{\psi}(\xi)}, \quad (38.12.1)$$

$$\left[ \psi(x), \frac{\delta S}{\delta \psi(y)} \right]_+ = \int [\psi(x), \bar{\psi}(\eta)]_+ d\eta \frac{\delta^2 S}{\delta \bar{\psi}(\eta) \delta \bar{\psi}(y)}. \quad (38.12.2)$$

We note that the commutators of the Fermi operators  $\psi, \bar{\psi}$  with functionals that are even in the Fermi fields (for example, variational derivatives of  $S$  of even order in  $\psi, \bar{\psi}$ ) have the structure given by (38.12.1). At the same time, a structure similar to that of (38.12.2) is exhibited by the anticommutators of the Fermi operators  $\psi, \bar{\psi}$  with functionals that are odd in the Fermi fields.

We now express the vacuum expectations

$$\langle T \{ u(\xi_1) u_2(\xi_2) \dots u(\xi_k) S \} \rangle_0, \quad (37.1)$$

used in §37 to construct the higher Green's functions, in terms of the variational derivatives of the  $S$ -matrix.

Using the generalized Wick theorem for  $u_1$ , together with (38.1) and (38.10), we obtain

$$\begin{aligned} \langle T u_1 \dots u_k S \rangle_0 &= \sum_{2 \leq l \leq k} \eta_{l-2} \overline{u_1 u_l} \langle T \{ u_2 \dots u_k S \} \rangle_0 + \\ &+ \eta_{k-1} \sum_m \int \overline{u_1 u_m}(\xi) d\xi \left\langle T \left\{ u_2 \dots u_k \frac{\delta S}{\delta u_m(\xi)} \right\} \right\rangle_0. \end{aligned} \quad (38.13)$$

Here  $T'_l$  represents the quantity obtained from (37.1) by deleting the operator  $u_l$ , and  $\eta_{l-2}$  and  $\eta_{k-1}$  are the sign factors representing the parity of commutations of Fermi operators when  $u_1$  is a field with half-integer spin. The formula given by (38.13) expresses the vacuum expectation of the chronological product of  $k$  operators and the  $S$ -matrix in terms of the sum of the analogous expectations of  $k-1$  operators and the  $S$ -matrix, or its variational derivative, i.e., it effectively reduces by unity the degree of linearity in  $u$ . If we apply (38.13) three times to  $\langle T\varphi_1\varphi_2\varphi_3S \rangle_0$ , which is related to the three-end Green's function as shown by (38.24), we obtain after some rearrangement

$$S_0^{-1} \langle T\{\varphi_1\varphi_2\varphi_3S\} \rangle_0 = \overline{\varphi_1\varphi_2}\Phi_3 + \overline{\varphi_1\varphi_3}\Phi_2 + \overline{\varphi_2\varphi_3}\Phi_1 + \\ + S_0^{-1} \int \overline{\varphi_1\varphi}(\tau) d\tau \overline{\varphi_2\varphi}(\rho) d\rho \left\langle \frac{\delta^3 S}{\delta\varphi(\sigma)\delta\varphi(\rho)\delta\varphi(\tau)} \right\rangle_0 d\sigma \overline{\varphi(\sigma)\varphi_3}. \quad (38.14)$$

Comparison of (37.24) and (38.14) yields the connection between  $\Delta_{123}$  and the vacuum expectation of the third variational derivative. This is of the form

$$\Delta_{123}(x, y, z|J) = \\ = \frac{i^2}{S_0} \int \overline{\varphi_1(x)\varphi}(\tau) d\tau \overline{\varphi_2(y)\varphi}(\rho) d\rho \left\langle \frac{\delta^3 S}{\delta\varphi(\sigma)\delta\varphi(\tau)\delta\varphi(\rho)} \right\rangle_0 d\sigma \overline{\varphi(\sigma)\varphi_3}(z) + \dots \quad (38.15)$$

The dots in this expression represent terms proportional to  $\Phi_k$ , which vanish when the sources are removed. For an  $n$ -end Green's function, the corresponding relation in the absence of external forces is

$$\Delta_{1, \dots, n}(x_1, \dots, x_n) = \lim_{J=0} \Delta_{1, \dots, n}(x_1, \dots, x_n|J) = \\ = \frac{i^{n-1}}{S_0} \int \prod_{1 \leq j \leq n} \{\overline{\varphi_j(x_j)\varphi}(\eta_j)\} d\eta_j \left\langle \frac{\delta^n S}{\delta\varphi(\eta_1) \dots \delta\varphi(\eta_n)} \right\rangle_0. \quad (38.16)$$

In the momentum representation,

$$\Delta_{1, \dots, n}(x_1, \dots, x_n) = \frac{1}{(2\pi)^{4n}} \int e^{-i \sum x_j k_j} \tilde{\Delta}(k_1, \dots, k_n) dk_1 \dots dk_n, \\ i\overline{\varphi_j(x)\varphi_j}(y) = \frac{1}{(2\pi)^4} \int e^{-ik(x-y)} D(k) dk, \quad (38.17)$$

we also have

$$\tilde{\Delta}_{1, \dots, n}(k_1, \dots, k_n) = \frac{1}{iS_0} \int \prod_{1 \leq j \leq n} \{D_j(k_j) e^{-ik_j x_j} dx_j\} \left\langle \frac{\delta^n S}{\delta\varphi_1(x_1) \dots \delta\varphi_n(x_n)} \right\rangle_0. \quad (38.18)$$

The formulas given by (38.16) and (38.18) have an obvious generalization to the case of Fermi fields.

38.2. *Reduction Formulas.* To establish the connection between the complete higher Green's functions introduced in §37 and the elements of the scattering matrix, we first solve the auxilliary problem in which the matrix elements of the operator  $S$  are expressed in terms of the vacuum expectations of its variational derivatives.

Consider the element of the  $S$ -matrix

$$\langle \Phi_{1,2,\dots,k}^* S \Phi_{k+1,\dots,n} \rangle, \quad (38.19)$$

where the state amplitudes are taken in the normalization defined by (25.11), i.e.,

$$\Phi_{1,2,\dots,k} = a_{\sigma_1}^+(p_1) \dots a_{\sigma_k}^+(p_k) \Phi_0. \quad (38.20)$$

As usual, we shall assume that, among particles in the initial ( $\Phi_{k+1,\dots,n}$ ) and final ( $\Phi_{1,\dots,k}$ ) states there are no identical particles (i.e., particles with identical momenta  $p$  and remaining quantum numbers  $\sigma$ ). In other words, any of the creation operators among  $\Phi_{k+1,\dots,n}$  will exactly (anti) commute with each of the annihilation operators from among  $\Phi_{1,\dots,k}$ .

We now shift the creation operators to the left, commuting them with the scattering matrix or its derivatives, and shift the annihilation operators to the right. For this we use (38.11) and (38.12), having transformed them to the momentum representation. For spinless Bose operators, we have from (38.11)

$$[a^\pm(p), S]_- = \frac{\pm 1}{(16\pi^3 p^0)^{1/2}} \int e^{\mp i p z} dz \frac{\delta S}{\delta \varphi(z)}. \quad (38.20b)$$

Instead of the  $S$ -matrix, these formulas contain its variational derivatives.

For the spinor field we have, correspondingly,

$$\left. \begin{aligned} [a_v^\pm(p), S] &= \frac{\bar{v}_v^\mp(p)}{(2\pi)^{3/2}} \int e^{\mp i p z} dz \frac{\delta S}{\delta \bar{\psi}(z)}, \\ [a^\pm(p), S] &= \frac{v^\mp(p)}{(2\pi)^{3/2}} \int e^{\mp i p z} dz \frac{\delta S}{\delta \psi(z)}. \end{aligned} \right\} \quad (38.20c)$$

Similar formulas can be written by replacing  $S$  with its variational derivatives. For derivatives containing an odd number of differentiations with respect to the field, the commutators on the left-hand side of (38.20b) must be replaced by the anticommutators.

If we now commute the operators in (38.19) with the  $S$ -matrix with the aid of (38.20), we obtain

$$\begin{aligned} & \langle \Phi_0^* a_1^-(p_1) \dots a_k^-(p_k) S a_{k+1}^+(q_{k+1}) \dots a_n^+(q_n) \Phi_0 \rangle_0 = \\ & = \eta \prod_{1 \leq i \leq k} f_i^-(p_i) \prod_{k+1 \leq j \leq n} f_j^-(q_j) \int \exp i \left( \sum p_i z_i - \sum q_j z_j \right) \times \\ & \quad \times \left\langle \frac{\delta^n S}{\delta u_1(z_1) \dots \delta u_n(z_n)} \right\rangle_0 dz_1 \dots dz_n. \end{aligned} \quad (38.21)$$

where

$$\left. \begin{aligned} f_i^{\pm}(k) &= \pm (16\pi^3 k^0)^{-1/2} \quad \text{— for spinless fields,} \\ f_i^{\pm}(p) &= \frac{v_{\mathbf{v}_i}^{\pm}(\rho)}{(2\pi)^{3/2}} \quad \text{— for fields with half-integer spin,} \end{aligned} \right\} \quad (38.22)$$

and  $\eta$  is the sign factor representing the change in sign during commutation of Fermi fields.

To relate the matrix element to the Green's function  $\Delta_{1,\dots,n}$ , we must now solve (38.16) for  $\langle \delta^n S \rangle$  and substitute the resulting expression into (38.21).

To do this, we use the equation satisfied by the free pairings on the right-hand side of (38.16). These equations can be written in the form

$$D_k^{-1}(x) \overline{u_k(x)} u_k(y) = \frac{1}{i} \delta(x-y).$$

where

$$\left. \begin{aligned} D_k^{-1}(x) &= m_k^2 - \square_x \quad \text{for Bose fields,} \\ &= m_k - i\hat{\partial}_x \quad \text{for Fermi fields.} \end{aligned} \right\} \quad (38.23)$$

If we apply the operators  $D^{-1}$  to (38.16)  $n$  times, we obtain

$$\frac{1}{S_0} \langle \overline{\delta^n S} \rangle_0 = i \prod_{1 \leq k \leq n} \{D_k^{-1}(z_k)\} \Delta_1, \dots, n(z_1 \dots z_n). \quad (38.24)$$

The relations given by (38.21) and (39.24) give the explicit connection between the scattering-matrix element and the corresponding connected vertex function. We recall that, formally, the matrix element on the left-hand side of (38.21) contains disconnected contributions which are represented by the singular phase factor  $S_0$ . As noted in Chapter V, we have agreed to omit such nonphysical contributions in perturbation-theory calculations and introduce the corresponding vacuum counterterms into the interaction Lagrangian. This procedure is equivalent to dividing the right-hand side of (38.21) by the vacuum expectation of the  $S$ -matrix, or to the multiplication of it by  $\overset{\uparrow}{S} = S^{-1}$  under the sign of vacuum expectation of the variational derivative

$$\langle \delta^n S \rangle_0 \rightarrow \frac{1}{S_0} \langle \delta^n S \rangle_0 = \langle \delta^n S \cdot \overset{\uparrow}{S} \rangle_0. \quad (38.25)$$

Using (38.25), we have from (38.21) and (38.24)

$$\begin{aligned}
\langle \Phi^* (p_1, \dots, p_n) S \Phi (q_1, \dots, q_m) \rangle_0 = \\
= i\eta \int \prod_{1 \leq k \leq n} \{f_k^+(p_k) D_k^{-1}(z_k) dz_k\} \prod_{1 \leq l \leq m} \{\bar{f}_l(q_l) D_l^{-1}(t_l) dt_l\} \times \\
| \times \exp i \left( \sum p_k z_k - \sum q_l t_l \right) \Delta_{1, \dots, n+m}(z_1, \dots, z_n, t_1, \dots, t_m). \quad (38.26)
\end{aligned}$$

Formulas of this type are called *reduction formulas*.

We now introduce the momentum representation for the higher Green's functions in accordance with (38.17), and rewrite (38.26) in the form

$$\begin{aligned}
\langle \Phi^* (p_1, \dots, p_n) S \Phi (q_1, \dots, q_m) \rangle_0 = \\
= i\eta \prod_{1 \leq k \leq n} \{f_k^+(p_k) \Delta_k^{-1}(p_k)\} \prod_{1 \leq l \leq m} \{\bar{f}_l(q_l) \Delta_l^{-1}(q_l)\} \times \\
\times \tilde{\Delta}_{1, \dots, n+m}(p_1, \dots, p_n; -q_1, \dots, -q_m). \quad (38.27)
\end{aligned}$$

In this expression,  $\Delta_k^{-1}$ ,  $\Delta_l^{-1}$  are the Fourier transforms of the differential operators (38.23), which are identical with the inverse single-particle free-field Green's functions.

**38.3. Schwinger's Equations.** We shall now consider the procedure for obtaining the equations in variational derivatives. We shall do this for spinor electrodynamics by introducing the external electromagnetic current into the interaction Lagrangian in accordance with (38.4). Sources of spinor fields will not be introduced.

The connected single-electron Green's function

$$G_{\sigma\rho}(x, y) = \frac{i}{S_0(J)} \langle T \{ \psi_\sigma(x) \bar{\psi}_\rho(y) S(J) \} \rangle_0 \quad (38.28)$$

is then a functional of  $J$ . We shall not explicitly indicate this on the left-hand side of (38.28). Using the generalized Wick theorem and (38.8), we can rewrite the numerator of (38.28), i.e.,

$$g_{\sigma\rho}(x, y) = i \langle T \{ \psi_\sigma(x) \bar{\psi}_\rho(y) S \} \rangle_0,$$

in the form

$$g(x, y) = i \overline{\psi(x) \bar{\psi}(y)} S_0 + i^2 \int d\tau \langle T \{ \overline{\psi(x) L(\tau) \bar{\psi}(y) S} \} \rangle_0. \quad (38.29)$$

Next, using (38.5) and applying the operator  $(i\hat{\partial} - m)$  to (38.29), we obtain after some rearrangement

$$(i\hat{\partial} - m) g(x, y) = -\delta(x - y) S_0 - i e \gamma^n \langle T \{ \psi(x) \bar{\psi}(y) A_n(x) S \} \rangle_0.$$

The last term on the right-hand side can be expressed in terms of the variational derivative of  $g(x, y)$ . We note that the variational differentiation of the operator  $S$  with respect to  $J^n(x)$  is equivalent to the multiplication of it by  $iA_n(x)$ . We therefore have

$$\frac{\delta}{\delta J^n(z)} g(x, y) = i^2 \langle T \{ \psi(x) \bar{\psi}(y) A_n(z) S \} \rangle_0.$$

Therefore

$$\left\{ i\gamma^n \left( \partial_n - e \frac{\delta}{\delta J^n(x)} \right) - m \right\} g(x, y) = -\delta(x - y) S_0.$$

Hence, transforming the Green's function  $G = gS_0^{-1}$ , we have

$$\left\{ i\gamma^n \left( \partial_n - e \frac{\delta}{\delta J^n(x)} \right) - m \right\} G(x, y) = -\delta(x - y) + \frac{ie}{S_0} \gamma^n \frac{\delta S_0}{\delta J^n(x)} G(x, y). \quad (38.30)$$

The quantity

$$\frac{1}{iS_0} \frac{\delta S_0}{\delta J^n(x)} = \frac{\langle T A_n(x) S \rangle_0}{S_0} = \mathfrak{A}_n(x) \quad (38.31)$$

on the right-hand side of the above equation is, by virtue of (37.22), the expectation value of the operator for the electromagnetic-field potential. In accordance with (37.26.1), it can also be written in the form

$$\mathfrak{A}_n(x) = \frac{\delta Z(J)}{\delta J^n(x)},$$

where the functional  $Z$  is related to  $S_0$  by (37.25).

Substituting (38.31) in (38.30), we obtain the required functional equation for the one-fermion Green's function:

$$\left\{ i\gamma^n \left( \partial_n - e \frac{\delta}{\delta J^n(x)} - ie\mathfrak{A}_n(x) \right) - m \right\} G(x, y) = -\delta(x - y). \quad (38.32)$$

We shall now obtain the corresponding equations for  $\mathfrak{A}_n$ . To do this, we use (38.1) and (38.8), and represent  $\mathfrak{A}_n$  in the form

$$\mathfrak{A}_n(x) = \frac{1}{S_0} \langle T \{ \overline{A_n(x)} S \} \rangle_0 = \frac{i}{S_0} \int \langle T \{ A_n(x) \mathcal{L}(\tau) S \} \rangle_0 d\tau.$$

Next, using (38.7) and (15.20), we obtain

$$\mathfrak{A}_n(x) = -\frac{1}{S_0} \int D_0^c(x - \tau) d\tau [e \langle T \{ j_n(\tau) S \} \rangle_0 + J_n(\tau) S_0], \quad (38.33)$$

where



$$j_n(x) = \bar{\psi}(x) \gamma_n \psi(x).$$

Recalling that the function  $D_0^c$  satisfies the equation

$$\square D_0^c(x) = -\delta(x),$$

and applying the operation  $\square$  to (38.33), we obtain

$$\square \mathfrak{A}_n(x) = J_n(x) + e \frac{\langle T \{j_n(x) S\} \rangle_0}{S_0}.$$

Finally, noting that

$$\begin{aligned} \langle T \{j_n(x) S\} \rangle_0 &= \langle T \{ \bar{\psi}_\sigma(x) (\gamma_n)_{\sigma\sigma'} \psi_{\sigma'}(x) \} S \rangle_0 = \\ &= - \sum_{\sigma, \sigma'} (\gamma_n)_{\sigma\sigma'} \langle T \{ \psi_{\sigma'}(x) \bar{\psi}_\sigma(x) S \} \rangle_0 = i \operatorname{Tr} (\gamma_n G(x, x)), \end{aligned} \quad (38.34)$$

we obtain the equation for  $\mathfrak{A}_n$  in the form

$$\square \mathfrak{A}_n(x) = J_n(x) + ie \operatorname{Tr} (\gamma_n G(x, x)). \quad (38.35)$$

Equations (38.32) and (38.35) form a system which can be used to determine the two unknowns  $G$  and  $\mathfrak{A}$  in terms of the given function  $J$ . These equations were first obtained by Schwinger (1951a, b), who based his derivation on a quantum dynamic principle specially formulated by him. Equations such as (38.32) and (38.35) are called *Schwinger's equations*.

**38.4. Dyson's Equations.** Schwinger's equations can be transformed into a set of integro-differential equations for the electron function  $G$  and the photon function  $B$ , which does not contain explicitly the variational differentiation operators. This can be conveniently done by transforming from  $J$  to a new functional argument  $\mathfrak{A}$ , i.e., by using the transformation considered in §37.4 in connection with vertex functions.

Let us introduce the connected (one-) photon Green's function  $D_{nm}(x, y)$ . From (37.23) and also (37.26.2) we have

$$D_{mn}(x, y) = \frac{i}{S_0(J)} \langle T \{A_m(x) A_n(y) S\} \rangle_0 - i \mathfrak{A}_m(x) \mathfrak{A}_n(y) = \frac{\delta \mathfrak{A}_m(x)}{\delta J^n(y)}. \quad (38.36)$$

The variational derivative with respect to  $J$  can therefore be expressed in terms of the derivative with respect to  $\mathfrak{A}$  by

$$\frac{\delta}{\delta J^n(y)} = \int \frac{\delta \mathfrak{A}_m(z)}{\delta J^n(y)} dz \frac{\delta}{\delta \mathfrak{A}_m(z)} = \int D_{nm}(y, z) dz \frac{\delta}{\delta \mathfrak{A}_m(z)}, \quad (38.37)$$

and, if we introduce this into (38.22), we obtain

$$\{\gamma^n (i\partial_n + e\mathfrak{A}_n(x)) - m\} G(x, y) + ie\gamma^n \int D_{nm}(x, z) dz \frac{\delta G(x, y)}{\delta \mathfrak{A}_m(z)} = -\delta(x - y). \quad (38.38)$$

On the other hand, differentiating (38.35) with respect to  $J^n(y)$ , and using (38.36) and (38.37), we obtain

$$\square_x D_{mn}(x, y) = -g_{mn}\delta(x - y) + ie \int \text{Tr} \left\{ \gamma_m \frac{\delta G(x, x)}{\delta \mathfrak{A}_l(z)} \right\} dz D_{ln}(z, y). \quad (38.39)$$

Equations (38.38) and (38.39) form a system in which Green's functions  $G$  and  $D$  must be looked upon as functionals of the effective field  $\mathfrak{A}$ . The current  $J$  is absent from these equations.

The variational derivatives in (38.38) and (38.39) can be expressed with the aid of relationships such as (37.26.3) and (37.34) in terms of the strongly connected vertex function

$$\Gamma^n(x, y; z) = -\frac{\delta G^{-1}(x, y)}{\delta \mathfrak{A}_n(z)}. \quad (38.40)$$

This formula is an analog of (37.36). We have

$$\frac{\delta G(x, y)}{\delta \mathfrak{A}_n(z)} = \int G(x, x') dx' \Gamma^n(x', y'; z) dy' G(y', y). \quad (38.41)$$

and substituting this in (38.38) and (38.39), we obtain

$$(i\hat{\partial} + e\hat{\mathfrak{A}}(x)) G(x, y) - \int M(x, z) dz G(z, y) = -\delta(x - y) \quad (38.42)$$

and

$$\square_x D_{mn}(x, y) - \int P_m{}^l(x, z) dz D_{ln}(z, y) = -g_{mn}\delta(x, y), \quad (38.43)$$

where the mass operator  $M$  and the polarization operator  $P$  are defined by

$$M(x, y) = m\delta(x - y) - ie\gamma^n \int G(x, x') dx' \Gamma^n(x', y; z') dz' D_{nm}(z', z), \quad (38.44)$$

$$P_m{}^l(x, z) = ie \text{Tr} \left[ \gamma_m \int G(x, x') dx' \Gamma^l(x', y; z) dy G(y, x) \right]. \quad (38.45)$$

We emphasize the operators  $\Gamma$ ,  $M$ , and  $P$  are similar to Green's functions  $G$  and  $D$  in that they are functionals of  $\mathfrak{A}$ .

To clarify the physical meaning of the operators  $\Gamma$ ,  $M$ , and  $P$ , it is convenient to transform from the integrodifferential equations (38.32) and (38.33) to the purely integral equations through an operation that is the inverse of the application of  $\square$  and  $(i\hat{\partial} - m)$ .

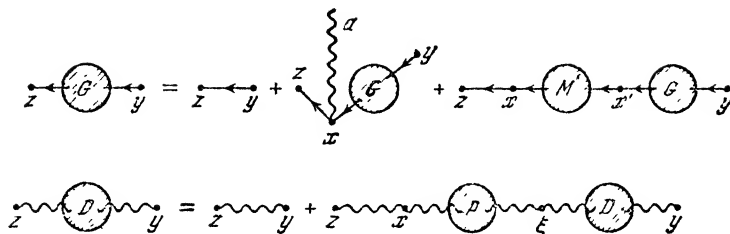


Fig. 57.

To do this, it is sufficient to multiply (38.32) by  $S^c(z-x)$  and (38.33) by  $D_0^c(z-x)$ , and integrate with respect to  $z$ . Integrating by parts, we obtain

$$G(z, y) = S^c(z-y) + e \int dx S^c(z-x) \mathfrak{M}(x) G(x, y) - \int S^c(z-x) dx M'(x, x') dx' G(x', y), \quad (38.46)$$

$$M'(x, y) = M(x, y) - m\delta(x-y), \quad (38.47)$$

$$D_{mn}(z, y) = g_{mn} D_0^c(z-y) - \int D_0^c(z-x) dx P_m{}^l(x, x') dx' D_{ln}(x', y), \quad (38.48)$$

or, in symbolically abbreviated form,

$$G = S^c + e S^c \mathfrak{M} G - ie S^c (\gamma G \Gamma D) G, \quad (38.49)$$

$$D = D_0 - ie D_0 \text{Tr} \backslash (\gamma G \Gamma G) D. \quad (38.50)$$

With (38.46)–(38.48) we can associate the graphs of Fig. 57. The elements  $M'$  and  $P$  in these schemes are interpreted with the aid of (38.44) and (38.45), and this is shown graphically in Fig. 58.

The integral equations for the complete Green's functions of the type given by (38.44)–(38.48) are sometimes called *Dyson's equations*. The essential point is that such equations do not form a closed system. Thus, (38.44)–(38.48) include the vertex function  $\Gamma$  which is related to  $G$  through the operation of variational differentiation. It is, of course, also

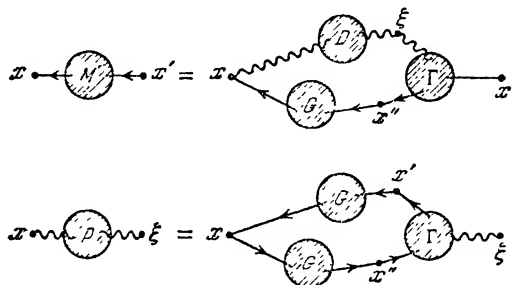


Fig. 58.

possible to obtain an integral equation for  $\Gamma$ . This can be done by differentiating (38.46) with respect to  $\mathfrak{A}_m$  and then transform to  $\Gamma$  in accordance with (38.40). The result of this will be the appearance of

$$\frac{\delta \Gamma^n(x, y; z)}{\delta \mathfrak{A}_m(t)} \langle T \{ \psi(x) \bar{\psi}(y) A_n(z) A_m(t) S \} \rangle_0,$$

on the right-hand side. This is connected with the four-tail vertex function  $K_{mn}$  of the Compton type. The set of equations (38.44)–(38.48), augmented by the “equation” for  $\Gamma$ , is thus again not a closed system. If we continue this process, we obtain equations containing increasingly higher Green’s functions. Equations (38.44)–(38.48), which depend functionally on  $\mathfrak{A}$ , are the generating equations for this type of infinite chain.

The method presented above may also be applied directly to obtain closed equations for more complicated expressions, for example, for Green’s functions for two fermions, two photons, and so on. By defining such functions in terms of the vacuum expectation values of the chronological product of the corresponding number of field operators, one may, by using the generalized Wick theorem, obtain for them equations similar to those obtained by Schwinger (1951b) for the two-fermion Green’s function.

*38.5. Taking Counterterms into Account.* As we have already noted, the “complete” Green’s functions contain divergences of the same type as the scattering matrix. To compensate these divergences, it is necessary to introduce certain counterterms into the interaction Lagrangian, i.e., to go over from the Lagrangian (38.4) to (33.88).

We recall that the counterterms (33.38) remove from the  $S$ -matrix all the divergences with the exception of the vacuum loops which, by definition, are compensated in Green’s functions by the factor  $(S_0)^{-1}$ .

Repeating the preceding argument for the Lagrangian (33.38), i.e., introducing the counterterms (33.38) into (38.29), (38.30), (38.33), (38.34), and so on, we obtain Schwinger’s equations for Green’s functions without divergences in the form

$$Z_2 \left( i \frac{\partial}{\partial x} - m \right) G(x, y) - i Z_1 e \gamma^k \left( \frac{\delta}{\delta J^k(x)} + i \mathfrak{A}_k(x) \right) G(x, y) = -\delta(x - y), \quad (38.51)$$

$$\begin{aligned} Z_3 \square D_{kl}(x, y) = & -g^{kl} \delta(x - y) + i Z_1 e \int d\xi \operatorname{Tr} \left\{ \gamma_k \frac{\delta G(x, \xi)}{\delta \mathfrak{A}_m(\xi)} \right\} D_{ml}(\xi, y) - \\ & - (Z_3 - 1) \frac{\partial^2}{\partial x^k \partial x_n} D_{nl}(x, y). \end{aligned} \quad (38.52)$$

Similarly, we may go over to equations such as (38.42) and (38.43) which in place of variational derivatives contain the operators  $M$  and  $P$ .

The essential difference between equation (38.52) for the function  $D$  and (38.39) is the presence in it of the term

$$\frac{\partial^2}{\partial x^k \partial x_n} D_{nl}(x, y).$$

The origin of this expression is connected with the transverse form of the counterterm  $(Z_3 - 1)$  which differs from the Lagrangian of the free electromagnetic field by the term  $(\partial A/\partial x)^2$ . Essentially, we have here encountered the consequences of the "incomplete renormalization" of the function  $D$  which contains a longitudinal component. It may be shown that to obtain an equation with counterterms which has the same form as the equation without counterterms, it is sufficient to utilize the purely transverse pairing instead of the diagonal pairing of the electromagnetic-field operators.

Equations (38.51) and (38.52) can be obtained from (38.38) and (38.39) [with allowance for (38.36)] with the aid of the formal *Dyson transformation*

$$G \rightarrow Z_2 G, \quad D^{\text{tr}} \rightarrow Z_3 D^{\text{tr}}, \quad D' \rightarrow D', \quad \mathfrak{A} \rightarrow Z_3^{1/2} \mathfrak{A}, \quad e \rightarrow Z_1 Z_2^{-1} Z_3^{-1/2} e. \quad (38.53)$$

If we now transform to the integral form of the Schwinger equations, we have instead of (38.49) and (38.50)

$$Z_2 G = S^c - Z_1 e \gamma S^c \mathfrak{A} G - i Z_1 e^2 S^c (\gamma G \Gamma D) G, \quad (38.54)$$

$$Z_3 D = D_0 - i Z_1 e^2 D_0 \text{Tr} (\gamma G \Gamma G) D - (Z_3 - 1) \partial_\mu \partial_\nu D_{\mu\nu}. \quad (38.55)$$

The last two equations are obtained from (38.49) and (38.50) through the same transformation (38.53). The transformation law for the vertex function

$$\Gamma \rightarrow Z_1^{-1} \Gamma \quad (38.56)$$

follows from (38.40) and (38.53).

It is also necessary to make the following remark with respect to equations (38.51) and (38.52). The direct motivation for Schwinger's equations lay in the hope of obtaining some information about complete Green's functions that would not be based on the perturbation-theory formalism. However, solutions of equations such as (38.38) and (38.39) contain divergences, while equations (38.51) and (38.52) for the Green's functions free of infinities contain divergent constants and are therefore of a formal nature. Thus, attempts to obtain any sort of information about complete Green's functions on the basis of Schwinger's equations encounter fundamental difficulties connected with the procedure used to remove the infinities.

Of course, if we regard these equations only as a source of formal expansions in powers of the coupling constant then the corresponding divergences will be compensated, but we shall obtain nothing new in comparison with perturbation theory. The problem of finding an effective method of solving these equations that is not based on perturbation theory is at present still far from any sort of satisfactory solution.

# Chapter 7

## SCHROEDINGER EQUATIONS AND DYNAMIC VARIABLES

### §39. Schroedinger Equation for the State Amplitude

*39.1. Equation for  $\Phi(g)$  in Terms of Variational Derivatives.* In studying interactions between free fields, we have so far been considering situations in which the interaction is effective over a finite space-time region. By means of an appropriate limiting transition in the course of which the “region of effectiveness” of the interaction was expanded indefinitely, we were able to determine the elements of the scattering matrix. The elements of the scattering matrix in turn permit us to calculate the cross sections for scattering processes in those cases where particles that may be regarded as free appear at the beginning and at the end of the process. However, such processes do not exhaust all the problems which the theory has to solve. We have in mind problems involving determinations of the energy and other characteristics of bound states, the lifetimes of excited states, cross sections for processes in which bound clusters of particles appear in the initial and final states, and so on.

In order to be able to attack such problems, it is not sufficient to have only the scattering matrix at one's disposal, and it turns out to be necessary to have an instrument for a more detailed description of the system, for example, an equation of the Schroedinger type. To approach the derivation of such an equation in a natural manner, we shall start in a purely formal way from the relation given by (20.13):

$$\Phi(g) = S(g) \Phi,$$

which we have established earlier, where  $\Phi$  is a constant,  $S(g)$  is the scattering matrix in the presence of an interaction switched on with an intensity  $g$ , and  $\Phi(g)$  is the amplitude

for the state of the system subjected to the interaction of the given intensity. By varying this relation with respect to the function  $g(x)$ , and by utilizing the condition that the matrix  $S(g)$  is unitary, we obtain

$$\frac{\delta \Phi(g)}{\delta g(x)} = \frac{\delta S(g)}{\delta g(x)} \Phi = \frac{\delta S(g)}{\delta g(x)} S^\dagger(g) \Phi(g).$$

Therefore, by introducing the operator (cf. §§21.2 and 31.2)

$$H(x; g) = i \frac{\delta S(g)}{\delta g(x)} S^\dagger(g), \quad (21.10)$$

we may write

$$i \frac{\delta \Phi(g)}{\delta g(x)} = H(x; g) \Phi(g) \quad (39.1)$$

or in integral form

$$i \delta \Phi(g) = \int H(x; g) \Phi(g) \delta g(x) dx. \quad (39.2)$$

In its form equation (39.2) is the variational analog of the Schroedinger equation, while the operator  $H(x; g)$  plays the role of the generalized Hamiltonian (or, more accurately, of the generalized Hamiltonian density). Moreover, the analogy with the Schroedinger equation becomes closer and, as we shall see, reduces from the formal point of view to an identity, if the function  $g(x)$  is made to approach the discontinuous function which is equal to unity for all points of four-space with time coordinate smaller than a certain fixed

$$x^0 = \tau$$

and which is zero for all points with  $x^0 > \tau$ . Such a  $g(x)$  evidently describes the instantaneous switching-off of the interaction over the whole three-dimensional space at time  $x^0 = \tau$ . The state amplitude  $\Phi(g)$  is in this case a function of the parameter  $\tau$  and may be denoted by  $\Phi_\tau$ .

However, in the usual theory, a concept such as the *wave function at a given time*  $\tau$  may also be defined by means of an instantaneous switching-off of the interaction at time  $\tau$ . After such a switching-off of the interaction, the wave function stops changing and we simply have

$$\Phi_\tau = \Phi_\infty.$$

If the switching-off of the interaction is carried out not over the four-plane  $x^0 = \tau$  but over the space-like four plane  $\sigma$ , so that

$$x^0 = T_\sigma(x)$$

we obtain the well-known covariant equation in the Tomonaga-Schwinger form. Both in the ordinary Schroedinger equation and in the equation in the Tomonaga-Schwinger form, the Hamiltonian density  $H(x; \sigma)$  depends on the behavior of the fields in the infinitesimal neighborhood of the point  $x$ . Our point of view differs from the usual one by the fact that we employ *continuous* rather than *sudden* switching-off of the interaction.

**39.2. Schroedinger Equation in the Interaction Representation and the Tomonaga-Schwinger Equation.** We shall investigate the behavior of the operator  $H(x; g)$  in the process of letting the continuous function  $g(x)$  tend to the discontinuous limit discussed above. We recall that as was shown in §21.2, the generalized Hamiltonian  $H(x; g)$  may be represented by a functional expansion in powers of the interaction

$$H(x; g) = H_0(x) + \sum_{n \geq 1} \frac{1}{n!} \int H_n(x, x_1, \dots, x_n) g(x_1) \dots g(x_n) dx_1 \dots dx_n, \quad (39.3)$$

where the  $H_n$  are expressed by the relations

$$H_0(x) = -\mathcal{L}(x), \quad (39.4)$$

$$\begin{aligned} H_n(x, x_1, \dots, x_n) &= \\ &= i \sum_{0 \leq k \leq n} P\left(\frac{x_1, \dots, x_k}{x_{k+1}, \dots, x_n}\right) S_{k+1}(x, x_1, \dots, x_k) \overset{\dagger}{S}_{n-k}(x_{k+1}, \dots, x_n) \end{aligned} \quad (21.12)$$

( $n = 1, 2, \dots$ ),

in terms of the expansion coefficients  $S_k(x_1, \dots, x_k)$  of the matrix  $S(g)$  and, by virtue of the condition of causality imposed on the matrix  $S(g)$ , have the property that

$$\left. \begin{aligned} H_n(x, x_1, \dots, x_n) &= 0 & (n = 1, 2, \dots), \\ \text{if for at least one } x_j &x_j \gtrsim x_j & (j = 1, \dots, n). \end{aligned} \right\} \quad (21.13)$$

Therefore the integration over each of the  $x_j$  in the terms of the expansion (39.3) actually takes place over the light cone at the point  $x$ , directed into the future. The convergence of the integrals appearing in (39.3) will be guaranteed if the sufficiently smooth weighting function  $g(x)$  falls off sufficiently rapidly (for example, exponentially) as  $x_j^0$  tends to infinity. The function  $g(x)$  will, evidently, satisfy this condition if it is defined as an approximation to the well-known step function  $\theta(\tau - x_0)$  [see (3.18)] which vanishes for values of  $x^0$  greater than a certain parameter  $\tau$ . We can take, for example,

$$\left. \begin{aligned} g(x) &= f(\tau - x^0), \\ f(t) - \theta(t) &= 0 \quad \text{for } |t| \geq \Delta t, \end{aligned} \right\} \quad (39.5)$$

where



if  $f(t)$  is chosen to be a sufficiently smooth function. It is clear that the integrals (39.3) will converge with this choice of  $g(x)$ , since  $g(x_j)$  falls sufficiently rapidly as  $x_j^0 \rightarrow \infty$ , while the integrals in (39.3) by virtue of the property (21.13) do not depend on the behavior of  $g(x_i)$  for infinite values of the spatial and the negative time variables.

Our choice of  $g(x)$  corresponds to the situation where the interaction is switched on over the whole three-space and for times from  $-\infty$  up to  $\tau - \Delta t$ , and is gradually switched off between  $\tau - \Delta t$  and  $\tau + \Delta t$ .

The state amplitude may now be regarded as a function of the parameter  $\tau$ :

$$\Phi(g) = \Phi_\tau.$$

Determining the variation  $\delta g$  from (39.5)

$$\delta g(x) = f'(\tau - x^0) \delta \tau,$$

we find with the aid of (39.2) that

$$i\delta_\tau \Phi_\tau = \int H(x; f) f'(\tau - x^0) dx \delta \tau \Phi_\tau,$$

from which it follows that

$$i \frac{\partial \Phi_\tau}{\partial \tau} = \int H(x; f) f'(\tau - x^0) dx \Phi_\tau. \quad (39.6)$$

By virtue of the definition of the function  $f$ , this integral is taken over the time "slab" between  $\tau - \Delta t$  and  $\tau + \Delta t$ . On the other hand, as was shown earlier, the integration in (39.3) is taken over the light cone of "height"

$$h = \tau + \Delta t - x^0.$$

Taking into account the fact that in (39.6),  $x^0$  is in any case greater than  $\tau - \Delta t$ , we find (see Fig. 59) that  $h \leq 2\Delta t$ . Thus, the integration in (39.3) is evaluated over the region satisfying the condition

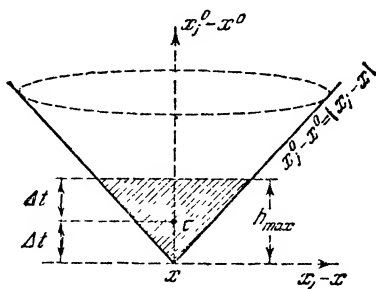


Fig. 59.

$$|x_i^0 - x^0| \leq 2\Delta t, \quad |x_j - x| \leq 2\Delta t. \quad (39.7)$$

Therefore the "effective Hamiltonian"  $H(x; f)$  appearing in (39.6) depends on the behavior of the field functions in the neighborhood of the point  $x$  of order  $2\Delta t$ . If it were possible to go to the limit  $\Delta t \rightarrow 0$  in (39.6), and to deal with  $f$  in place of  $\theta$ , then we would obtain in its place the equation

$$i \frac{\partial \Phi_\tau}{\partial \tau} = \int H(x) d\mathbf{x} \Phi_\tau, \quad (39.8)$$

in which the Hamiltonian density  $H(x)$  depends on the behavior of the fields within an infinitesimal neighborhood of the point  $x$ . Equation (39.8) is essentially the *Schroedinger equation in the interaction representation*.

Until now, we have considered only the special case of switching off the interaction along the plane  $x^0 = \tau$ . However, it is not difficult to generalize the argument. Consider the space-like surface  $\sigma$

$$x^0 = T_\sigma(x),$$

for which the condition of being space-like is fulfilled in the strong sense:

$$|T_\sigma(x) - T_\sigma(y)| < \lambda |x - y| \quad (x \neq y),$$

with constant  $\lambda < 1$ . We introduce the step function  $\theta_\sigma(x)$  such that

$$\begin{cases} \theta_\sigma(x) = 1 & \text{for } x^0 > T_\sigma(x), \\ \theta_\sigma(x) = 0 & \text{for } x^0 < T_\sigma(x), \end{cases} \quad (39.9)$$

i.e.,  $\theta_\sigma(x) = \theta(x^0 - T_\sigma(x))$ , and consider a sufficiently smooth function  $g(x)$  that differs from  $\theta_\sigma(-x)$  only within the time interval

$$|x^0 - T_\sigma(x)| \leq \Delta t.$$

It is clear that the effective region of integration over each  $x_j$  in the expression

$$\int H_n(x, x_1, \dots, x_n) g(x_1) \dots g(x_n) dx_1 \dots dx_n \quad (39.10)$$

is then determined, similarly to the case considered earlier, by the intersection of the upper light cone at the point  $x$  with a "slab" of thickness  $2\Delta t$  on the surface  $\sigma$ , placed at an angle to the cone axis not smaller than

$$\varphi = \cot^{-1} \lambda. \quad (39.11)$$

It may be seen from Fig. 60 that the height of that part of the cone which contains the

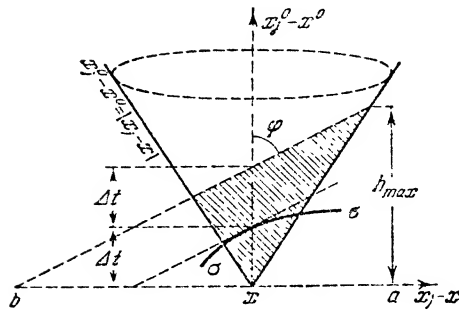


Fig. 60.

region of integration (shaded on the diagram) does not exceed  $2\Delta t(1 - \lambda)^{-1}$ .

The region of integration is therefore restricted by the inequalities

$$x_j^0 - x^0 \leq \frac{2\Delta t}{1 - \lambda}, \quad |x_j - x| \leq \frac{2\Delta t}{1 - \lambda}. \quad (39.12)$$

We see that also in this case the time and the spatial dimensions of the effective region of integration in (39.10) do not exceed a quantity proportional to  $\Delta t$ . The operator  $H(x; g)$  will therefore depend on the state of the fields only in the neighborhood of the point  $x$ , with the dimensions of this neighborhood approaching zero as the "layer of smearing-out" defined by the function  $g(x)$  is reduced in thickness.

We now make the formal transition in (39.2) to the limit corresponding to an indefinite reduction in the thickness of the "smeared-out layer"  $g(x)$ :

$$\Delta t \rightarrow 0, \quad g(x) \rightarrow \theta(T_\sigma(x) - x_0) = \theta_\sigma.$$

The variation  $\delta g(x)$  will then correspond to the variation in the surface  $\sigma$ , where

$$\delta g(x) = g'(T_\sigma - x_0) \delta T_\sigma(x).$$

We recall the definition of the variational derivative with respect to  $\sigma$ :

$$\frac{\delta F(\sigma)}{\delta \sigma(x)} = \lim_{\Omega(x) \rightarrow 0} \frac{F(\sigma') - F(\sigma)}{\Omega(x)},$$

where  $\Omega(x)$  is the volume between the surfaces  $\sigma'$  and  $\sigma$ , and  $\sigma'$  is obtained by a small deformation  $\delta T_\sigma$  of the surface  $\sigma$  near the point  $x$ . It is clear that

$$\Omega(x) = \int_{\sigma}^{\sigma'} \delta T_\sigma dx'$$

if the four-dimensional volume of the smeared-out region.

The result of the preceding limiting transition is that (39.2) becomes

$$i\delta_\sigma \Phi(\sigma) = \lim_{g \rightarrow \theta_g} \int H(x, g) g'(T_\sigma(x) - x_0) dx_0 \Omega(x) \Phi(\sigma)$$

or

$$i \frac{\delta \Phi(\sigma)}{\delta \sigma(x)} = H(x, \sigma) \Phi(\sigma), \quad (39.13)$$

where the operator

$$H(x, \sigma) = \lim_{g \rightarrow \theta_g} \int H(x, g) g'(T_\sigma(x) - x_0) dx_0 \quad (39.14)$$

depends on the behavior of the fields only in the infinitesimal neighborhood of the point  $x$  on the surface  $x_0 = T_\sigma(x)$ . We have thus obtained the well-known *Tomonaga-Schwinger equation*.

Thus, by reducing the size of the region in which the function  $g(x)$  varies, and by letting it approach the discontinuous limit, we can in fact obtain both the usual Schroedinger equation (39.8) and the Tomonaga-Schwinger equation (39.13) from (39.1). The effective Hamiltonian  $H(x; \sigma)$  or  $H(x; f)$  is then given by (39.8) and (39.14).

**39.3. Singularities of the Generalized Hamiltonian.** We must now consider in greater detail the limiting transition (39.14). To avoid additional complications connected with the problem of ultraviolet divergences, we shall start with the regularized expression for  $H(x, g)$  with finite auxiliary masses  $M$ .

We start with the second-order term in  $H(x, g)$ . From (21.12) and (21.41), we have

$$H_1(x, x') = -\Lambda_2(x, x') + i\mathcal{L}(x)\mathcal{L}(x') - iT(\mathcal{L}(x)\mathcal{L}(x')).$$

If regularization has been performed for each internal line, then, for finite  $M$ , the expression

$$\mathcal{L}(x) \propto (x') - T(\mathcal{L}(x)\mathcal{L}(x'))$$

will be regular and the integral of its product with  $g(x')$  over the region defined by (39.12) will be zero as  $\Delta t \rightarrow 0$  because the region itself then contracts to a point. At the same time, the corresponding integral, i.e.,

$$-\int \Lambda_2(x, x') g(x') dx'$$

with the quasilocal operator containing  $\delta$ -function will be nonzero.

In a completely analogous fashion, we can establish that in the more general case, we have the limiting equation

$$\begin{aligned} \int H_n(x, x_1, \dots, x_n) g(x_1) \dots g(x_n) dx_1 \dots dx_n = \\ = - \int \Lambda_{n+1}(x, x_1, \dots, x_n) g(x_1) \dots g(x_n) dx_1 \dots dx_n. \end{aligned}$$

To prove this, we write  $H_n$  as the sum of two terms:

$$H_n(x, x_1, \dots, x_n) = -\Lambda_{n+1}(x, x_1, \dots, x_n) + \tilde{H}_n(x, x_1, \dots, x_n),$$

and it is then sufficient to verify that the operator  $\tilde{H}_n$  contains  $\delta$ -functions whose number is smaller than  $n$ , so that its  $n$ -fold integral over an infinitesimal region will also be zero.

Thus, the limiting (for  $\Delta t \rightarrow 0$ ) expression for the density of the Hamiltonian has the form [Sukhanov (1962a)]

$$\begin{aligned} H(x, \sigma) = - \lim_{g \rightarrow \theta_\sigma} \int dx_0 g'(T_\sigma - x_0) \{ \mathcal{L}(x) + \\ + \sum_{n=1} \frac{1}{n!} \int \Lambda_{n+1}(x, x_1, \dots, x_n) g(x_1) \dots g(x_n) dx_1 \dots dx_n \} = \\ = - \lim_{g \rightarrow \theta_\sigma} \int dx_0 g'(T_\sigma - x_0) \frac{\delta \int \mathcal{L}(y, g) dy}{\delta g(x)}. \quad (39.15) \end{aligned}$$

We shall illustrate the properties of this expression by two examples.

In the theory with the interaction  $\lambda : \varphi^3(x) :$ , the quasilocal operators  $\Lambda_{n+1}$ , which contain derivatives, refer only to the vacuum terms that do not contain the field operators. Their contribution to the limiting Hamiltonian (39.15) is proportional to the integral

$$\begin{aligned} -2 \lim_{g \rightarrow \theta_\sigma} \int dx_0 g'(T_\sigma - x_0) \int dy \square \delta(x - y) g(T_\sigma - y_0) = \\ = \lim_{g \rightarrow \theta_\sigma} \int dx_0 \frac{\partial}{\partial x_0} (g'(T_\sigma - x_0))^2, \quad (39.16) \end{aligned}$$

which vanishes because the function  $g'$  decreases at infinity. In the remaining terms, the quasilocal operators do not contain derivatives and the limiting transition (39.15), performed with allowance for the symmetry properties of the operators  $\Lambda_{n+1}$  [see Sukhanov (1962a) for more details], yields

$$H(x, \sigma) = -\mathcal{L}(x; 1)|_{x^0 = \tau_\sigma(x)}, \quad (39.17)$$

where the effective Lagrangian  $\mathcal{L}(x; g)$  of the  $\lambda : \varphi^3 :$  theory is given by (36.7).

In the  $\lambda : \varphi^4 :$  theory, derivatives are contained in  $\Lambda_{n+1}$  even in the operator terms. When the integrals with respect to  $x_i$  are evaluated in (39.15), these derivatives act both on the fields  $\varphi(x_i)$  and the functions  $g(x_i)$ . In terms of the first type, the limiting transition  $g(x) \rightarrow \theta_\sigma$  can be performed and, together with the limit of the terms without the derivatives, gives an expression analogous to (39.17). For terms of the second type, on the other

hand, the formal limit  $g \rightarrow \theta_\sigma$  does not exist. In fact, in second order in  $g(x)$ , the Hamiltonian density  $H(x, \sigma)$  will contain a contribution due to the expression

$$\begin{aligned} -2 \lim_{g \rightarrow \theta_\sigma} \int dx_0 g'(T_\sigma - x_0) : \varphi^2(x) : g''(T_\sigma - x_0) = \\ = 2 \lim_{g \rightarrow \theta_\sigma} \int dx_0 \frac{\partial}{\partial x_0} [ : \varphi^2(x) : (g'(T_\sigma - x_0))^2 ] - \\ - \lim_{g \rightarrow \theta_\sigma} \int dx_0 \left[ \frac{\partial}{\partial x_0} : \varphi^2(x) : \right] [g'(T_\sigma - x_0)]^2. \quad (39.18) \end{aligned}$$

For the same reason as in (39.16), the first term in the difference will be zero, whereas the second term will contain a nonintegrable expression as  $g \rightarrow \theta_\sigma$  because  $g'(x) \rightarrow \delta(x_0 - T_\sigma)$ . We emphasize that this difficulty arises even for finite auxiliary masses  $M$ , and is connected with the sudden removal of the interaction on the space-like surface  $x_0 = T_\sigma$ . The existence of divergences of this type was first noted by Stueckelberg (1951) who called them *surface divergences*.

Formally,  $\lim_{g \rightarrow \theta_\sigma} [g'(T_\sigma - x_0)]^2$  can be redefined as an extension of the functional  $[\delta(x)]^2$  defined on the basic functions which vanish at  $x = 0$ , to all the basic functions of the class  $C(q, r, n)$ . This extension [see, for example, Vladimirov (1964)] has the form

$$[\delta(x)]^2 = C \delta(x), \quad (39.19)$$

where  $C$  is an arbitrary constant. The expression given by (39.18) then assumes the form

$$-C \left[ \frac{\partial}{\partial x_0} : \varphi^2(x) : \right] \Big|_{x_0 = T_\sigma}.$$

The higher orders in  $g(x)$  provide analogous contributions (with new arbitrary constants).

Thus, in the  $\lambda : \varphi^4(x) :$  theory, the formal redefinition (39.19) of the limiting transition  $g \rightarrow \theta_\sigma$  reduces the generalized Hamiltonian to the form [Sukhanov (1962b)]

$$H(x, \sigma) = -\mathcal{L}(x, 1) \Big|_{x_0 = T_\sigma} - C(M) \left[ \frac{\partial}{\partial x_0} : \varphi^2(x) : \right] \Big|_{x_0 = T_\sigma}, \quad (39.20)$$

where the effective Lagrangian  $\mathcal{L}(x, g)$  is defined by (21.39) and  $C(M)$  is the sum of contributions to the renormalizing factor  $(Z_3 - 1)$  due to different orders of perturbation theory, multiplied by the arbitrary constants  $c$  (these contributions diverge as  $M \rightarrow \infty$ ).

We must now consider how this arbitrariness is reflected in the solution of the Tomonaga-Schwinger equation (39.13) which is naturally sought in the form

$$\Phi(\sigma) = S(\sigma) \Phi. \quad (39.21)$$

We have

$$S(\sigma) = T \exp \left\{ -i \int_{-\infty}^{\sigma} H(y, \sigma) dy \right\} \equiv S(\sigma, -\infty), \quad (39.22)$$

where  $H(y, \sigma)$  is given by (39.20). It is clear that the contribution of the second term to (39.20) is localized on the surface  $x_0 = T_\sigma$ . Since, moreover,  $\mathcal{L}(x, 1)$  and  $:\varphi^2(x):$  commute at equal times, this contribution can be taken outside the  $T$ -exponent sign and can be segregated into a separate exponential factor on the left:

$$\exp \left\{ iC(M) \int d\sigma(x) : \varphi^2(x) : \right\} \Big|_{x_0=T_\sigma}. \quad (39.23)$$

The entire arbitrariness in  $S(\sigma)$ , which appears after redefinition of the limiting transition  $g \rightarrow \theta_\sigma$ , can be segregated into a unitary operator factor localized on the surface  $\sigma$ . We note that this arbitrariness does not effect the complete  $S$ -matrix.

The analysis that led us to (39.15) and (39.20) is valid only when regularization is performed for each internal line of the diagram. This regularization is, of course, "excessive." To remove the divergences it is sufficient to regularize only the strongly connected parts of the diagrams. However, the advantage of the above procedure is the simplicity of formulas such as (39.15) and (39.20).

In the case of the minimum regularization, on the other hand, expressions of the form

$$\mathcal{L}(x) \mathcal{L}(x') - T(\mathcal{L}(x) \mathcal{L}(x')),$$

are not, in general, regular and contain the quasilocal components if the Lagrangian contains derivatives. For example, two-time derivatives acting on  $D^c(x - x')$  give a quasilocal term proportional to  $\delta(x - x')$ . (In our previous discussion, involving nonminimum regularization, the factor in front of this term was the sum of the Pauli-Villars coefficients which was zero.)

To summarize, in each order in  $g(x)$ , as  $\Delta t \rightarrow 0$  we have in addition to the contribution due to the operators  $\Lambda_{n+1}$  the further quasilocal contribution to  $H(x; g)$ :

$$- \int \Delta_{n+1}(x, x_1, \dots, x_n) g(x_1) \dots g(x_n) dx_1 \dots dx_n, \quad (39.24)$$

so that instead of (39.15) we have [Sukhanov (1961)]

$$H(x, \sigma) = - \lim_{g \rightarrow \theta_\sigma} \int dx^0 g'(T_\sigma - x^0) \frac{\delta}{\delta g(x)} \int [\mathcal{L}(y, g) + \Delta(y, g)] dy, \quad (39.25)$$

where

$$\Delta(x_1; g) = \sum_{n \geq 2} \frac{1}{n!} \int \Delta_n(x_1, \dots, x_n) g(x_1) \dots g(x_n) dx_2 \dots dx_n. \quad (39.26)$$

We note that the limiting transition  $g \rightarrow \theta_\sigma$  in the term involving  $\Delta(y, g)$  will also lead to surface divergences because both  $\Delta_n$  and  $\Lambda_n$  may contain derivatives.

In the case of the  $\lambda : \varphi^4 :$  theory, the limiting expression for the generalized Hamiltonian in the case of minimum regularization assumes the form

$$H(x, \sigma) = -\mathcal{L}(x, 1) \Big|_{x_0=T_\sigma} - \Delta(x, 1) \Big|_{x_0=T_\sigma} - C'(M) \left[ \frac{\partial}{\partial x_0} : \varphi^2(x) : \right]_{x_0=T_\sigma}. \quad (39.27)$$

The last term leads, as before, to the appearance in the matrix  $S(\sigma)$  of a (different) unitary factor localized on the surface  $\sigma$ . The second term, which distinguishes  $H(x, \sigma)$  from  $\mathcal{L}(x; 1)$ , is always present in theories involving derivatives. This distinction was mentioned at the end of §21 in connection with the Wick and Dyson forms of the  $S$ -matrix.

We emphasize that our entire discussion was confined to finite auxiliary masses. In the limit as  $M \rightarrow \infty$ , the renormalizing constants in the effective Lagrangian  $\mathcal{L}(x, g)$  are found to diverge and from this it follows that, in some cases, it is possible to write down directly the "running" Schroedinger equation (with instantaneous switching off), but then divergent terms automatically appear in the equation.

We thus see that the scattering matrix and the Schroedinger equation have the *complementarity property in the sense of the presence of infinities*, even after the segregation of the surface terms into unitary factors. Thus, in the usual theory, the Hamiltonian (i.e., the Schroedinger equation) is regular, but the  $S$ -matrix contains infinite terms. On the other hand, we have just shown that regularization of the scattering matrix is accompanied by the automatic appearance of divergences in the Schroedinger equation.

At this point, we introduce the following assumption about the physical origin of these divergences. In the case of instantaneous switching-off of the interaction, the influence of this process on the system is so large that it is practically impossible to determine any of its characteristics prior to the switching-off. From the formal point of view, there is here a certain analogy with the Heisenberg uncertainty relationships which, as is well known, are intimately connected with the most general properties of wave processes.

We thus arrive at the conclusion that, in general, a finite equation can be written down relatively simply for the amplitude  $\Phi(g)$  where  $g$  is a sufficiently smooth function, but one cannot go to the limit as  $g \rightarrow \theta_\sigma$  and obtain a meaningful amplitude  $\Phi(\sigma)$  when regularization is removed, i.e., for  $M \rightarrow \infty$ .

Thus, generally speaking, the Schroedinger and the Tomonaga-Schwinger equations have only a purely formal significance in quantum theory.

Thus, to avoid infinities, we must work with sufficiently smooth  $g(x)$ . Moreover, we note that to ensure the convergence of the integrals under consideration, we do not have to demand that  $g'(x)$  be zero everywhere outside the slab of thickness  $2\Delta t$ . It is enough to demand that  $g(x)$  should fall sufficiently rapidly, for example, exponentially, outside this slab, and that it should be sufficiently smooth everywhere. It is evident that the class  $G$  of such functions will be relativistically invariant under any Lorentz transformation  $L$ , i.e., if  $g(x) \in G$  then also  $g(Lx) \in G$ .

**39.4. Fundamental Properties of the Generalized Hamiltonian.** We shall now establish a number of fundamental properties of the generalized Hamiltonian  $H(x; g)$  introduced above. It is clear that the operator  $H(x; g)$  is Hermitian (as was already shown in §21.2), i.e.,

$$\dot{H}(x; g) = H(x; g). \quad (39.28)$$

The transformation properties of  $H(x; g)$  may be obtained from the condition of covariance of the matrix  $S(g)$ :



$$S(Lg) = U_L S(g) \dot{U}_L. \quad (20.21)$$

Recalling that  $Lg(x) = g(L^{-1}x)$ , we obtain from the above

$$H(x; Lg) = i \frac{\delta S(Lg)}{\delta g(L^{-1}x)} \dot{S}(Lg) = i U_L \frac{\delta S(g)}{\delta g(L^{-1}x)} \dot{U}_L U_L \dot{S}(g) U_L,$$

i.e.,

$$\left. \begin{aligned} H(x; Lg) &= U_L H(L^{-1}x; g) \dot{U}_L \\ H(x; g) &= U_L H(L^{-1}x; L^{-1}g) \dot{U}_L \end{aligned} \right\} \quad (39.29)$$

It is now readily seen that the covariance of (39.1) is a consequence of the transformation laws of the state amplitude (20.18) and of the Hamiltonian (39.29), as was to be expected. Indeed, by going over from  $g$  to  $L^{-1}g$  in (39.2) and by using (20.18) we obtain in turn

$$\begin{aligned} i\delta\Phi'(g) &= iU_L \delta\Phi(L^{-1}g) = U_L \int H(x; L^{-1}g) \delta g(Lx) dx \Phi(L^{-1}g)_{Lx \rightarrow x} = \\ &= U_L \int H(L^{-1}x; L^{-1}g) \delta g(x) dx \dot{U}_L U_L \Phi(L^{-1}g) \end{aligned}$$

or, after taking (39.17) into account,

$$i\delta\Phi'(g) = \int H(x; g) \delta g(x) dx \Phi'(g).$$

This completes the proof of covariance.

It is readily seen that the definition of the Hamiltonian (21.10), taking the unitary property of the matrix  $S(g)$  into account, automatically guarantees the compatibility of the equations in (39.1). For this it is sufficient to calculate the second variation of the state amplitude

$$\frac{\delta^2 \Phi}{\delta g(x) \delta g(y)}$$

and to show that its value does not depend on the order of variation for  $x \sim y$  (for greater details see §40.2).

We thus see that starting with the scattering matrix defined by the conditions of unitarity, causality, and covariance, one may uniquely obtain an equation of the Schroedinger type, which satisfies the conditions of covariance, compatibility, and localizability of the Hamiltonian. One can also show that the converse procedure is likewise possible.

As has already been pointed out, neither formalism (i.e., neither the Schroedinger equation nor the matrix  $S(g)$ ) contains the usual divergences, provided  $g$  belongs to the class  $G$ . Therefore, the question arises of the formulation of the whole theory for the case where the interaction is switched on with an intensity  $g(x) \in G$ . In other words, we must be able to

define the principal physical quantities with the aid of the state amplitude  $\Phi(g)$  in a manner similar to the way in which they are defined in the usual theory by means of the amplitude  $\Phi_\tau$  or  $\Phi_\sigma$ . Since the particular form of the function  $g$  has no physical significance, it is necessary in the construction of the theory to guarantee that observable values of physical quantities should not depend on the specific choice of  $g(x)$ .

#### §40. Dynamic Variables of a System of Interacting Fields

Let us now consider the problem of constructing the dynamic variables of a system of interacting fields such as the energy-momentum four-vector, the angular-momentum tensor, and the current four-vector. In the theory of noninteracting fields, the corresponding expressions are obtained directly with the aid of Noether's theorem, starting from the principle of extremal action. In the case of interacting fields, we cannot proceed in this way, since we do not have at our disposal exact solutions of the equations for the interacting fields, which make the total action of the system as a whole an extremum. We have to employ the operator fields functions for the fields and to construct the various physical quantities from them.

*40.1. Energy, Momentum, and the Angular-Momentum Tensor.\** In order to construct the energy-momentum four-vector and the angular-momentum tensor, we consider an infinitesimal unitary transformation of the state amplitude under the infinitesimal Poincaré transformations

$$x \rightarrow x' = Lx = x + \delta x; \quad \delta x^k = a^k + \omega^{kn} x_n. \quad (9.16)$$

According to (20.18), we then have

$$\Phi(g) \rightarrow \Phi'(g) = U_L \Phi(L^{-1}g). \quad (40.1)$$

We now evaluate the infinitesimal variation

$$\delta_L \Phi(g) = \Phi'(g) - \Phi(g).$$

Using (9.17) and (20.19), we have

$$\Phi'(g) = U_L S(L^{-1}g) \Phi = U_L S(L^{-1}g) \dot{S}(g) \Phi(g).$$

Thus, the variation  $\delta_L \Phi$  is due to both the unitary transformation  $U_L$  and the transformation of the interaction region

$$\delta_L \Phi(g) = [U_L S(L^{-1}g) \dot{S}(g) - 1] \Phi(g). \quad (40.2)$$

We now recall that in accordance with (9.17) and (9.18),

\*This subsection is based on the work of Bogolyubov (14).

$$U_L = 1 + \delta U_L, \quad \delta U_L = iP_0^k a_k + \frac{i}{2} M_0^{kl} \omega_{kl},$$

where  $P_0$  and  $M_0$  have the form of energy-momentum and angular-momentum operators in the absence of interaction, and also

$$S(L^{-1}g) = S(g + \delta g) = S(g) + \int \frac{\delta S(g)}{\delta g(x)} \delta g(x) dx = \left(1 - i \int H(x; g) \delta g(x) dx\right) S(g),$$

where, by virtue of (20.17), we have

$$\begin{aligned} L^{-1}g(x) &= g(Lx) = g(x) + \delta g(x), \\ \delta g(x) &= \frac{\partial g}{\partial x^k} a^k + \frac{1}{2} \left( x^k \frac{\partial g}{\partial x^l} - x^l \frac{\partial g}{\partial x^k} \right) \omega_{kl}. \end{aligned}$$

Substituting this in (40.2), we obtain

$$\begin{aligned} \delta_L \Phi(g) &= [\delta U_L - i \int H(x; g) \delta g(x) dx] \Phi(g) = \\ &= i \left\{ P^k(g) a_k + \frac{1}{2} M^{kl}(g) \omega_{kl} \right\} \Phi(g), \end{aligned} \quad (40.3)$$

where

$$P^k(g) = P_0^k - \int H(x; g) \frac{\partial g(x)}{\partial x^k} dx \quad (40.4)$$

and

$$M^{kl}(g) = M_0^{kl} - \int H(x; g) \left( x^k \frac{\partial g(x)}{\partial x^l} - x^l \frac{\partial g(x)}{\partial x^k} \right) dx. \quad (40.5)$$

We have thus obtained the desired expressions for the components of the energy-momentum four-vector and of the angular-momentum tensor for the dynamic system in the presence of interaction. It is evident from the form of these expressions that these quantities have the correct transformation properties.

By taking

$$g(x) = f(\tau - x \cdot \xi),$$

where  $\tau$  is the surface parameter while  $\xi$  is a certain four-vector which characterizes the geometry of the surface we obtain

$$\frac{\partial g}{\partial x^k} = -f'(\tau - x \cdot \xi) \frac{\partial (x \cdot \xi)}{\partial x^k} = -\xi_k f'(\tau - x \cdot \xi),$$

so that

$$P^k(\tau) = P_0^k + \xi^k \int f'(\tau - \xi x) H(x; f) dx. \quad (40.6)$$

In particular, if one chooses for  $\xi$  the unit vector along the time axis

$$\xi^k = \delta_{k0},$$

one obtains

$$P^\alpha(\tau) = P_0^\alpha \quad (\alpha = 1, 2, 3), \quad (40.7)$$

$$P^0(\tau) = P_0^0 + \int f'(\tau - x^0) H(x; f) dx. \quad (40.8)$$

It may be seen that, in this case, the expression for the total momentum will be the same as in the absence of interaction, while the expression for the total energy is given by the sum of the "self-energy" of the particles  $P_0^0$  and the interaction energy

$$\int f'(\tau - x^0) H(x; f) dx.$$

We have here a complete analogy with the usual formulation of quantum mechanics, and it is clear that to a degree of accuracy which depends on the extent of "smearing-out" of the function  $f$ , we may regard  $H(x; f)$  as the interaction energy density.

We shall now show that the expectation values

$$\Phi^*(g) P^i(g) \Phi(g), \quad \Phi^*(g) M^{ik}(g) \Phi(g)$$

do not depend on the form of  $g$  when  $g$  is varied within the class of allowable functions under consideration.

To prove this, we start from the property established earlier (see §39.4) that the transformed state amplitude

$$\Phi'(g) = U_L \Phi(L^{-1}g)$$

also satisfies the Schroedinger equation in integral form. We therefore have

$$i\delta \{U_L \Phi(L^{-1}g) - \Phi(g)\} = \int H(x; g) \delta g(x) dx \{U_L \Phi(L^{-1}g) - \Phi(g)\}.$$

This relation holds for any Lorentz transformation, and, in particular, for an infinitesimal one. But, as we have already shown, for an infinitesimal Lorentz transformation

$$U_L \Phi(L^{-1}g) - \Phi(g) = J(g) \Phi(g),$$

where

$$J(g) = i(P(g) \cdot a) + \frac{i}{2} M_{kl}(g) \omega^{kl}.$$

We therefore have

$$i\delta[J(g)\Phi(g)] = \int H(x; g) \delta g(x) dx J(g)\Phi(g),$$

from which on taking into account the fact that

$$i\delta^*\Phi(g) = -\Phi^*(g) \int H(x; g) \delta g(x) dx,$$

we obtain

$$i\delta[\Phi^*(g)J(g)\Phi(g)] = i\delta^*\Phi(g)J(g)\Phi(g) + i\Phi^*(g)\delta[J(g)\Phi(g)] = 0,$$

i.e.,

$$\delta(\Phi^*(g)J(g)\Phi(g)) = 0.$$

Recalling that  $J(g)$  is a linear combination of  $P(g)$  and  $M(g)$  with arbitrary coefficients  $a, \omega$ , we see that the following relations hold:

$$\delta(\Phi^*(g)P^i(g)\Phi(g)) = 0 \quad \text{and} \quad \delta(\Phi^*(g)M^{ik}(g)\Phi(g)) = 0, \quad (40.9)$$

which is what we wanted to prove.

We shall now show that the eigenvalues of the operators  $P^i(g)$  and  $M^{ik}(g)$  also do not change as  $g$  is varied within the class  $G$ . We take, for example, the energy operator. We assume that for a certain acceptable  $g = g_0$ , the eigenfunction of this operator is  $\Phi_E$  while the corresponding eigenvalue is  $E$ . Then

$$(P^0(g_0) - E)\Phi_E = 0.$$

Let  $\Phi(g)$  be the state amplitude satisfying the fundamental equation which goes over into  $\Phi_E$  for  $g = g_0$ , i.e.,

$$(P^0(g) - E)\Phi(g) = 0 \quad \text{for} \quad g = g_0. \quad (40.10)$$

On the other hand, as we have already shown, if  $\Phi(g)$  satisfies the Schroedinger equation (39.1), then  $J(g)\Phi(g)$  also satisfies the same equation. But since  $J(g)$  is a linear combination of the operators  $P(g)$  and  $M(g)$  with arbitrary coefficients, then each of them separately will have this property. Therefore,  $P^0(g)\Phi(g)$  and, consequently, also  $\Psi(g) \equiv (P^0(g) - E)\Phi(g)$ , satisfy (39.1). From this we conclude that, since  $\Psi(g)$  vanishes for  $g = g_0$  in accordance with (40.10), then  $\Psi(g)$  remains equal to zero everywhere within  $G$ . Thus, everywhere within  $G$ ,

$$(P^0(g) - E) \Phi(g) = 0,$$

which proves the invariance of the eigenvalues. We also see that it is always possible to choose for the eigenfunction the state amplitude satisfying the fundamental equation.

The above invariance properties of expectation values and eigenvalues of the operators under consideration in our theory represent, at the same time, the laws of conservation, covariance, and the independence of the foregoing physical quantities of the extent of "smearing-out" of the allowable space-like surfaces. Thus, invariance under a change in  $g$ , determined by a translation in time, evidently corresponds to the usual conservation laws. Invariance under changes in  $g$  brought about by Lorentz rotations guarantees actual covariance. Finally, invariance under changes in  $g$  due to changes in the process of smoothing out the function  $g$  expresses the independence of the physical quantities under consideration of the character of the process of switching off the interaction.

We note in this connection that, in general, if the relation

$$\delta_g (\hat{\Phi}^*(g) O(g) \Phi(g)) = 0,$$

holds for some  $O(g)$  then, in our representation, this means that the quantity  $O(g)$  is an integral of the motion.

**40.2. Local Dynamic Variables.\*** In the preceding subsection we introduced the energy-momentum four-vector and the angular-momentum tensor that characterize the system of quantized fields as a whole. However, certain other more detailed characteristics of the system of interacting fields, for example, the current density, which depend not only on the region in which the interaction  $g$  is switched on, but also on the point  $x$ , are also of interest. The Hamiltonian density  $H(x; g)$  considered before is also a quantity of a similar local type.

We now turn to the general properties of local dynamic variables of the aforementioned type,  $B(x; g)$ . We first formulate the physical requirements that they must satisfy by writing  $B(x; g)$  in the form of a functional expansion in "powers" of  $g$ :

$$B(x; g) = B(x) + \sum_{n \geq 1} \frac{1}{n!} \int B_n(x, x_1, \dots, x_n) g(x_1) \dots g(x_n) dx_1 \dots dx_n. \quad (40.11)$$

We begin with the property of locality. The property of locality of the dynamic variable  $B(x; g)$  corresponds to the fact that, in the limit as the function  $g(x)$  approaches the step function  $\sigma(x)$  (see §39),  $B(x; g)$  turns out to depend on the behavior of the field only in an infinitesimal neighborhood of the point  $x$ . It is readily shown that the corresponding condition imposed on the expansion coefficients  $B_n$  has the form

$$\left. \begin{aligned} B_n(x, x_1, \dots, x_n) &= 0, \\ \text{if for at least one of the } x_j (j = 1, \dots, n) \\ x &\not\gtrsim x_j. \end{aligned} \right\} \quad (40.12)$$

\*The material of this subsection is based on the work of Stepanov (1955).

Indeed, in §39.2, the locality of the Hamiltonian  $H(x; g)$  was obtained directly from condition (40.12) in the form (21.13). Condition (40.12) may, evidently, also be written in the form

$$\frac{\delta B(x; g)}{\delta g(y)} = 0 \quad \text{for} \quad x \gtrless y. \quad (40.13)$$

An essential requirement is the condition that, in the limit  $g \rightarrow 0$  of the interaction being switched off,  $B(x; g)$  should coincide with the corresponding expression  $B(x)$  taken from the theory of free fields

$$B(x; 0) = B(x). \quad (40.14)$$

Finally, an important requirement is the condition that the observable value

$$\Phi^*(g) B(x; g) \Phi(g) \quad (40.15)$$

should be independent of the behavior of the function  $g$  at times later than  $x$ :

$$\frac{\delta (\Phi^*(g) B(x; g) \Phi(g))}{\delta g(y)} = 0 \quad \text{for} \quad y \gtrless x. \quad (40.16)$$

This condition is a consequence of the principle of causality and expresses the fact that the result of a measurement made at the time  $x$  and described by the quantity (40.15) cannot depend on the behavior of the system at later times.

By making use of the Schroedinger equation (39.1), we find from the foregoing the equation that

$$i \frac{\delta B(x; g)}{\delta g(y)} = [H(y; g), B(x; g)] \quad \text{for} \quad y \gtrless x, \quad (40.17)$$

which in the special case  $B(x; g) = H(x; g)$  gives

$$i \frac{\delta H(x; g)}{\delta g(y)} - i \frac{\delta H(y; g)}{\delta g(x)} = [H(y; g), H(x; g)], \quad (40.18)$$

which is the condition of compatibility for the Schroedinger equation (39.1).

We note that the definition of local dynamic variables which we gave is a natural generalization of the corresponding concepts in the local theory of Schwinger (1948). We shall investigate the behavior of the quadratic forms  $(\Phi^*(g)B(x; g)\Phi(g))$  as the function  $g$  approaches the step function  $\sigma$  passing through the point  $x$ . If this limiting transition turned out to be possible, we would have obtained the expression

$$\Phi^*(\sigma) B_\sigma(x) \Phi(\sigma),$$

which was considered by Schwinger. Schwinger shows that this form does not depend on the form of the surface  $\sigma$  passing through the point  $x$ . In our case, the corresponding condition has the more general form of (40.16).

We also note that, by virtue of (40.17), the following relation holds, which is more general than (40.16):

$$\frac{\delta(\Phi_1^*(g) B(x; g) \Phi_2(g))}{\delta g(y)} = 0 \quad (y \gtrsim x), \quad (40.19)$$

where  $\Phi_1$  and  $\Phi_2$  are two different solutions of Schroedinger's equation (39.1).

By starting with relations (40.11), (40.12), (40.14), and (40.16), we may determine all the coefficients  $B_n$  by an appropriate modification of the argument used to determine the expansion coefficients of the scattering matrix  $S_n$ . However, since this procedure has already been employed in the case of the operator  $H(x; g)$  of the type under consideration, we shall proceed differently, and shall reduce the present problem to the preceding one. We introduce into the interaction Lagrangian the additional term

$$\mathcal{L}(x) \rightarrow \mathcal{L}(x) - B(x) b(x), \quad (40.20)$$

where  $B(x)$  is the free-field operator which corresponds to  $B(x; g)$ , while  $b(x)$  is a certain classical function of the same tensor dimensionality as  $B(x)$ . The matrix  $S(g)$ , like the Hamiltonian  $H(x; g)$ , becomes a functional which depends on  $b$ :

$$S(g) \rightarrow S(g; b), \quad H(x; g) \rightarrow H(x; g; b) = i \frac{\delta S(g, b)}{\delta g(x)} \dot{S}(g; b).$$

We shall now show that the operator  $B(x; g)$  may be represented in the form

$$\begin{aligned} B(x; g) &= \left. \frac{\partial H(x; g; b)}{\partial b(x)} \right|_{b=0} \equiv i \left. \frac{\delta S(g; b)}{\delta [b(x) g(x)]} \dot{S}(g; b) \right|_{b=0} = \\ &= T(B(x) S(g)) \dot{S}(g). \end{aligned} \quad (40.21)$$

Indeed, by carrying out the functional differentiation of (40.21), and by taking into account the fact that the operations of ordinary and of functional differentiation commute, we obtain

$$\frac{\delta B(x; g)}{\delta g(y)} = \frac{\partial}{\partial b(x)} \left[ i \frac{\delta}{\delta g(y)} \left( \frac{\delta S(g, b)}{\delta g(x)} \dot{S}(g, b) \right) \right]_{b=0},$$

which, on taking into account the condition of causality for the matrix  $S(g, b)$ , yields:

$$\frac{\delta B(x; g)}{\delta g(y)} = 0 \quad \text{for} \quad x \gtrsim y.$$



From this it follows that the property of locality of the operator  $B(x, g)$  is a direct consequence of the condition of causality for the scattering matrix. Further, recalling that

$$H(x; 0) = H_0(x) = -\mathcal{L}(x),$$

we find, by differentiating with respect to  $b$ , that

$$B(x; 0) = B(x).$$

Finally, by differentiating (40.18) with respect to  $b$  for  $b \neq 0$ , and by setting  $b = 0$ , we obtain, on taking (40.20) into account, the condition of causality (40.16).

We also note that, if  $B(x)$  (and consequently also  $b(x)$ ) in (40.20) are chosen to be Hermitian, then  $B(x; g)$  in (40.20) will also be Hermitian, since  $H(x; g)$  is Hermitian. Moreover, it is evident that  $B(x; g)$  is covariant.

*40.3. The Current Vector.* As an example, consider the expression for the current density in electrodynamics. By noting that the operator  $j(x)$  always appears in the interaction Lagrangian in the combination

$$j(x) A(x),$$

where  $A(x)$  is the potential of the electromagnetic field, we see that the auxiliary classical field with respect to which  $H(x; g)$  must be differentiated in order to obtain  $j(x; g)$  is simply an addition to  $A(x)$ , and that, therefore, one may in general set

$$j^k(x; g) = -\frac{\partial H(x; g)}{\partial A_k(x)} = T(j^k(x) S(g)) \overset{\dagger}{S}(g). \quad (40.22)$$

As has just been shown, the quantity given by (40.22) will satisfy the conditions of locality and of causality, and when the interaction is switched off, it will reduce to the current operator for the free field. We shall now show that it will also satisfy the equation of continuity

$$\operatorname{div}_x {}^{(4)}j(x; g) = 0. \quad (40.23)$$

By expressing (40.23) in terms of the coefficient functions  $H_n$ , we obtain

$$\operatorname{div}_x {}^{(4)}\frac{\partial H_n(x, x_1, \dots, x_n)}{\partial A(x)} = 0. \quad (40.24)$$

Expressing  $H_n$  with the aid of (21.11) as a quadratic form in  $S_k$  in which the  $S_k$  depending on  $x$  appear linearly, we find that gauge invariance of the  $S$ -matrix in the form (33.12)

$$\operatorname{div}_{x_i}^{(4)} \frac{\partial S_n(x_1, \dots, x_n)}{\partial A(x_i)} = 0$$

guarantees that (40.24) is satisfied, and that, consequently, it also guarantees that the equation of continuity for the current, defined by expression (40.22), is also satisfied.

We now turn to the problem of constructing an expression for the total charge of the system. In the usual theory, this expression has the form

$$Q = \int d\mathbf{x} \cdot \mathbf{j}^0(x), \quad (2.29)$$

where  $j^0$  is the zero component of the current density. However, (2.29) is evidently non-invariant under Lorentz transformations. By subjecting (2.29) to a transformation opposite to that in (2.8), we obtain the Lorentz expression

$$Q = \int d\sigma_k j^k(x), \quad (40.25)$$

where  $d\sigma_k$  is an element of the three-dimensional surface orthogonal to the  $x^k$  axis, while the integration is taken over the whole three-dimensional surface  $\sigma$ . This expression may evidently be rewritten in the following form:

$$\int d\sigma_k j^k = \int d\Sigma \cdot \cos(n \cdot x^k) \cdot j^k = \int \frac{\partial \theta_\sigma(x)}{\partial x^k} j^k(x) dx,$$

where  $d\Sigma$  is an element of area of the surface  $\sigma$ ,  $\cos(n \cdot x^k)$  is the cosine of the angle between the  $x^k$  axis and the normal  $n$  to the surface  $\sigma$  at the point  $x$ , and  $\theta_\sigma(x)$  is the step function introduced in §39

$$\theta_\sigma(x) = \begin{cases} 1 & \text{for } x^0 > T_\sigma(x) \\ 0 & \text{for } x^0 < T_\sigma(x) \end{cases} \quad (\theta_\sigma(x) = \theta(x^0 - T_\sigma(x))). \quad (39.9)$$

The expression for the charge now assumes the form

$$Q = \int dx j^k(x) \frac{\partial \theta_\sigma(x)}{\partial x^k}. \quad (40.26)$$

In our theory, the natural generalization of this relation is

$$Q(g) = - \int dx j^k(x; g) \frac{\partial g(x)}{\partial x^k}, \quad (40.27)$$

which formally reduces to it in the course of the limiting transition  $g(x) \rightarrow \theta_\sigma(-x)$ . We shall now show that (40.27) is an integral of motion, i.e., that the observable value of  $Q$  does not depend on the form of the function  $g(x)$ , and that its variation

$$\delta_g \langle Q(g) \rangle = \delta_g \left( \Phi^*(g) Q(g) \Phi(g) \right) \quad (40.28)$$

vanishes for an arbitrary variation  $\delta g(x)$ .

In order to prove (40.28), we start with (40.18) differentiated with respect to  $A(x)$ :

$$-i \left( \frac{\delta j(x; g)}{\delta g(y)} + \frac{\partial}{\partial A(x)} \frac{\delta H(y; g)}{\delta g(x)} \right) = -[H(y; g), j(x; g)],$$

which may be written in the form

$$i \frac{\delta j(x; g)}{\delta g(y)} - [H(y; g), j(x; g)] = -i \frac{\partial}{\partial A(x)} \frac{\delta H(y; g)}{\delta g(x)}. \quad (40.29)$$

From (40.23) it also follows in virtue of (40.29) that

$$\text{div}_x^* \frac{\partial}{\partial A(x)} \frac{\delta H(y; g)}{\delta g(x)} = 0. \quad (40.30)$$

By evaluating the variation (40.28) and taking into account Schroedinger's equation and (40.27) and (40.29), we obtain

$$\begin{aligned} \delta_{g(y)} \langle Q(g) \rangle &= \Phi^*(g) \left\{ \frac{\delta Q(g)}{\delta g(y)} + i[H(y; g), Q(g)] \right\} \Phi(g) \delta g(y) = \\ &= - \int dx \Phi^*(g) \left[ \left\{ \frac{\delta j^k(x; g)}{\delta g(y)} + i[H(y; g), j^k(x; g)] \right\} \frac{\partial g(x)}{\partial x^k} + \right. \\ &\quad \left. + j^k(x; g) \frac{\partial}{\partial x^k} \frac{\delta g(x)}{\delta g(y)} \right] \Phi(g) \delta g(y) = \\ &= - \int dx \Phi^*(g) \left[ \frac{\partial}{\partial A_k(x)} \left( \frac{\delta H(y; g)}{\delta g(x)} \right) \frac{\partial g(x)}{\partial x^k} + j^k(x; g) \frac{\partial \delta(x-y)}{\partial x^k} \right] \Phi(g) \delta g(y). \end{aligned}$$

Integrating the second term by parts, we find that

$$- \int dx j^k(x; g) \frac{\partial \delta(x-y)}{\partial x^k} = \frac{\partial j^k(y; g)}{\partial y^k} = 0$$

by virtue of (40.23), and also

$$\int dx \frac{\partial}{\partial A_k(x)} \frac{\delta H(y; g)}{\delta g(x)} \frac{\partial g(x)}{\partial x^k} = 0$$

by virtue of (40.30) and the fact that  $g(x)$  vanishes for positive infinite time and  $\delta H(y; g)/\delta g(x)$  vanishes for negative time and space infinities because of the localization condition (40.13).

**40.4. The Lorentz Condition.** In conclusion, we must also mention the problem of the subsidiary condition on the allowable states of the system of interacting fields. It is well

known in the theory of free fields that this condition has the form

$$\left( \frac{\partial A^-(x)}{\partial x} \right) \Phi \equiv \frac{\partial A_k^-(x)}{\partial x_k} \Phi = 0. \quad (12.8)$$

For a system with interaction, this condition must take the form

$$\Omega^-(x; g) \Phi(g) = 0, \quad (40.31)$$

where the operator  $\Omega^-(x; g)$  must, on the one hand, reduce to  $\partial A^-(x)/\partial x$  for  $g \rightarrow 0$ , and on the other hand, must be an integral of motion, since (40.31) is an equation with time-independent zero eigenvalue of the operator  $\Omega^-$ . The above requirement on the operator  $\Omega^-$  is a condition of compatibility of equation (40.31) with the Schroedinger equation of motion.

We shall omit the corresponding calculations and shall merely state that the previous two conditions completely determine the operator  $\Omega^-$  in the form

$$\Omega^-(x; g) = \left( \frac{\partial A^-(x)}{\partial x} \right) - \int D_0^-(x-y) j^k(y; g) \frac{\partial g(y)}{\partial y^k} dy$$

and that the condition (40.31) assumes the form

$$\left\{ \left( \frac{\partial A^-(x)}{\partial x} \right) - \int D_0^-(x-y) j^k(y; g) \frac{\partial g(y)}{\partial y^k} dy \right\} \Phi(g) = 0. \quad (40.32)$$

Here  $D_0^-(x)$  is the negative-frequency part of the Pauli-Jordan photon function

$$D_0^-(x) = \frac{i}{8\pi^3} \int dk \delta(k^2) \theta(-k^0) e^{ikx}.$$

**40.5. Wave Field Operators.** As in the definition of the operator  $j(x; g)$ , we may also introduce the operators

$$A(x; g), \quad \psi(x; g), \quad \bar{\psi}(x; g),$$

which are generalizations of the corresponding local quantities in the theory of free fields. For this purpose, by following the general method of constructing local quantities in interaction theory, we introduce the following additional terms into the interaction Lagrangian of spinor electrodynamics  $\mathcal{L}$ :

$$\bar{\eta}(x) \psi(x) + \bar{\psi}(x) \eta(x) + J(x) A(x). \quad (40.33)$$

Here  $J(x)$  is an unquantized function that may be identified with an external current, and  $\bar{\eta}(x)$  and  $\eta(x)$  are unquantized spinor functions that anticommute with each other and

with the fields  $\bar{\psi}$ ,  $\psi$ . These “classical spinor fields” were introduced in §37.2 as the generators of a Grassman algebra.

We now introduce the expressions

$$A(x; g) = -\frac{\partial H(x; g)}{\partial J(x)} = -\frac{i}{g(x)} \left( \frac{\delta S}{\delta J(x)} \dot{S} \right), \quad (40.34)$$

$$\psi(x; g) = -\frac{\partial H(x; g)}{\partial \eta(x)} = -\frac{i}{g(x)} \left( \frac{\delta S}{\delta \eta(x)} \dot{S} \right), \quad (40.35)$$

$$\bar{\psi}(x; g) = -\frac{\partial H(x; g)}{\partial \bar{\eta}(x)} = -\frac{i}{g(x)} \left( \frac{\delta S}{\delta \bar{\eta}(x)} \dot{S} \right) \quad (40.36)$$

We note that, in contrast to  $\eta$ ,  $\bar{\eta}$ , the auxiliary “field”  $J(x)$  may have a physical meaning. Suppose, for example, that we wish to investigate the influence on the quantum system under consideration of some external sources of electric field that are characterized by an unquantized external current-density vector  $J_0(x)$ . It is then evident that, in place of (40.34), one should take

$$A(x; g) = -\frac{i}{g(x)} \left( \frac{\delta S(g)}{\delta J(x)} \dot{S} \right) \Big|_{\eta, \bar{\eta}=0, J=J_0}. \quad (40.37)$$

The operators  $A(x; g)$ ,  $\psi(x; g)$  and  $\bar{\psi}(x; g)$  introduced in this manner may be regarded as generalizations of the free-field operators to the interaction of intensity  $g$ . These operators will be used in §§41 and 42 to investigate the anomalous magnetic moment of the electron and the atomic level shifts.

## §41. Vacuum Polarization and the Anomalous Magnetic Moment of the Electron

*41.1. Vacuum Polarization.* We now turn to an examination of some applications of the general formal theory presented above. We shall first consider the question of the change in the state of the vacuum under the action of an external unquantized current  $J(x)$  or, the equivalent of this, under the action of the given external potential  $A^{\text{ext}}(x)$  related to the current  $J$  by

$$\square A^{\text{ext}}(x) = -J(x). \quad (41.1)$$

It might seem that this problem is only of purely theoretical interest, since changes in the properties of the vacuum are not directly observable. However, the point is that the change in these properties under the action of an external electromagnetic field is a component of a number of observable effects (shift of the levels of atomic electrons, scattering of light by light, and so on).

It is convenient to investigate the average observable value of the operator for the electromagnetic potential. In accordance with the previously given general prescription (§40.5), we introduce into the Lagrangian the additional term

$$J(x) A(x). \quad (41.2)$$

Then, on the basis of (40.37), the average observable value of the electromagnetic potential is given by

$$A^{\text{eff}}(x) = -\Phi_0^*(g; J) \frac{i}{g(x)} \frac{\delta S(g; J)}{\delta J(x)} \overset{\dagger}{S}(g, J) \Phi_0(g; J). \quad (41.3)$$

Here  $\Phi_0(g, J)$  is the state amplitude for the real-particle vacuum altered by the action of the external current  $J(x)$ , while the function  $g(y)$  is equal to unity for all  $y \lesssim x$ . In accordance with the general theory,  $A^{\text{eff}}$  does not depend on the particular choice of  $g$ . A physical meaning may evidently be ascribed only to the case where this external current may be considered to be time independent, since in the opposite case it would give rise to real processes of mutual transformation of particles. We shall therefore consider the case when

$$J(x) = J(x). \quad (41.4)$$

Then  $\Phi_0(g, J)$  may be defined as the lowest energy state of the dynamic system in the presence of  $J(x)$ .

We note first of all that

$$\overset{\dagger}{S}(g; J) \Phi_0(g; J) = \Phi_0, \quad (41.5)$$

where  $\Phi_0$  is the vacuum state amplitude for free fields. By making use of the fact that (41.3) is independent of the particular form of the function  $g$ , we make the usual transition to the limit  $g(x) \rightarrow 1$  and obtain

$$A^{\text{eff}}(x) = -i\Phi_0^* \overset{\dagger}{S} \frac{\delta S}{\delta J(x)} \Phi_0, \quad (41.6)$$

where the quantity

$$S = S(1; J)$$

represents the scattering matrix when the interaction is equal to zero both in the remote past and in the remote future.

Since real processes of creation of particles cannot occur in the case of such an adiabatic switching-on and switching-off of the interaction, we also have

$$S\Phi_0 = a\Phi_0, \quad (41.7)$$

where evidently

$$a = \langle S \rangle_0 = S_0, \quad (41.8)$$

with

$$a a^* = 1.$$

Therefore

$$\Phi_0^* S_0^* = \Phi_0^* a^* = \frac{\Phi_0^*}{S_0^*}. \quad (41.9)$$

Substituting (41.9) into (41.6), we obtain

$$A^{\text{eff}}(x) = -i \frac{\Phi_0^* \frac{\delta S}{\delta J(x)} \Phi_0}{\Phi_0^* S_0^*} = -\frac{i}{S_0^*} \langle \frac{\delta S}{\delta J(x)} \rangle_0. \quad (41.10)$$

We shall further find it convenient to express the right-hand side of expression (40.10) in terms of the complete photon Green's function. In order to do this, consider the expression

$$\frac{\delta}{\delta J(y)} (S_0 \cdot A^{\text{eff}}(x)).$$

With the aid of (40.10), we obtain

$$\frac{\delta}{\delta J^m(y)} [S_0 \cdot A_n^{\text{eff}}(x)] = \frac{1}{i} \langle \frac{\delta^2 S}{\delta J^m(y) \delta J^n(x)} \rangle_0. \quad (41.11)$$

We note the meaning of the operation of iterated variation with respect to the external current in the right-hand side of (40.11). The matrix  $S(1; J)$  may be represented by a set of diagrams which, in addition to the ordinary vertices, contain vertices corresponding to the term  $J(x)A(x)$  in the Lagrangian from each of which one photon line emerges (we have already encountered vertices of this type in the case of bremsstrahlung in §26). As a result of variation with respect to  $J$ , the corresponding photon line acquires a free end, and the operator  $A(x)$  appears within the  $T$ -product. Therefore, by explicitly carrying out the operation of iterated variation on the right-hand side of (41.11), we find that the quantity

$$\bar{D}_{nm}(x, y; J) = i \langle \frac{\delta^2 S}{\delta J^m(y) \delta J^n(x)} \rangle_0 = \frac{1}{i} \langle T(A_m(y) A_n(x) S) \rangle_0 \quad (41.12)$$

represents the sum of coefficient functions corresponding to the internal lines of all possible diagrams with two free-photon line ends. Thus, to within the factor  $S_0^{-1}$ , the preceding

expression for  $\bar{D}$  represents Green's function  $D(x, y; J)$  for a photon moving in the external field of the current  $J$ . We can now write

$$\frac{\delta}{\delta J^m(y)} (S_0 \cdot A_n^{\text{eff}}(x)) = -S_0 D_{nm}(x, y; J). \quad (41.13)$$

This formula allows us to write the *functional expansion of  $A^{\text{eff}}$  "in powers" of  $J$* , starting with the corresponding expansions for  $S_0$  and  $D(x, y; J)$ . With later applications in mind, we shall now perform this expansion explicitly, treating the external current  $J$  as a small quantity and retaining only the leading terms in (41.13).

By letting  $J \rightarrow 0$  in (41.13), we obtain

$$S_0 \frac{\delta A_n^{\text{eff}}(x)}{\delta J^m(y)} = -S_0 D_{mn}(x, y). \quad (41.14)$$

Here we have made use of the fact that as  $J \rightarrow 0$ , the expression

$$\frac{\delta S_0}{\delta J^m(y)} = i \langle A_m(y) S \rangle_0$$

corresponds to diagrams with one external photon line and is equal to zero. We also note that the expression

$$D_{nm}(x, y) = D_{nm}(x, y; J)|_{J=0}$$

represents the complete photon Green's function in the absence of external currents.

Cancelling out the factor  $S_0$  in (41.14), and integrating over  $\delta J$ , we obtain:

$$A_n^{\text{eff}}(x) = - \int D_{nm}(x, y) J^m(y) dy. \quad (41.15)$$

By going over to the momentum representation

$$\left. \begin{aligned} A_n^{\text{eff}}(x) &= \frac{1}{(2\pi)^4} \int e^{ikx} A_n^{\text{eff}}(k) dk, \\ J^m(y) &= \frac{1}{(2\pi)^4} \int e^{iqy} J^m(q) dq, \\ D_{mn}(x, y) &= - \frac{1}{(2\pi)^4} \int e^{ik(x-y)} \left\{ \frac{d(k^2)}{k^2} \left( g^{nm} - \frac{k^m k^n}{k^2} \right) + \frac{d_l}{k^2} \frac{k_n k_m}{k^2} \right\} dk \end{aligned} \right\} \quad (41.16)$$

and by taking into account the equation of continuity for the external current

$$k_m J^m(k) = 0,$$

we obtain



$$A_n^{\text{eff}}(k) = g^{nm} \frac{d(k^2)}{k^2} J^m(k) = \frac{d(k^2)}{k^2} J_n(k). \quad (41.17)$$

In the particular case (41.4) under consideration, when  $J$  does not depend on the time, we may set

$$J_n(k) = 2\pi\delta(k^0) J_n(k^2). \quad (41.18)$$

Substituting this expression into (41.16) and (41.15), we obtain

$$A_n^{\text{eff}}(x) = \frac{1}{(2\pi)^3} \int e^{-ikx} \frac{d(-k^2)}{-k^2} J_n(k^2) dk. \quad (41.19)$$

It follows from (41.19) that the “space part”  $d(-k^2)/(-k^2)$  of the photon Green’s function represents the effective potential due to a unit charge as a result of its interaction with the electron-positron and photon vacuum (for this it is sufficient to set  $J_m(y) = \delta_{m0}\delta(y)e$ , i.e.,  $J(k^2) = e$ ). By utilizing the fact that for a free field, when  $d(-k^2) = 1$ , the potential (41.19) reduces to the Coulomb potential, we shall in the general case represent it in the form

$$V(r) = \frac{e}{(2\pi)^3} \int e^{-ikr} \frac{d(-k^2)}{-k^2} dk = -e \frac{\varrho(r)}{4\pi r} \quad (r = |\mathbf{r}|). \quad (41.20)$$

The deviation from unity of the function  $\rho(r)$  introduced above corresponds to the deviation of  $V(r)$  from the Coulomb potential (the function  $d$  plays an analogous role in the momentum representation). This deviation of the effective potential from the Coulomb potential is a consequence of vacuum polarization and may be visualized as the effect of the *screening* of the charge introduced into the vacuum by the creation in the vacuum of virtual electron-positron pairs. The function  $\rho$  describes the relative degree of screening. In order to obtain its specific form, we must turn to the corresponding part of the photon propagation function.

At present, the theory does not provide a complete expression for the photon propagation function. However, for our purposes, we may use the expansion of  $d$  into a series in powers of  $e^2$  whose first term was obtained in §34.

We first consider the case of large  $r$ . By introducing the new variable  $|\mathbf{k}|r = x$  in the integral (41.20) and by carrying out the integration over the angles, we bring (41.20) into the form

$$\frac{\varrho(r)}{r} = \frac{1}{\pi r} \int_0^\infty dx d\left(-\frac{x^2}{r^2}\right) \int_{-1}^{+1} d\mu e^{-ix\mu} = \frac{2}{\pi r} \int_0^\infty dx \frac{\sin x}{x} d\left(-\frac{x^2}{r^2}\right). \quad (41.21)$$

From this it follows that in order to investigate the asymptotic behavior at large  $r$ , it is necessary to know the behavior of the function  $d$  for small  $k^2$ . From (35.6) we find that

$$d(k^2) |_{k^2 \ll m^2} = 1 + e^2 O(k^2).$$

Substituting this expression into (41.21), we find that for large  $r$  (in comparison with  $1/m$ , or in ordinary units in comparison with  $r_0 = \hbar/mc = 3.862 \times 10^{-11}$  cm), the function  $\rho(r)$  tends to unity\*:

$$\varrho(r) = 1 + O\left(\frac{1}{r^2}\right) \quad \left(r \gg \frac{1}{m}\right). \quad (41.22)$$

Therefore the constant  $e$  should be identified with the value of the observable charge. The form of the function  $\rho(r)$  at small  $r$  may be obtained in accordance with (41.21) by investigating the asymptotic form of  $d(k^2)$  for large  $k^2$ . Making use of formula (35.15), we have

$$d(k^2) |_{k^2 \gg m^2} = 1 + \frac{e^2}{12\pi^2} \ln \frac{k^2}{m^2} + \dots$$

By direct integration, we obtain from this the asymptotic behavior of  $\rho(r)$  for small  $r$  in the form (Schwinger (1949a))

$$\varrho(r) = 1 + \frac{e^2}{6\pi^2} \left[ \ln \left( \frac{1}{rm} \right) \right] + \dots \quad (41.23)$$

Thus for small  $r \ll m^{-1}$ , the effective density  $\rho(r)$  increases, and has a logarithmic singularity in the limit as  $r \rightarrow 0$ . However, it is important to emphasize that (41.23) was obtained by means of perturbation theory and is therefore valid, roughly speaking, only in the region in which the second term is considerably smaller than the first. Hence, well-founded conclusions on the true behavior of  $\rho(r)$  for  $r \rightarrow 0$  may be reached only by arguments independent of perturbation theory.

*41.2. Anomalous Magnetic Moment of the Electron.* As a second illustration of the application of the general theory to specific problems, we shall calculate the magnetic moment of the electron, taking radiative corrections into account.

To do this, we shall consider the interaction of a single electron with an external current, and determine the energy due to this interaction (taking radiative corrections into account). In doing this, we shall, of course, assume that the external field (the external current) is constant in time, since the electron will have a definite energy only in this case.

We have considered how the properties of vacuum are affected when a constant external current

$$J(x) = J(x),$$

\*This property remains unchanged when higher-order perturbation theory approximations are taken into account.

is present, and we must now investigate the properties of the one-electron state. By repeating the preceding arguments, we find with the aid of (40.37) that

$$\frac{\delta E(J)}{\delta J(x)} = -A_1^{\text{eff}}(x) = i\Phi_1^* \frac{\delta S(J)}{\delta J(x)} \Phi_1 \frac{1}{\langle S \rangle_0}, \quad (41.24)$$

where

$$S(J) = S(1; J),$$

while the matrix element is taken between two single free-electron states normalized to a unit volume

$$\Phi_1^* = (2\pi)^{1/2} \Phi_0^* a_v^-(k'), \quad \Phi_1 = (2\pi)^{1/2} a_v^+(k) \Phi_0 \quad (41.25)$$

with the same energy  $\sqrt{k^2 + m^2} = \sqrt{k'^2 + m^2}$ .

Taking (41.25) into account, and evaluating the variational derivative  $\delta S/\delta J(x)$ , we obtain

$$\frac{\delta E(J)}{\delta J^n(x)} = -\frac{(2\pi)^3}{S_0} \Phi_0^* a_v^-(k') T(A_n(x) S) a_v^+(k) \Phi_0. \quad (41.26)$$

Since the term in the expression for the energy  $E(J)$  corresponding to the magnetic moment is linear in the external field (current), we may set  $J = 0$  on the right-hand side of (41.26).

We now note that, if the operators  $a^-$  and  $a^+$  in (41.26) were inside the  $T$ -product, then such an expression would correspond to the sum of connected diagrams of different orders with one external photon line and two external electron lines, and could be represented in the form

$$G' \Gamma^n D_{nk} G. \quad (41.27)$$

Here the first electron Green's function  $G'$  corresponds to the external electron line  $k'$ , the second Green's function  $G$  corresponds to the external electron line  $k$ , and  $D_{nk}$  is the photon Green's function corresponding to the operator  $A_k(x)$ . Here, as usual, the vertex part  $\Gamma^n$  represents the sum of strongly connected diagrams with external ends corresponding to  $G'$ ,  $G$ , and  $D$ . In accordance with the foregoing remark, we may regard  $G'$ ,  $G$ ,  $\Gamma$ , and  $D$  as corresponding to free particles (for  $J = 0$ ).

However, in view of the fact that the operators  $a^-$  and  $a^+$  in expression (41.26) are not inside the  $T$ -product, it differs from (41.27). The difference consists of the fact that the "last external" pairings in  $G'$  and  $G$  of (41.27) are replaced in expression (41.26) by ordinary pairings of the form

$$\underbrace{\psi(x) a_v^+(k)} = \frac{v^{v,-}(k)}{(2\pi)^{1/2}} e^{-ikx},$$

i.e.,

$$\frac{1}{\hat{k} - m} \rightarrow \frac{v^-(k)}{(2\pi)^{3/2}}, \quad \frac{1}{\hat{k}' - m} \rightarrow \frac{\bar{v}^+(k')}{(2\pi)^{3/2}}$$

or

$$\left. \begin{aligned} G(k) &\rightarrow G(k) (\hat{k} - m) \frac{v^-(k)}{(2\pi)^{3/2}}, \\ G'(k') &\rightarrow \frac{\bar{v}^+(k')}{(2\pi)^{3/2}} (\hat{k}' - m) G'(k'). \end{aligned} \right\} \quad (41.28)$$

The expressions appearing on the right-hand side of (41.28) may not be interpreted directly since  $(\hat{k} - m)v^-(k) = \bar{v}^+(k')(\hat{k}' - m) = 0$ , while the function  $G(k)$  has a pole at  $\hat{k} = m$ . In order to remove the ambiguity, we shall express the Green's function  $G$  in terms of the self-energy operator  $\Sigma$ :

$$G(k) = \frac{1}{m - \hat{k} + \Sigma(k)}.$$

We shall consider that the infrared catastrophe has been removed from  $\Sigma$  by the introduction of a small photon mass  $\lambda_0$  (as in §35.2) and that  $\Sigma(p)$  satisfies the condition

$$\left. \frac{\Sigma(k)}{\hat{k} - m} \right|_{\hat{k}=m} = 0.$$

We then obtain

$$(m - \hat{k}) G(k) \Big|_{\hat{k}=m} = \frac{m - \hat{k}}{m - \hat{k} - \Sigma(k)} \Big|_{\hat{k}=m} = 1, \quad (41.29)$$

and, consequently, taking conservation of energy into account ( $k_0 = k'_0$ ),

$$\begin{aligned} (2\pi)^3 \Phi_0 a_{\bar{v}}^-(k') T \left\{ A_k(x), \frac{S}{S_0} \right\}^* a_v^+(k) \Phi_0 &= \\ &= e \int \bar{v}^{v,+}(k') \Gamma^n(k', k | q) v^{v,-}(k) D_{nk}(q) e^{iqx} dq \delta(k' - k - q) = \\ &= e \bar{v}^{v,+}(k') \Gamma^n(k', k | k' - k) v^{v,-}(k) D_{nk}(k' - k) e^{i(k-k')x}. \end{aligned}$$

Substituting this expression into (41.24) and integrating over  $J(x) = J(x)$ , we obtain

$$\begin{aligned} e \int d\mathbf{x} J^k(\mathbf{x}) e^{i\mathbf{x} \cdot (\mathbf{k} - \mathbf{k}')} \bar{v}^{v,+}(k') \Gamma^n(k', k | k' - k) v^{v,-}(k) D_{nk}(k' - k) &= \\ = e \bar{v}^{v,+}(k') \Gamma^n(k', k | k' - k) v^{v,-}(k) D_{nk}(k' - k) J^k(k - k'). \end{aligned}$$

By expressing the current  $J$  in terms of the external field  $A^{\text{ext}}$ , and taking into account the expression

$$D_{nk}(k) = -\frac{d(k^2)}{k^2} \left( g_{nk} - \frac{k_n k_k}{k^2} \right) - \frac{k_n k_k}{k^2} \frac{d_l}{k^2}$$

and the Lorentz condition for the external field

$$(k \cdot A^{\text{ext}}(k)) = 0$$

we bring the foregoing expression to the form

$$e\bar{v}^{\nu} + (k') \Gamma^n(k', k | k' - k) v^{\nu} - (k) A_n^{\text{ext}}(k - k') d(-(k - k')^2). \quad (41.30)$$

In order to find the magnetic moment, we shall have to pick out from (41.30) the term that is proportional to the magnetic field  $H$ :

$$H_1 = H_{23}, \quad H_2 = H_{31}, \quad H_3 = H_{12},$$

$$H_{ik} = \frac{\partial A_k^{\text{ext}}}{\partial x^i} - \frac{\partial A_i^{\text{ext}}}{\partial x^k} \rightarrow i(q_i A_k^{\text{ext}}(q) - q_k A_i^{\text{ext}}(q)).$$

We first consider the main term in (41.30) when

$$\Gamma^n = \gamma^n \quad \text{and} \quad d = 1.$$

The energy of interaction with the external field (41.30) assumes the form

$$E = -j^n A_n^{\text{ext}}, \quad (41.31)$$

where

$$j^n = e\bar{v}^{\nu} + (k') \gamma^n v^{\nu} - (k). \quad (41.32)$$

To obtain the magnetic moment from the above expression, we have to go to the non-relativistic limit. For  $k \rightarrow 0$ ,  $k^0 > 0$  the positron components  $v_1$  and  $v_2$  are small in comparison with the electron components  $v_3$  and  $v_4$  (cf. (7.41)) and are of order  $|k|/2k^0$ . We denote the electron components by  $v_{\text{II}}$  and the positron components by  $v_{\text{I}}$ , i.e.,

$$\begin{aligned} v^- &= (v_1, v_{11}), \\ v^+ &= \begin{pmatrix} * \\ v_1, v_{11} \end{pmatrix}, \end{aligned} \quad (41.33)$$

and note that, in accordance with (6.19),

$$j^n = e(\bar{v}^{\nu} + (k') \gamma^n v^{\nu} - (k)) = e\bar{v}^{\nu} + (k') \alpha_n v^{\nu} - (k) \quad \text{for} \quad n = 1, 2, 3,$$

and that since in the “split” representation (41.33)

$$\alpha = \begin{pmatrix} 0 & \sigma \\ \sigma & 0 \end{pmatrix}, \quad (41.34)$$

where  $\sigma = (\sigma_1, \sigma_2, \sigma_3)$  are the two-row Pauli spin matrices, then

$$\mathbf{j} = e \left( \overset{*}{v}_1^\nu(\mathbf{k}') \sigma v_{11}^\nu(\mathbf{k}) \right) + e \left( \overset{*}{v}_{11}^\nu(\mathbf{k}') \sigma v_1^\nu(\mathbf{k}) \right). \quad (41.35)$$

Now, noting that the relations

$$\begin{aligned} (k^0 + m) v_1 &= (\sigma \cdot \mathbf{k}) v_{11}, \\ \overset{*}{v}_1 (k^0 + m) &= \overset{*}{v}_{11} (\sigma \cdot \mathbf{k}), \end{aligned}$$

follow from the Dirac equations, and taking into account the fact that in the limit of small  $k$

$$k^0 + m \simeq 2m,$$

we obtain

$$\mathbf{j}_\alpha = \frac{e}{2m} \overset{*}{v}_{11}^\nu [(\sigma \cdot \mathbf{k}') \sigma_\alpha + \sigma_\alpha (\sigma \cdot \mathbf{k})] v_{11}^\nu. \quad (41.36)$$

Substituting (41.36) into (41.31), and taking into account the fact that, because of the purely magnetic nature of the external field,  $A_0^{\text{ext}} = 0$ , and that, in accordance with (6.19),

$$\sigma_1 \sigma_2 = i \sigma_3, \quad \sigma_2 \sigma_3 = i \sigma_1, \quad \sigma_3 \sigma_1 = i \sigma_2,$$

we obtain

$$E = \mathbf{j} A^{\text{ext}} = -\frac{e}{2m} \overset{*}{v}_{11}^\nu \sigma H v_{11}^\nu - \frac{e}{m} v_{11}^\nu (\mathbf{k}' A^{\text{ext}}(q)) v_{11}^\nu. \quad (41.37)$$

The second term contains the external potential and has no relation to the magnetic moment, while the first term corresponds to the magnetic moment of the electron

$$\mathbf{M} = \mu_0 \overset{*}{v}_{11} \sigma v_{11}, \quad (41.38)$$

where  $\mu_0$  is the Bohr magneton

$$\mu_0 = \frac{e}{2m}. \quad (41.39)$$

We now turn to the radiative corrections. To calculate the first radiative correction to  $\mu_0$ , it is necessary to examine terms of order  $e^2$  in the product  $\Gamma^n d$  that appears in (41.30). However, it is clear that, since we are interested in terms linear in the derivatives of  $A^{\text{ext}}$ , while the expansion of  $d(q)$  about  $q = 0$  starts with quadratic terms, we actually need only to examine the correction to  $\Gamma^n$ .

According to (35.26), the expression

$$\bar{v}^{v,+}(k') \Gamma(k', k | k' - k) v^{v,-}(k)$$

in the limit of small  $q = k' - k$  may be represented to within terms of order  $e^2$  in the form

$$\bar{v}^{v,+}(k') \gamma^n v^{v,-}(k) + \frac{e^2}{16\pi^2 m} \bar{v}^{v,+}(k') \gamma^n \hat{q} v^{v,-}(k).$$

As we have just shown, the first term yields the Bohr magneton while the second one gives

$$\Delta E = \frac{e}{2m} \frac{e^2}{8\pi^2} \bar{v}^{v,+}(k') \hat{A}^{\text{ext}}(q) \hat{q} v^{v,-}(k) = -\mu_0 \frac{e^2}{8\pi^2} \bar{v}^{v,+}(k') (\sigma H(q)) v^{v,-}(k).$$

In going to the nonrelativistic limit in the foregoing expression, we shall make use of the fact that, in the "split" representation (41.33), the four-row matrices  $\sigma$  are expressed diagonally in terms of the two-row Pauli matrices. The result is

$$\Delta E = -\mu_0 \frac{e^2}{8\pi^2} \bar{v}_{11} (\sigma H) v_{11} = -\mu_0 \frac{e^2}{8\pi^2} (MH).$$

According to this expression, the radiative correction of order  $e^2$  gives an *additional electron magnetic moment* equal to

$$\Delta\mu = \mu_0 \frac{e^2}{8\pi^2} = \mu_0 \frac{\alpha}{2\pi} = 0.001161\mu_0.$$

This result was first obtained by Schwinger (1949b). Similar calculations were subsequently performed in the next two approximations in  $\alpha = e^2/4\pi$ . The final result is [see the paper by Levine and Wright (1973)]:

$$\mu_{\text{theor}} = \mu_0 \left[ 1 + \frac{\alpha}{2\pi} - 0.3285 \frac{\alpha^2}{\pi^2} + (1.21 \pm 0.07) \frac{\alpha^3}{\pi^3} \right]. \quad (41.40)$$

The uncertainty in the last term is due to the approximate evaluation of the corresponding Feynman integrals. The theoretical value given by (41.40) must be compared with the experimental value. The most accurate modern experimental result is [Wesley and Rich (1971), Granger and Ford (1972)]

$$\mu_{\text{exp}} = \mu_0 (1.0011596567 \pm 35 \cdot 10^{-10}) \quad (41.41)$$

can be written in the form

$$\mu_{\text{exp}} = \mu_0 \left[ 1 + \frac{\alpha}{2\pi} - 0.3285 \frac{\alpha^2}{\pi^2} + (1.60 \pm 0.33) \frac{\alpha^3}{\pi^3} \right]. \quad (41.42)$$

In transforming from (41.41) to (41.42), we have used the fine structure constant [Taylor, Parker, and Langenberg (1969)]

$$\alpha^{-1} = 137.03608 \pm 26 \cdot 10^{-5}, \quad (41.43)$$

where the uncertainty in the last term in (41.42) represents both the uncertainty in (41.41) and in the value of the fine structure constant (41.43). It is clear that the measured and calculated values are in excellent agreement.

## §42. Dirac Equation with Radiative Corrections

*42.1. Generalization of the Electron Wave Function.* We now consider the problem of obtaining the equation describing the motion of a single electron in a given external electromagnetic field  $A^{\text{ext}}(x)$ .

In classical theory, the equation for the unquantized spinor wave function  $\psi(x)$  of the electron-positron field may be obtained with the aid of the variational principle from the complete Lagrangian of the system, equal to the sum of the Lagrangian of the free electron-positron field and the interaction Lagrangian between the field  $\psi(x)$  and the external field  $A^{\text{ext}}(x)$ :

$$\mathcal{L}(x) = \frac{i}{2} \left( \bar{\psi}(x) \gamma^\mu \frac{\partial \psi(x)}{\partial x^\mu} - \frac{\partial \bar{\psi}(x)}{\partial x^\mu} \gamma^\mu \psi(x) \right) - m \bar{\psi}(x) \psi(x) + e \bar{\psi}(x) \hat{A}^{\text{ext}}(x) \psi(x). \quad (42.1)$$

For such a Lagrangian, the variational principle leads to the well-known Dirac equation

$$(i\hat{\partial} + e\hat{A}^{\text{ext}} - m) \psi(x) = 0, \quad (42.2)$$

on the basis of which quantum mechanics in its time achieved considerable success in the explanation of the magnetic properties of the electron, in the calculation of the fine structure of the levels of hydrogen-like atoms, and so on.

However, this equation does not take into account such specifically quantum-field effects as vacuum polarization, creation of virtual pairs, and so on, so that it must be suitably generalized in quantum field theory. As a preliminary to the solution of this problem, we shall examine the description in quantum theory of a free, noninteracting electron-positron field. Here the state with a single electron of definite momentum  $p$  and definite spin direction will be described by the function



$$\Phi_1 = (2\pi)^{3/2} \hat{a}_v^+ (\mathbf{p}) \Phi_0. \quad (41.25)$$

The general single-electron state may be expressed by a superposition of such "pure" states

$$\Phi_1 = (2\pi)^{3/2} \sum_v \int c_v (\mathbf{p}) \hat{a}_v^+ (\mathbf{p}) d\mathbf{p} \Phi_0. \quad (42.3)$$

We introduce the expression

$$\hat{\Phi}_0 \psi (x) \Phi_1 = (2\pi)^{3/2} \sum_v \int c_v (\mathbf{p}) \hat{\Phi}_0 \psi (x) \hat{a}_v^+ (\mathbf{p}) \Phi_0 d\mathbf{p} \quad (42.4)$$

and note that, by virtue of the commutation relations,

$$\psi (x) \hat{a}_v^+ (\mathbf{p}) + \hat{a}_v^+ (\mathbf{p}) \psi (x) = \frac{1}{(2\pi)^{3/2}} v^{v,-} (\mathbf{p}) e^{-i\mathbf{p}x} \quad (p_0 = \sqrt{\mathbf{p}^2 + m^2}).$$

Therefore

$$\hat{\Phi}_0 \psi (x) \Phi_1 = \int e^{-i\mathbf{p}x} \sum_v v^{v,-} (\mathbf{p}) c_v (\mathbf{p}) d\mathbf{p}. \quad (42.5)$$

Since, by definition,

$$(\hat{\mathbf{p}} - m) v^{v,-} (\mathbf{p}) = 0,$$

we see that the unquantized spinor

$$\psi (x) = \hat{\Phi}_0 \psi (x) \Phi_1 \quad (42.6)$$

satisfies the Dirac equation for the free field  $\psi(x)$ :

$$(i\hat{\partial} - m) \psi (x) = 0.$$

From the representation (32.5) it further follows that the spinor  $\psi(x)$  completely determines the amplitudes  $c_v(\mathbf{p})$  and thereby also the state  $\Phi_1$ . In particular, when the electron in this state has a definite energy  $E > 0$ , then

$$c_v (\mathbf{p}) = f_v (\mathbf{p}) \delta (\sqrt{\mathbf{p}^2 + m^2} - E)$$

and we obtain from (42.5)

$$\psi (x) = e^{-iEx^0} \chi (\mathbf{x}). \quad (42.7)$$

After these preliminary remarks, we now turn to the problem of interest to us and, in order to obtain the unquantized spinor characterizing the single-electron states, we shall generalize (38.6).

The state amplitudes  $\Phi_0, \Phi_1$  must, evidently, be replaced by  $\Phi_0(g), \Phi_1(g)$ . It is natural to take  $\psi(x; g)$  as the local operator which is the generalization of  $\psi(x)$ . To eliminate the dependence on  $g$ , we shall deal only with such  $g = g_x(y)$  for which

$$g_x(y) = 1 \quad \text{for} \quad y \lesssim x. \quad (42.8)$$

Then, by virtue of the general results of §40, the expression

$$\varphi(x) = \Phi_0^*(g) \psi(x; g) \Phi_1(g) \quad (42.9)$$

will be an unquantized spinor that does not depend on the particular choice of the function  $g$  satisfying (42.8).

We first consider the situation where there is no external electromagnetic field and, consequently, invariance under translation holds. We consider a certain translation  $L$ ,

$$x \rightarrow x + a,$$

and note that, in accordance with the transformation law (20.21) for the matrix  $S(g)$ ,

$$S(Lg) = U_L S(g) \dot{U}_L$$

and the transformation law for the operator  $\psi(x)$  corresponding to a translation [see (9.23)]

$$\psi(Lx) = U_L \psi(x) \dot{U}_L$$

and, in accordance with the definition given by (40.35), the following relation holds:

$$\psi(Lx, Lg) = U_L \psi(x, g) \dot{U}_L$$

or

$$\psi(Lx, g) = U_L \psi(x, L^{-1}g) \dot{U}_L.$$

However, for an infinitesimal translation  $a = \delta a$

$$U_L = 1 + \delta U_L, \quad \delta U_L = i(P_0 \cdot \delta a).$$

Therefore, in accordance with (9.24),

$$i \frac{\partial \psi(x; g)}{\partial x_n} = [\psi(x; g), P_0^n] + i \int \frac{\partial \psi(x; g)}{\partial g(y)} \frac{\partial g(y)}{\partial y_n} dy.$$

As a result of (42.8),

$$\frac{\partial g(y)}{\partial y^n} = 0 \quad \text{for} \quad y \lesssim x,$$

and from (40.17)

$$i \frac{\delta \psi(x; g)}{\delta g(y)} = [H(x; g), \psi(x; g)] \quad \text{for} \quad y \gtrsim x.$$

From this we have

$$i \frac{\partial \psi(x; g)}{\partial x_n} = \left[ \psi(x, g), \left( P_0^n - \int H(y; g) \frac{\partial g(y)}{\partial y_n} dy \right) \right].$$

Consequently, by virtue of (40.4) we obtain

$$i \frac{\partial \psi(x; g)}{\partial x_n} = [\psi(x; g), P^n(g)],$$

and, therefore,

$$i \frac{\partial \varphi(x)}{\partial x_n} = \hat{\Phi}_0(g) \psi(x; g) P^n(g) \Phi_1(g) - \hat{\Phi}_0^*(g) P^n(g) \psi(x; g) \Phi_1(g).$$

Consider the case where the electron in the state  $\Phi_1(g)$  has a definite energy-momentum four-vector  $p^n$ . Then

$$P^n(g) \Phi_1(g) = p^n \Phi_1(g).$$

On the other hand, for the vacuum state we always have

$$P^n(g) \Phi_0(g) = 0, \quad \hat{\Phi}_0^*(g) P^n(g) = 0.$$

Therefore

$$\frac{\partial \varphi(x)}{\partial x^n} = -i p_n \varphi(x)$$

and

$$\varphi(x) = C e^{-i p x} \quad (C = \text{const}, \quad p^0 = \sqrt{p^2 + m^2}).$$

Since, moreover,  $\varphi(x)$  has the transformation properties of a spinor, we see that

$$(i\hat{\partial} - m) \varphi(x) = 0.$$

This equation is satisfied when the single-electron state  $\Phi_1(g)$  is characterized by a definite energy-momentum four-vector. Since any arbitrary single-electron state may be obtained as a superposition of states of this type, we see that the Dirac equation is satisfied in the general case as well.

We finally turn to the case where an external electromagnetic field is present. If this field does not depend on the time, we retain invariance under time translation and, therefore, by repeating the preceding argument, we see that if the single-electron state is characterized by a definite energy  $E$ , then

$$\varphi(x) = e^{-iEx_0} \chi(x). \quad (42.10)$$

Thus, the energy levels of stationary single-electron states must be found by solving an eigenvalue problem.

*42.2. Generalization of the Dirac Equation.* We shall obtain this solution by substituting (42.10) into the generalized Dirac equation. For the actual construction of this equation we return to (42.9). By substituting this into (40.35), we obtain

$$\varphi(x) = \frac{1}{ig(x)} \tilde{\Phi}_0^\dagger S(g) \frac{\delta S(g)}{\delta \eta(x)} \Phi_1,$$

where  $\Phi_0$  and  $\Phi_1$  are the vacuum and the single-electron states in the absence of interaction.

By carrying out the usual transition to the limit as  $g \rightarrow 1$ , we obtain

$$\varphi(x) = \frac{1}{i} \tilde{\Phi}_0^\dagger S \frac{\delta S}{\delta \eta(x)} \Phi_1. \quad (42.11)$$

But

$$\frac{\delta \mathcal{L}(x)}{\delta \eta(x)} = i\psi(x),$$

from which it follows that

$$\frac{\delta S}{\delta \eta(x)} = iT(\psi(x) S). \quad (42.12)$$

To exclude the usual divergences right at the outset, we suppose that the  $T$ -product is redefined in such a way that (42.12) does not contain any ultraviolet infinities. We consequently obtain

$$\varphi(x) = \tilde{\Phi}_0^\dagger \hat{S} T(\psi(x) S) \Phi_1 = \frac{\tilde{\Phi}_0^\dagger T(\psi(x) S) \Phi_1}{S_0},$$

where  $\Phi_1$  is given by (42.37).

From this we see that the diagrams determining  $\varphi(x)$  are obtained directly from the corresponding diagrams for Green's function free of divergences

$$G(x, y | A^{\text{ext}}) = i \frac{\Phi_0^* T(\psi(x) \bar{\psi}(y) S) \Phi_0}{S_0}$$

by replacing the factor for the entering line

$$i \Phi_0^* T(\psi(x) \bar{\psi}(z)) \Phi_0 = S^c(x - z)$$

by the "external" factor

$$\Phi_0^* \psi(x) \Phi_1 = \int e^{-ipx} \sum_{\mathbf{v}} v^{\mathbf{v}} \cdot \mathbf{c}_{\mathbf{v}}(\mathbf{p}) d\mathbf{p} = \psi(x).$$

Therefore, the relation between  $\varphi(x)$  and  $G$  may be written in the following form:

$$\varphi(x) = \int G(x, y | A^{\text{ext}}) [S^c(y - z)]^{-1} \psi(z) dy dz. \quad (42.13)$$

In §38 we used the generalized Wick theorem to obtain Schwinger's equation for the electron Green's function, the integral form (38.46) of which may be written symbolically

$$G = S^c - e S^c \hat{\mathcal{A}} G + S^c M' G.$$

By substituting this expression into equation (42.13), written in the form

$$\varphi = G \frac{1}{S^c} \psi,$$

we obtain

$$\varphi = (S^c - e S^c \hat{\mathcal{A}} G + S^c M' G) \frac{1}{S^c} \psi = \psi - e S^c \hat{\mathcal{A}} \varphi + S^c M' \varphi,$$

i.e.,

$$\varphi(x) = \psi(x) - e \int S^c(x - y) \hat{\mathcal{A}}(y) \varphi(y) dy + \int S^c(x - y) M'(y, z) \varphi(z) dy dz. \quad (42.14)$$

Applying the Dirac operator to (42.14) and taking into account the fact that  $\psi(x)$  satisfies the Dirac equation for the free field, we obtain

$$(i \hat{\partial}_x + e \hat{\mathcal{A}}(x)) \varphi(x) - \int M(x, y) \varphi(y) dy = 0, \quad (42.15)$$

which was first given by Schwinger (1951b). We recall that  $M(x, y)$  is the mass operator introduced in §38 which, in accordance with (38.44), has the form

$$M(x, y) = m\delta(x - y) - ie^2\gamma^k \int G(x, z) \Gamma'(z, y | \xi) D_{kl}(\xi, x) d\xi dz \quad (42.16)$$

and that, moreover,

$$\mathfrak{A}(x) = \frac{1}{S_0} \langle T[(A(x) + A^{\text{ext}}(x)) S] \rangle_0. \quad (42.17)$$

It is now clear that in the limit as the interaction with the quantized electromagnetic field is switched off ( $e \rightarrow 0$ ,  $eA^{\text{ext}}$  finite)\* we obtain from (42.15) the Dirac equation for the classical electron in an external field:

$$(i\hat{\partial}_x + e\hat{A}^{\text{ext}}(x) - m)\varphi(x) = 0.$$

Thus (42.15) represents the Dirac equation with radiative corrections included. It may be seen that these radiative corrections are of two different types. The term  $\mathfrak{A}$  represents the potential of the external field  $A^{\text{ext}}$  added to the effective average potential of the field "induced" in the vacuum. This term contains corrections to  $A^{\text{ext}}$  associated with vacuum polarization. The quantity  $M(x, y)$  represents a mass operator which includes the self-energy effects. In the lowest order in  $e^2$ , these quantities may be written in the form (assuming that  $e^2 \ll 1$ ,  $eA^{\text{ext}}$  finite)

$$M(x, y) = m\delta(x - y) - ie^2\gamma^k \int S^c(x, z | A^{\text{ext}}) \gamma' \delta(z - y) \delta(z - \xi) \times \\ \times D_{kl}^c(\xi - x) d\xi dz = m\delta(x - y) - ie^2\gamma^k S^c(x, y | A^{\text{ext}}) \gamma' D_{kl}^c(y - x) \quad (42.18)$$

and

$$\mathfrak{A}_k(x) = A_k^{\text{ext}}(x) - ie^2 \int \text{Sp}[S^c(y - \tau) \gamma^m S^c(\tau - y) \gamma^n] D_{nk}^c(y - x) A_m^{\text{ext}}(\tau) dy d\tau. \quad (42.19)$$

Thus, after taking into account corrections of order  $e^2$ , the generalized Dirac equation (42.15) may be written in the form

$$\left(i \frac{\partial}{\partial x} + e\hat{A}^{\text{ext}}(x) - m\right)\varphi(x) - \\ - ie^2\gamma^k \varphi(x) \int dy d\tau D_{kn}^c(x - y) \text{Sp}[S^c(y - \tau) \gamma^m S^c(\tau - y) \gamma^n] eA_m^{\text{ext}}(\tau) + \\ + ie^2 \int dy \gamma^k S^c(x, y | A^{\text{ext}}) \gamma' D_{lk}^c(y - x) \varphi(y) = 0. \quad (42.20)$$

\*Here we have in mind the case where the interaction with the external electron field  $eA^{\text{ext}}$  is not small in spite of the smallness of the charge  $e$ . The simplest example of such a situation is the problem of the motion of an electron in the Coulomb field of a nucleus with a large value of  $Z$ , so that

$$eA^{\text{ext}} \sim Ze^2 \sim 1 \quad \text{for} \quad Z \gg 1.$$

Here  $S^c(x, y|A^{\text{ext}})$  is Green's function for the classical electron moving in a given external field  $A^{\text{ext}}$ . This function by definition satisfies the equation

$$\left(i \frac{\partial}{\partial x} + e \hat{A}^{\text{ext}} - m\right) S^c(x, y|A^{\text{ext}}) = -\delta(x - y) \quad (42.21)$$

and is represented by the sum of diagrams with two external electron lines and any number of external photon lines corresponding to the given field  $A^{\text{ext}}$  (Fig. 61). The corresponding expansion for  $S^c(A^{\text{ext}})$  may be obtained formally by expanding the following expression into a series in powers of  $A^{\text{ext}}$ :

$$S^c(x, y|A^{\text{ext}}) \simeq -\frac{1}{i\hat{\partial} - m + e\hat{A}^{\text{ext}}} = \frac{1}{(S^c)^{-1} - e\hat{A}^{\text{ext}}}$$

and has the form

$$S^c(x, y|A^{\text{ext}}) = S^c(x - y) + e \int S^c(x - z) \hat{A}^{\text{ext}}(z) S^c(z - y) dz + \\ + e^2 \int S^c(x - z) \hat{A}^{\text{ext}}(z) S^c(z - t) \hat{A}^{\text{ext}}(t) S^c(t - y) dz dt + \dots \quad (42.22)$$

Equation (42.20) enables us to calculate radiative corrections to the energy levels of bound states.

**42.3. Lamb Shift.** An important example of this type is the problem of calculating the radiation shift of the electron levels in a hydrogen-like atom—the so-called *Lamb shift*. To calculate the hyperfine structure of electron levels in a hydrogen-like atom, we consider the motion of an electron in the Coulomb field of a nucleus. In this case

$$\left. \begin{aligned} A_n^{\text{ext}}(x) &= \delta_{n0} A_0(x), \\ A_0(x) &= \frac{Ze}{4\pi|x|} \end{aligned} \right\} \quad (42.23)$$

or, in the momentum representation,

$$A(x) = \frac{1}{(2\pi)^{3/2}} \int e^{-iqx} A_0(q) dq, \quad (42.24)$$

$$A_0(q) = \frac{Ze}{(2\pi)^{3/2} |q|^2}. \quad (42.25)$$

In this case, the function  $\varphi(x)$  has the form (42.10) where the energy  $E$  after substituting (42.10) into (42.15) may be determined from the equation

$$(E\gamma^0 - i\partial\gamma + e\hat{A}(x)) \chi(x) = \int e^{iE(x^0 - y^0)} M(x, y) \chi(y) dy, \quad (42.26)$$

where it is convenient to write the energy  $E$  in the form of a series in powers of  $\alpha = e^2/4\pi$ :

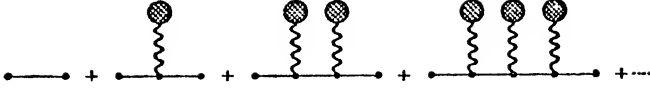


Fig. 61.

$$E = E_0 + \alpha E_1 + \dots, \quad (42.27)$$

taking for the zero-order approximation the case where radiative corrections are absent. In this approximation, the equation

$$(E_0 \gamma^0 - i \partial \gamma + e \gamma^0 A_0(x) - m) \chi_0(x) = 0 \quad (42.28)$$

is the usual Dirac equation for the motion of an electron in the Coulomb field of a nucleus, while the functions  $\chi_0$  and the eigenvalues  $E$  correspond to the fine structure of hydrogen-like levels. It is clear that the terms added to  $E_0$  are just the desired Lamb shift of the levels due to the radiative corrections.

The first correction  $E_1$  may be calculated on the basis of equation (42.20). Substituting (42.25) and (42.27) into this equation, taking (42.28) into account, equating terms of order  $e^2$ , going over to momentum representation

$$\begin{aligned} \chi(x) &= \frac{1}{(2\pi)^{3/2}} \int e^{ipx} \chi(p) dp, \\ M(x, y) &= \frac{1}{(2\pi)^3} \int e^{-ipx + iqy} M(p, q) dp dq, \\ S^c(x, y | A^{\text{ext}}) &= \frac{1}{(2\pi)^3} \int e^{-ipx + iqy} S^c(p, q | A^{\text{ext}}) dp dq \end{aligned}$$

and multiplying  $\chi(x)$  on the left by the adjoint spinor  $\bar{\chi}(p)$  such that

$$\frac{1}{(2\pi)^3} \int \bar{\chi}(p) \gamma^0 \chi(p) dp = \frac{1}{(2\pi)^3} \int |\chi(p)|^2 dp = 1,$$

we find that  $E_1$  may be expressed as the sum of two terms, one of which,  $E'_1$ , corresponds to the radiative correction to the mass operator and the second,  $E''_1$ , corresponds to vacuum polarization:

$$\alpha E_1 = E'_1 + E''_1, \quad (42.29)$$

where

$$\begin{aligned} E'_1 &= \frac{1}{(2\pi)^3} \int \bar{\chi}(p) M(p, q) \chi(q) dp dq = \\ &= \frac{e^2}{i(2\pi)^7} \int \bar{\chi}(p) \gamma^\alpha S^c(p-k, q-k | A^{\text{ext}}) \gamma^\mu D_{nm}^c(k) \chi(q) dk dp dq \end{aligned} \quad (42.30)$$



$$E_1' = -\frac{e}{(2\pi)^3} \int \bar{\chi}(p) \gamma^k D_{km}^c(p-q) \Pi^{mn}(p-q) A_n^{\text{ext}}(p-q) \chi(q) dp dq, \quad (42.31)$$

and

$$p^0 = q^0 = E \approx E_0. \quad (42.32)$$

Here  $\Pi^{nm}(p-q)$  is the second-order polarization operator that we have already considered in Chapter V. By virtue of (35.3) and (35.8), this operator has the following form after the removal of divergences:

$$\Pi^{nm}(k) = \frac{e^2}{4\pi^2} (g^{mn} k^2 - k^n k^m) I(k^2), \quad (42.33)$$

where

$$I(k^2) = -\frac{k^2}{4\pi^2} \int_0^1 \frac{z^2 \left(1 - \frac{z^2}{3}\right) dz}{1 - \frac{k^2}{4m^2} (1 - z^2)} \bigg|_{k^2 \ll m^2} \approx -\frac{1}{15} \frac{k^2}{m^2} + \dots \quad (42.34)$$

Expression (42.30) gives the main contribution to the radiative shift of the levels. Substituting the expansion (42.22) for  $S^c(x, y|A^{\text{ext}})$  into (42.30), we find that the first term leads to a divergent expression of the type of the electron self-energy, which is completely compensated by a corresponding redefinition of the  $T$ -product (compare (42.12)). The second term leads to the expression

$$\begin{aligned} E_1'^{(s)} &= \frac{e^3}{i(2\pi)^7} \int \bar{\chi}(p) \gamma^k S^c(p-k) \hat{A}^{\text{ext}}(p-q) S^c(q-k) \gamma^l D_{kl}^c(k) \chi(q) dk dp dq = \\ &= -\frac{e}{(2\pi)^3} \int \bar{\chi}(p) \Lambda^n(p, q, p-q) A_n^{\text{ext}}(p-q) \chi(q) dp dq, \end{aligned} \quad (42.35)$$

in which  $\Lambda^n(x, y | z)$  is the vertex function introduced in §28.1. After the removal of the ultraviolet divergence, the latter is given by (35.28) which, as we have already shown, contains an infrared singularity. In order to remove this singularity, one has to examine the next terms of the mass operator. We refer the reader who is interested in the details of the calculation to the original papers\* and merely note here that the principal contribution to  $E_1'$  is given by the nonrelativistic term corresponding to small momenta  $q^2 \ll m^2$  of the external field  $A_0(q)$ , which was first calculated by Bethe (1947).

An important feature of (42.35) is its explicit dependence on the form of the functions  $\chi, \bar{\chi}$ . This leads to different Lamb shifts for different electron states in the hydrogen-like

\*Karplus, Klein and Schwinger (1952), Baranger, Bethe, and Feynman (1953).

atom. It is precisely this factor that leads to the observed *splitting* of the  $2S_{1/2}$  and  $2P_{1/2}$  levels which coincide in the one-electron Dirac theory.

The vacuum polarization term (42.31) in the lowest approximation leads to the shift of only the  $S$ -levels. In point of fact, substituting (42.23), (42.24), (42.32), and (42.33) in (42.31), we obtain

$$E_1^* = -e \frac{\alpha}{\pi} \frac{1}{(2\pi)^3} \int \chi^*(p) I(-(p-q)^2) A_0(p-q) \chi(q) dp dq \simeq \\ \simeq 4\alpha (Z\alpha) \left. \frac{dI(k^2)}{dk^2} \right|_{k^2=0} |\chi^{nr}(x=0)|^2 = -\frac{4\alpha(Z\alpha)}{15m^2} |\chi^{nr}(0)|^2, \quad (42.36)$$

where  $I(k^2)$  is given by (42.34) with  $k^2 = -k^2$  and  $\chi^{nr}$  is the nonrelativistic Coulomb wave function.

Finally, we note that the calculated relative shift of the  $2S_{1/2}$  and  $2P_{1/2}$  hydrogen levels [Peterman (1972) and Faustov (1972)], which was obtained to within terms of order  $\alpha(Z\alpha)^2$  and  $\alpha^2$ , and was corrected for the fine-structure effects associated with the interaction between the electron and the magnetic dipole moment of the nucleus, and also for structural effects and the finite mass of the nucleus, turned out to be  $1057.91 \pm 0.01$  MHz. This is in complete agreement with the experimental result  $1057.90 \pm 0.06$  MHz obtained by Robiskoe and Shyn (1970).

**42.4. Concluding Remarks.** We note, finally, that the application of the method given in § 42.1 is, of course, not restricted to the examination of the Lamb shift of hydrogen levels. This method may easily be extended to systems in which the number of electrons and positrons is greater than unity.

For example, the wave function describing a positron has the form

$$\bar{\varphi}(x) = \frac{1}{i} \bar{\Phi}_0 S \frac{\delta S}{\delta \eta(x)} \Phi_1 = \frac{\bar{\Phi}_0 T(\bar{\psi}(x) S) \Phi_1}{S_0}, \quad (42.37)$$

where  $\Phi_1$  is the amplitude of the single-positron state. By noting that  $\bar{\varphi}(x)$  differs from the one-electron Green's function by the replacement of the factor  $S^c$  for an emerging line by  $\bar{\psi}(x)$ , and by using arguments similar to those employed earlier, we obtain for  $\bar{\varphi}$  the Dirac equation with the radiative corrections.

In a similar manner, we can introduce the wave function for two electrons or positrons. In the case of positronium, it is convenient to introduce the quantity

$$\Psi(x, y) = \frac{1}{i^2} \bar{\Phi}_0 S \frac{\delta^2 S}{\delta \eta(x) \delta \eta(y)} \Phi_2 = \frac{\bar{\Phi}_0 T(\psi(x) \bar{\psi}(y) S) \Phi_2}{S_0}, \quad (42.38)$$

where  $\Phi_2$  is the amplitude of the state containing one electron and one positron. The equation for  $\varphi(x, y)$  may be obtained by appropriately modifying the equation for the two-electron Green's function

$$G(x, x', y, y') = \frac{\Phi_0^* T(\psi(x) \psi(x') \bar{\psi}(y) \bar{\psi}(y') S) \Phi_0}{S_0}.$$

It is clear that by considering matrix elements of higher-order variational derivatives of the  $S$ -matrix, we may generalize the foregoing results to any arbitrary many-electron system.

# Chapter 8

## METHOD OF FUNCTIONAL AVERAGING

### §43. Functional Integrals in Quantum Field Theory

*43.1. Introduction.* We shall now consider the various attempts that have been made to extend the analysis beyond the framework of perturbation theory. The method based on the use of the functional integral, originally introduced by Feynman (1948a) and described in this chapter, is of particular interest among these attempts. It will be shown below that it leads to closed expressions for complete Green's functions including all the radiative corrections. These expressions may be regarded as averages of classical Green's functions for particles moving in a given external field, evaluated over the quantum fluctuations of this field. The functional integrals taken over functional space arise in this averaging process.

It is important to note that investigations in this area are far from complete, and there are many unresolved problems of both technical and fundamental nature.

Representations of Green's functions in the form of functional integrals can be obtained in various ways. One of them\* is based on the formal integration of the equations in terms of variational derivatives considered in §38. However, we shall follow a different and, in our view, simpler route [Bogolyubov (1954)] which starts with the representation of Green's functions in terms of the vacuum expectations of chronological products, in which the vacuum averaging operation is interpreted as a functional integral.

We shall start with the expressions obtained in §37 for the complete Green's functions

$$\Delta(x, y) = i \frac{\langle T \varphi(x) \varphi(y) S \rangle_0}{S_0}$$

\*Edwards and Peierls (1954), Gel'fand and Minlos (1954), Fradkin (1954), and Khalatnikov (1955).

in the form of variational derivatives of the generating functional  $S_0(J)$  [see (37.12)]

$$\langle T \varphi(x) \varphi(y) S(J) \rangle_0 = (-i)^2 \frac{\delta^2 S_0(J)}{\delta J(x) \delta J(y)}.$$

For the sake of simplicity, we shall confine our attention for the time being to the theory with one scalar field:

$$S_0(J) = \langle T \exp i \left( \int \mathcal{L}(\varphi) dx + \varphi J \right) \rangle_0. \quad (37.10)$$

In the more general case, the generating vacuum functional  $Sp(\eta, \bar{\eta}, J, \dots)$  contains Fermi arguments, and the operation  $\langle \dots \rangle_0$  contains averaging over both boson and fermion vacuum. However, when the vacuum expectation values are evaluated, averaging over the Bose and Fermi vacuum can be performed independently because the corresponding expressions are linear forms of the products of the vacuum averages obtained for each vacuum separately.

We begin with the procedure for the evaluation of the average over the Bose vacuum. This operation is exhaustive for the class of quantum-field models that do not contain Fermi fields. In view of the foregoing, it is also an essential independent stage in the analysis of models containing both Bose and Fermi fields. Finally, we shall show that it is the logical precursor of the evaluation of expectation values over the Fermi vacuum.

We now consider the determination of the vacuum expectation value  $\langle T(F(\varphi)) \rangle_0$  of an arbitrary functional  $F$  that depends on operator functions of a real field,  $\varphi$ , quantized according to the Bose-Einstein rules. We shall suppose that the chronological pairings of the operators  $\varphi$  are given and have the following form in momentum representation:

$$\overline{\varphi(p) \varphi(p')} = -i \delta(p + p') \Delta^c(p). \quad (43.1)$$

We shall start with the special case where  $F$  has the exponential form, and determine the quantity

$$I = \langle T \left( \exp i \int v(p) \varphi(p) dp \right) \rangle_0, \quad (43.2)$$

where  $v(p)$  is an arbitrary function.

**43.2. Evaluation of  $\langle T(\exp i \int v(p) \varphi(p) dp) \rangle_0$ .** To determine  $I$ , we note that any operator  $T$ -functional  $F$  which depends on the Bose field  $\varphi$  can be reduced to the normal form by the following procedure.\* The field  $\varphi$  in  $F$  is replaced by  $\varphi'$  which, in the momentum representation, has the form\*\*

\*See, for example, Anderson (1954).

\*\*Accordingly, in the  $x$ -representation,

$$\varphi'(x) = \varphi(x) - i \int \Delta^c(x - x') \frac{\delta}{\delta \varphi(x')} dx'.$$

$$\varphi'(k) = \varphi(k) + \int \overline{\varphi(k) \varphi(k')} \frac{\delta}{\delta \varphi(k')} dk' = \varphi(k) - i \Delta^c(k) \frac{\delta}{\delta \varphi(-k)}. \quad (43.3)$$

When this has been done, the functional  $F(\varphi')$  should be looked upon as an ordinary  $C$ -functional of nonquantized function  $\varphi$  and its functional derivatives, and the necessary functional differentiation should be performed in it, assuming that  $F$  is multiplied by unity on the right. The expression obtained as a result of this operation can be regarded as the functional  $F$  of the operator functions  $\varphi$ , written in the normal form. For example,

$$\begin{aligned} T \left( \int \varphi(k) a(k) dk \int \varphi(k') b(k') dk' \right) &\rightarrow \\ \rightarrow \int \left[ \varphi(k) - i \Delta^c(k) \frac{\delta}{\delta \varphi(-k)} \right] a(k) dk \int \left[ \varphi(k') - i \Delta^c(k') \frac{\delta}{\delta \varphi(-k')} \right] b(k') dk' \cdot 1 = \\ = \int \varphi(k) a(k) dk \int \varphi(k') b(k') dk' - i \int \Delta^c(k) b(-k) a(k) dk, \end{aligned}$$

i.e.,

$$\begin{aligned} T \left( \int \varphi(k) a(k) dk \int \varphi(k') b(k') dk' \right) &= \\ = \int a(k) dk \int b(k') dk' : \varphi(k) \varphi(k') : - i \int a(k) \Delta^c(k) b(-k) dk, \end{aligned}$$

which can be verified by direct substitution.

The validity of the above recipe can also be readily verified in the case where  $F$  has a polynomial form, and from this it follows that it is also valid for any  $F$  that can be expanded into power series. Moreover, it is clear that to obtain the vacuum expectation value, it is sufficient to set  $\varphi = 0$  in the result, i.e.,

$$\langle T(F(\varphi)) \rangle_0 = \left\{ F \left( \varphi - i \Delta^c \frac{\delta}{\delta \varphi} \right) \cdot 1 \right\}_{\varphi=0}. \quad (43.4)$$

Applying the prescription (33.4) to the integral in (43.2), we find that the latter can be written in the form

$$I = \left[ \left\{ \exp i \int v(\rho) \left( \varphi(\rho) - i \Delta^c(\rho) \frac{\delta}{\delta \varphi(-\rho)} \right) d\rho \right\} \cdot 1 \right]_{\varphi=0}. \quad (43.5)$$

To evaluate this integral, we introduce the auxiliary quantity

$$U(\lambda | \varphi) = \exp \left( i \lambda \int v(\rho) \left( \varphi(\rho) - i \Delta^c(\rho) \frac{\delta}{\delta \varphi(-\rho)} \right) d\rho \right) \cdot 1, \quad (43.6)$$

where, obviously,

$$I = U(1 | 0).$$

Differentiating (43.6) with respect to  $\lambda$ , we obtain the following differential equations for  $U$ :

$$\frac{\partial U(\lambda | \varphi)}{\partial \lambda} = i \int v(p) \left( \varphi(p) - i \Delta^c(p) \frac{\delta}{\delta \varphi(-p)} \right) d\rho U(\lambda | \varphi), \quad (43.7)$$

and the solution of this satisfying the initial condition

$$U(\lambda | \varphi)|_{\lambda=0} = U(0 | \varphi) = 1,$$

will be sought in the form

$$U(\lambda | \varphi) = e^{s(\lambda | \varphi)}. \quad (43.8)$$

Substituting (43.8) in (43.7), we obtain the following "partial differential equation" for  $s$ :

$$\frac{\partial s}{\partial \lambda} = \int v(p) \left( i\varphi(p) + \Delta^c(p) \frac{\delta s}{\delta \varphi(-p)} \right) d\rho.$$

This can be solved by writing

$$s(\lambda | \varphi) = i\lambda \int v(p) \varphi(p) d\rho + r(\lambda),$$

where  $r$  is independent of  $\varphi$  and satisfies the equation

$$\frac{\partial r}{\partial \lambda} = i\lambda \int v(p) \Delta^c(p) v(-p) d\rho.$$

Integration of this subject to the boundary condition given above yields

$$r(\lambda) = \frac{i\lambda^2}{2} \int v(p) \Delta^c(p) v(-p) d\rho.$$

Therefore

$$U(\lambda | \varphi) = \exp i \left( \lambda \int v(p) \varphi(p) d\rho + \frac{\lambda^2}{2} \int v(p) \Delta^c(p) v(-p) d\rho \right),$$

and, in accordance with (43.4) and (43.5),

$$I = \left\langle T \left( \exp i \int v(p) \varphi(p) d\rho \right) \right\rangle_0 = \exp \left[ \frac{i}{2} \int v(p) \Delta^c(p) v(-p) d\rho \right]. \quad (43.9)$$

43.3. *Functional Integrals.* To transform from (43.9) to the functional integral, we introduce a discrete momentum "lattice," based on a finite set of points  $p_j$  in momentum space, which is symmetric under the transformation

$$p \rightarrow -p.$$

Let  $\Delta_p$  be the four-dimensional volume element of the lattice. Substituting

$$\varphi(p_j) = \varphi_j, \quad \Delta^c(p_j) = \Delta_j^c, \quad v(p_j) \Delta p = v_j, \quad (43.10)$$

we then have from (43.9), after taking into account the parity of  $\Delta^c$ ,

$$\begin{aligned} \left\langle T \left( \exp i \sum_j v_j \varphi_j \right) \right\rangle_0 &= \exp \left( \frac{i}{2\Delta p} \sum_j v_j \Delta_j^c v_{-j} \right) = \prod_j \left( \exp \frac{i}{2\Delta p} v_j \Delta_j^c v_{-j} \right) = \\ &= \prod_{j>0} \left( \exp \frac{i}{\Delta p} v_j \Delta_j^c v_{-j} \right). \end{aligned} \quad (43.11)$$

Noting that the condition for the field  $\varphi(x)$  to be real in the discrete momentum representation is

$$\varphi^*(p_j) = \varphi(-p_j),$$

and choosing the notation so that

$$-p_j = p_{-j},$$

we substitute

$$\begin{aligned} \varphi_j &= x_j + iy_j, \\ \varphi_{-j} &= x_j - iy_j, \end{aligned}$$

where  $j > 0$  and all the  $x_j, y_j$  are real.

Using the usual Gaussian integral

$$\exp \left( i \frac{\lambda^2 - \mu^2}{4a} \right) = \lim_{\epsilon \rightarrow +0} \frac{ia}{\pi} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \exp i \{ (-a + i\epsilon)(x^2 + y^2) + \lambda x + i\mu y \}$$

we can write the factors on the right-hand side of (43.11) in the form ( $a = \Delta p / \Delta_j^c$ ,  $\lambda = v_j + v_{-j}$ ,  $\mu = v_j - v_{-j}$ ):

$$\exp \left( \frac{i}{\Delta p} v_j \Delta_j^c v_{-j} \right) = \lim_{\epsilon \rightarrow +0} \frac{i\Delta p}{\pi \Delta_j^c} \int_{-\infty}^{\infty} dx_j \int_{-\infty}^{\infty} dy_j \exp i \left\{ \varphi_j \left( -\frac{\Delta p}{\Delta_j^c(\epsilon)} \right) \varphi_{-j} + v_j \varphi_j + v_{-j} \varphi_{-j} \right\},$$

where



$$\frac{1}{\Delta_j^c(\epsilon)} = \frac{1}{\Delta_j^c} - i\epsilon,$$

and this corresponds to the usual definition of the causal function  $\Delta^c$ . Henceforth, we shall follow common procedure and will not explicitly indicate the limiting transition in  $\epsilon$ , and instead of  $\Delta_j^c(\epsilon)$  we shall write  $\Delta_j^c$ .

It will be convenient to use the new notation

$$\delta\varphi_j = \frac{i\Delta p}{\pi\Delta_j^c} \exp\left(-i\varphi_j \frac{\Delta p}{\Delta_j^c} \varphi_{-j}\right) dx_j dy_j = \frac{\exp\left(-i\varphi_j \frac{\Delta p}{\Delta_j^c} \varphi_{-j}\right) dx_j dy_j}{\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp\left(-i\varphi_j \frac{\Delta p}{\Delta_j^c} \varphi_{-j}\right) dx_j dy_j}, \quad (43.12)$$

so that (43.11) can be rewritten in the form

$$\begin{aligned} \langle T \exp\left(i \sum_j v_j \varphi_j\right) \rangle_0 &= \prod_{i>0} \left\{ \int \exp\left(i(v_j \varphi_j + v_{-j} \varphi_{-j})\right) \delta\varphi_j \right\} = \\ &= \int \exp\left(i \sum_j v_j \varphi_j\right) \prod_{i>0} \delta\varphi_j. \end{aligned} \quad (43.13)$$

The product  $\prod_{i>0} \delta\varphi_j$  is the weighted volume element in the discrete functional space. Since it is independent of  $v_j$ , we can use the Fourier transformation with respect to the variables  $v_k$

$$F(\dots \varphi_k \dots) = \int \prod_k (e^{i v_k \varphi_k} dv_k) F(\dots v_k \dots),$$

to conclude that a relation such as (43.13) holds for any function  $F$  of the variables  $\dots \varphi_k \dots$ , i.e.,

$$\langle T [F(\dots \varphi_k \dots)] \rangle_0 = \int F(\dots \varphi_k \dots) \prod_{i>0} \delta\varphi_j. \quad (43.14)$$

We now go back to continuous momentum space by reducing the lattice element  $\Delta p$  and by increasing without limit the region of the four-dimensional momentum space occupied by the lattice. We define the *functional integral* as the corresponding limit of the right-hand sides of (43.13) and (43.14), denoting it by

$$\int \dots \delta\varphi,$$

i.e., instead of (43.13) and (43.14), we write

$$\begin{aligned}\langle T \exp \left( i \int v(p) \varphi(p) dp \right) \rangle_0 &= \exp \left( \frac{i}{2} \int v(p) \Delta^c(p) v(-p) dp \right) = \\ &= \int \exp \left( i \int v(p) \varphi(p) dp \right) \delta\varphi\end{aligned}\quad (43.15)$$

and

$$\langle TF(\varphi) \rangle_0 = \int F(\varphi) \delta\varphi. \quad (43.16)$$

On the left-hand sides of (43.15) and (43.16),  $\varphi$  is an operator, whereas on the right-hand sides it is a  $c$ -function.

We shall now use (43.12) to introduce another form of the functional integral. Let

$$\delta\varphi = \exp(i\mathcal{A}_0(\varphi)) d\varphi, \quad (43.17)$$

where the functional

$$\begin{aligned}\mathcal{A}_0(\varphi) &= -\frac{1}{2} \int \varphi(p) [\Delta^c(p)]^{-1} \varphi(-p) dp = \frac{1}{2} \int \varphi(p) (p^2 - m^2 + i\varepsilon) \varphi(-p) dp = \\ &= \int [\mathcal{L}_0(p) + i\varepsilon \varphi(p) \varphi(-p)] dp\end{aligned}$$

turns out to be equal to the free-field action. Instead of (43.15) and (43.16), we now have

$$\exp \left( \frac{i}{2} \int v(p) \Delta^c(p) v(-p) dp \right) = \int \exp i [\mathcal{A}_0(\varphi) + \int v(p) \varphi(p) dp] d\varphi, \quad (43.18)$$

$$\langle TF(\varphi) \rangle_0 = \int F(\varphi) e^{i\mathcal{A}_0(\varphi)} d\varphi. \quad (43.19)$$

We note that the functional differential  $d\varphi$  introduced above, “includes” the limit of the product of integrals in the denominator on the right-hand side of (43.12). Because of this, for example,

$$\int \exp(i\mathcal{A}_0(\varphi)) d\varphi = \int \delta\varphi = 1. \quad (43.20)$$

The formula given by (43.17) may now be checked directly by means of the linear substitution

$$\varphi(p) = \varphi'(p) + v(-p) \Delta^c(p).$$

The above transition to continuous functional space was carried out in a purely formal fashion. The question of the restrictions that must be imposed on the class of functionals and on the space of functions  $\varphi$  to ensure the convergence of this transition requires separate examination.

However, all this can be avoided by considering (43.15)–(43.20) as *definitions* of functional integrals. These definitions can be interpreted as formulas establishing the connection

between the vacuum chronological expectation value  $\langle T(\dots) \rangle_0$  and the functional average of the corresponding expressions, which depend on the classical  $c$ -fields, over the quantum fluctuations of these fields. The structure of the fluctuations is determined by the classical action.

In this most general *axiomatic* introduction of functional integrals we totally ignore existence questions connected, in the first place, with ultraviolet divergences. Here, we may narrow down the class of quantum-field functionals  $F(\varphi)$  under consideration by confining our attention to the sums of perturbation-theory series. This is, basically, a very serious limitation, but it turns out to be sufficient for the currently considered range of applications, which is restricted by the quasi-Gaussian quadrature of the form given by (43.15). Within the framework of perturbation theory, we can eliminate ultraviolet divergences either by introducing suitable counterterms or by regularization (through a modification of the pairing of  $\Delta^c$ ).

This program of systematic development of the axiomatic approach to the functional integral within the framework of perturbation theory has recently been carried through by Slavnov (1975) who used as a definition the following formula for the quasi-Gaussian integral:

$$\int \exp \left[ i \int \varphi(x) \frac{K(x-y)}{2} \varphi(y) dx dy + i \int \varphi(x) \eta(x) dx \right] \varphi(x_1) \dots \varphi(x_n) d\varphi = \\ = (-i)^n \frac{\delta^n}{\delta \eta(x_1) \dots \delta \eta(x_n)} \exp \left[ -\frac{i}{2} \int \eta(x) K^{-1}(x-y) \eta(y) dx dy \right], \quad (43.21)$$

which generalizes (43.17). Here, the kernel  $K$  is a covariant function which has an inverse defined by

$$\int K(x-z) dz K^{-1}(z-y) = \int K^{-1}(x-z) dz K(z-y) = \delta(x-y).$$

If, moreover,  $K$  or  $K^{-1}$  are not unambiguously defined (the Fourier transform  $\tilde{K}^{-1}(p)$  has a pole in  $p^2$ ), then it must be assumed that the pole is bypassed in accordance with the rule  $p^2 - m^2 + i\epsilon$ .

It can be shown that the object introduced in (43.21) has the algebraic properties of an ordinary integral, namely, (a) it permits integration by parts, and (b) a change of the variables of integration can be carried out.

It can also be shown that multiple integrals exist, the order of integration can be changed, the Fourier transform exists, and there are certain other properties which, at any rate, are sufficient to justify operations involving functional integrals in perturbation theory.

We shall not explicitly formulate and prove all these properties. Readers interested in this question are referred to the original paper by Slavnov (1975). We shall only give the following formula defining the *functional delta function*, which is particularly useful in applications:

$$\int \exp [i \{ f(x|\varphi) - y(x) \} \eta(x)] d\eta = \delta \{ f(x|\varphi) - y(x) \}. \quad (43.22)$$

where  $y(x)$  is a certain explicit function of  $x$ , and  $f(x|\varphi)$  is a certain functional of  $\varphi$ . This definition implies that

$$\int F(\varphi) \exp [i (f(x|\varphi) - y(x)) \eta(x)] \det \left( \frac{\delta f}{\delta \varphi} \right) d\varphi d\eta = F(\tilde{\varphi}), \quad (43.23)$$

where  $\tilde{\varphi}$  is a solution of

$$f(x|\tilde{\varphi}) - y(x) = 0.$$

**43.4. Functional Integral over Fermi Fields.** We now consider the case of Fermi fields. Proceeding by analogy with the foregoing, we consider the vacuum expectation value of the exponential linear functional

$$I(\eta, \bar{\eta}) = \langle T \exp i \int (\bar{\eta}(p) \psi(p) + \bar{\psi}(p) \eta(p)) dp \rangle_0.$$

where  $\psi, \bar{\psi}$  are the quantized Fermi fields whose chronological pairings in momentum representation have the form

$$\overline{\psi(p) \bar{\psi}(q)} = - \overline{\bar{\psi}(q) \psi(p)} = \frac{\delta(p+q)}{i} S^c(p),$$

and  $\eta, \bar{\eta}$  are the "classical anticommuting fields" which may be looked upon as the generators of a Grassman algebra; see §37.2), satisfying the commutation relations given by (37.15).

The vacuum expectation value  $I(\eta, \bar{\eta})$  can be determined by the method used in connection with the analogous quantity (43.2) in the Bose case. The result is

$$\langle T \exp i \int (\bar{\eta} \psi + \bar{\psi} \eta) dp \rangle_0 = \exp \left( i \int d\rho \bar{\eta}(\rho) S^c(\rho) \eta(-\rho) \right). \quad (43.24)$$

To transform from (43.24) to the functional integral, we must first introduce the momentum-space lattice

$$\psi(p_j) = \psi_j, \quad S^c(p_j) = S_j^c, \quad \eta(p_j) \Delta p = \eta_j,$$

and then consistently define the rule for integration in the space of complex anticommuting variables (the space of generators of the Grassman algebra with involution\*).

This space consists of the two sets of variables

$$\chi_1, \dots, \chi_n, \quad \varkappa_1, \dots, \varkappa_n \quad (43.25)$$

\*Such rules were first systematically introduced by Berezin (1961). Our account follows that given in Berezin's book (1965); see also §18 in the book by Konopleva and Popov (1972).

satisfying the commutation relations of Grassman algebra

$$\{\chi_i, \chi_k\}_+ = \{\chi_i, \kappa_k\}_+ = \{\kappa_i, \kappa_k\}_+ = 0. \quad (43.26)$$

So long as we do not require the operation of complex conjugation, we can look upon  $\chi_i$  and  $\chi_k$  as completely independent variables. By virtue of (43.26), the square of any of the generators is zero:

$$\chi_i^2 = \kappa_k^2 = 0.$$

Therefore, in particular, an arbitrary function of the generators (43.25) can be written in the form of a finite polynomial

$$\Phi(\chi, \kappa) = \sum_{(\alpha, \beta = 0, 1)} a(\alpha_1, \dots, \alpha_n; \beta_1, \dots, \beta_n) (\chi_1)^{\alpha_1} \dots (\chi_n)^{\alpha_n} (\kappa_1)^{\beta_1} \dots (\kappa_n)^{\beta_n}. \quad (43.27)$$

We now introduce the symbols for infinitesimal increments

$$d\chi_i, \quad d\kappa_k$$

on the generators (43.25), and subject them to the anticommutation relations

$$\{d\chi_i, d\chi_k\}_+ = \{d\chi_i, d\kappa_k\}_+ = \{d\kappa_i, d\kappa_k\}_+ = 0, \quad (43.28)$$

$$\{\chi_i, d\chi_k\}_+ = \{\chi_i, d\kappa_k\}_+ = \{\kappa_i, d\chi_k\}_+ = \{\kappa_i, d\kappa_k\}_+ = 0. \quad (43.29)$$

Single definite integrals will be defined by

$$\int d\chi_i = \int d\kappa_k = 0, \quad (43.30)$$

$$\int \chi_i d\chi_i = \int \kappa_k d\kappa_k = 1. \quad (43.31)$$

The somewhat unexpected definition given by (43.30) is the only one that is compatible with (43.28). On the other hand, the integrals in (43.31) might have been naturally expressed in terms of the square of the volume of integration. However, we have already shown in §43.3 that this (infinite) volume cancels out in the final results. This explains the choice of normalization in (43.31).

The above definition ensures that the integral of the function (43.27) turns out to be

$$\int \Phi(\chi, \kappa) d\chi_1 \dots d\chi_n d\kappa_1 \dots d\kappa_n = a(1, \dots, 1; 1, \dots, 1). \quad (43.32)$$

It thus turns out that the operations of integration and differentiation are equivalent.

For the integrals over anticommuting fields we can introduce a linear change of variables. Because of the definition given by (43.31), the variables and the differentials are then transformed by mutually inverse matrices:

$$\left. \begin{aligned} \chi_i &= a_{ik} \rho_k, & d\chi_i &= a_{ik}^{-1} d\rho_k, \\ \kappa_i &= b_{ik} \sigma_k, & d\kappa_i &= b_{ik}^{-1} d\sigma_k. \end{aligned} \right\} \quad (43.33)$$

This means that

$$\chi_1 \chi_2 \dots \chi_n = \det a \cdot \rho_1 \dots \rho_n, \quad \kappa_1 \dots \kappa_n = \det b \cdot \sigma_1 \dots \sigma_n$$

and

$$d^n \chi \equiv d\chi_1 \dots d\chi_n = \det a^{-1} d^n \rho, \quad d^n \kappa = \det b^{-1} d^n \sigma,$$

and also

$$\int \Phi(\chi, \kappa) d^n \chi d^n \kappa = \det a^{-1} \cdot \det b^{-1} \int \Phi(a\rho, b\sigma) d^n \rho d^n \sigma. \quad (43.34)$$

Finally, we give the formulas for the integration of the “quasi-Gaussian” exponential

$$\int e^{\kappa A \chi} d^n \chi d^n \kappa = \det A \quad (43.35)$$

and the exponential with linear terms

$$\int \exp(\kappa A \chi + \nu \chi + \eta \kappa) d^n \chi d^n \kappa = \det A \exp(-\nu A^{-1} \eta). \quad (43.36)$$

The formula given by (43.35) can be proved by the linear substitution  $A\chi = \chi'$  which diagonalizes the argument of the exponential, and the formula given by (43.36) can be proved by the following additive transformation of integration variables:

$$\chi = \chi' - A^{-1} \eta, \quad \kappa = \kappa' - \nu A^{-1}.$$

The formula given by (43.36) is similar to (43.13) for the Bose case, and is the basic expression in the introduction of the functional integral over the Fermi field.

If we make the formal substitutions

$$A_{kl} = \delta_{kl} (iS_k^c)^{-1} \Delta p, \quad \kappa_k = \bar{\psi}_k, \quad \chi_i = \psi_i, \quad \eta_k = i\mu_k, \quad \nu_k = i\bar{\mu}_k,$$

and pass to the limit as  $n \rightarrow \infty$ , we find from (43.36) that

$$\begin{aligned} \int \exp \left\{ i \int [\bar{\psi}(p) (\hat{p} - m + i\epsilon) \psi(p) + \bar{\mu}(p) \psi(p) + \bar{\psi}(p) \mu(p)] dp \right\} d\psi d\bar{\psi} = \\ = \exp \left( i \int \bar{\mu}(p) S^c(p) \mu(-p) dp \right). \end{aligned} \quad (43.37)$$

Here the symbols  $d\psi$ ,  $d\bar{\psi}$  are proportional to the limits of the products

$$d\psi \sim \lim_{n \rightarrow \infty} d^n \psi, \quad d\bar{\psi} \sim \lim_{n \rightarrow \infty} d^n \bar{\psi}.$$

Comparison of (43.37) and (43.24) gives

$$\begin{aligned} \langle T \exp \{ i \int [\bar{\eta}(p) \psi(p) + \bar{\psi}(p) \eta(p)] dp \} \rangle_0 &= \\ &= \int \exp \{ i \int [\bar{\eta}(p) \psi(p) + \bar{\psi}(p) \eta(p)] dp \} \delta\psi \delta\bar{\psi}, \end{aligned} \quad (43.38)$$

which is analogous to (43.15). Here, by definition,

$$\delta\psi \delta\bar{\psi} = e^{i\mathcal{A}_0(\psi, \bar{\psi})} d\psi d\bar{\psi}, \quad (43.39)$$

where

$$\mathcal{A}_0(\psi, \bar{\psi}) = \int dp \bar{\psi}(p) (\hat{p} - m) \psi(p) = \int dx \bar{\psi}(x) (i\hat{\partial} - m) \psi(x)$$

is the action corresponding to the free spinor field.

The formula given by (43.38) may be looked upon as a definition of the functional integral over Fermi fields. It tells us that averaging of an operator expression over the fermion vacuum can be represented by a functional integral, i.e., the operation of vacuum averaging can be represented by functional averaging:

$$\langle TF(\psi, \bar{\psi}) \rangle_0 = \int F(\psi, \bar{\psi}) \delta\psi \delta\bar{\psi}. \quad (43.40)$$

#### §44. Generating Functionals and Green's Functions

*44.1. Basic Quantities Expressed in Terms of Functional Integrals.* The formulas for functional averages obtained in the last subsection can be used to obtain "closed" expressions for the basic and higher Green's functions in the form of functional integrals. Such expressions are conveniently established by a unified procedure, using the idea of generating functionals introduced in Chapter VI.

For example, consider the set of interacting fermion and boson fields for which the Lagrangian is

$$\mathcal{L} = \mathcal{L}_0(\psi, \bar{\psi}) + \mathcal{L}_0(\varphi) + \mathcal{L}_1(\psi, \bar{\psi}, \varphi). \quad (44.1)$$

We add source terms to the interaction Lagrangian  $\mathcal{L}_1$  [see (37.8) and (37.14)]

$$\mathcal{L}_{\text{source}} = \eta\psi + \bar{\psi}\bar{\eta} + \varphi J, \quad (44.2)$$

and write the vacuum expectation value of the scattering matrix in the presence of sources

$$S_0(\eta, \bar{\eta}, J) = \langle T \exp i \int (\mathcal{L}_1 + \mathcal{L}_{\text{source}}) dx \rangle_0 \quad (44.3)$$

in the form of a functional integral. Using (43.16) and (43.40) we have

$$S_0(\eta, \bar{\eta}, J) = \int e^{i\mathcal{A}_1(\eta, \bar{\eta}, J)} \delta\psi \delta\bar{\psi} \delta\varphi, \quad (44.4)$$

where

$$\mathcal{A}_1(\eta, \bar{\eta}, J) = \int (\mathcal{L}_1 + \mathcal{L}_{\text{source}}) dx$$

is the action corresponding to the interaction in the given system.

By means of (43.17) and (43.39), the integral in (44.4) can be written in the form

$$S_0(\eta, \bar{\eta}, J) = \int e^{i\mathcal{A}(\eta, \bar{\eta}, J)} d\psi d\bar{\psi} d\varphi. \quad (44.5)$$

where  $\mathcal{A}(\eta, \bar{\eta}, J) = \mathcal{A}_0 + \mathcal{A}_1(\eta, \bar{\eta}, J)$  is the complete (classical) action for the system, corresponding to the sum of the complete physical Lagrangian (44.1) and terms with sources (44.2), whereas  $\mathcal{A}_0$  is the action of free fields.

It was shown in §37 that the functional  $S_0$  is the generating functional for the vacuum expectation values of the form

$$\langle Tu(\xi_1) \dots u(\xi_n) S \rangle_0.$$

The generating functional  $Z$  for connected Green's functions can be obtained for  $S_0$  by taking the logarithm:

$$Z(\eta, \bar{\eta}, J) = -i \ln \left[ \int e^{i\mathcal{A}(\eta, \bar{\eta}, J)} d\psi d\bar{\psi} d\varphi \right]. \quad (44.6)$$

Differentiating (44.6) twice with respect to  $J$ , we find that

$$\begin{aligned} \frac{1}{i} \Delta(x, y | \eta, \bar{\eta}, J) &= \\ &= \frac{\int e^{i\mathcal{A}(\eta, \bar{\eta}, J)} \varphi(x) \varphi(y) d\psi d\bar{\psi} d\varphi}{S_0(\eta, \bar{\eta}, J)} - \Phi(x | \eta, \bar{\eta}, J) \Phi(y | \eta, \bar{\eta}, J), \end{aligned} \quad (44.7)$$

$$\Phi(x | \eta, \bar{\eta}, J) = \frac{\int e^{i\mathcal{A}(\eta, \bar{\eta}, J)} \varphi(x) d\psi d\bar{\psi} d\varphi}{S_0(\eta, \bar{\eta}, J)} \quad (44.8)$$

which is the representation for the one-particle meson Green's function in the presence of sources in the form of a functional integral. In the limit  $\eta = \bar{\eta} = J = 0$ , instead of (44.7) we have

$$\Delta(x, y) = i \frac{\int e^{i\mathcal{A}} \varphi(x) \varphi(y) d\psi d\bar{\psi} d\varphi}{\int e^{i\mathcal{A}} d\psi d\bar{\psi} d\varphi}, \quad (44.9)$$

where

$$\mathcal{A} = \mathcal{A}(\eta = 0, \bar{\eta} = 0, J = 0)$$

is the complete action of the system in the absence of sources.



On the other hand, differentiating the general functional  $Z$  with respect to  $\eta(x)$  and  $\bar{\eta}(y)$ , we obtain (on substituting  $\eta = \bar{\eta} = J = 0$ ) the one-particle fermion Green's function in the absence of external sources

$$G(x, y) = i \frac{\int e^{i\mathcal{L}} \psi(x) \bar{\psi}(y) d\psi d\bar{\psi} d\varphi}{\int e^{i\mathcal{L}} d\psi d\bar{\psi} d\varphi}. \quad (44.10)$$

Similarly, for a weakly connected three-vertex function,

$$\Delta_3(x, y|z) = \frac{\int e^{i\mathcal{L}} \psi(x) \bar{\psi}(y) \varphi(z) d\psi d\bar{\psi} d\varphi}{\int e^{i\mathcal{L}} d\psi d\bar{\psi} d\varphi}. \quad (44.11)$$

*44.2. Representations for Green's Functions in the Form of Functional Bose Integrals.* Formulas of the form of (44.9)–(44.11) contain functional quadratures over both Bose and Fermi fields. If we evaluate (symbolically) the functional integrals over the Fermi fields, we can obtain expressions for Green's functions that may be looked upon as averages of the corresponding Green's functions for particles moving in a classical external field  $\varphi$ , evaluated over the quantum fluctuations of this field.

To do this, we note that the integration over the Fermi arguments  $\psi, \bar{\psi}$  in the numerators and denominators of (44.9)–(44.11) can be carried out first. According to (43.20), this integration reduces to averaging over the Fermi-operator vacuum:

$$\int F(\psi, \bar{\psi}, \varphi) \delta\psi \delta\bar{\psi} = \langle TF(\psi, \bar{\psi}, \varphi) \rangle_{F_0}. \quad (44.12)$$

Here the symbol  $\langle \dots \rangle_{F_0}$  represents the evaluation of the vacuum expectation value only over the Fermi operators  $\bar{\psi}, \psi$ . The Bose arguments  $\varphi$  are regarded as classical functions in the procedure.

To perform the Fermi-averaging operation, it is convenient to specify explicitly the dependence of the interaction Lagrangian on the Fermi fields. We shall substitute

$$\mathcal{L}_1(x) = g\bar{\psi}(x) \Gamma \psi(x) K(\varphi) + M(\varphi), \quad (44.13)$$

where  $\Gamma$  is a vertex function and  $K$  and  $M$  depend only on the Bose field  $\varphi$ . (The more general case of the four-fermion interaction of the Fermi type (8.13) can also be included in this analysis.)

Using (44.12) in the numerator and the denominator of (44.10), and taking (44.13) into account, we obtain

$$G(x, y) = i \frac{\int e^{i \int M(\varphi) dx} \delta\varphi \langle T\psi(x) \bar{\psi}(y) e^{ig \int \bar{\psi} \Gamma \psi K(\varphi) dx} \rangle_{F_0}}{\int e^{i \int M(\varphi) dx} \delta\varphi \langle T \exp(ig \int \bar{\psi} \Gamma \psi K(\varphi) dx) \rangle_{F_0}}. \quad (44.14)$$

On the other hand, the ordinary Green's function for a single fermion interacting with a classical external field  $\varphi_{cl}$  with

$$\mathcal{L}_{\text{int}} = g \bar{\psi} \Gamma \psi K(\varphi_{\text{cl}}),$$

has the form

$$G(x, y | \varphi, g) = i \frac{\langle T \bar{\psi}(x) \bar{\psi}(y) e^{ig \int \bar{\psi} \Gamma \psi K(\varphi) dx} \rangle_{F_0}}{S_0(\varphi, g)}, \quad (44.15)$$

where

$$S_0(\varphi, g) = \langle T \exp \left( ig \int \bar{\psi}(z) \Gamma \psi(z) K(\varphi(z)) dz \right) \rangle_0 \quad (44.16)$$

can be evaluated in an explicit form. Differentiating  $S_0(\varphi, g)$  with respect to  $g$ , and using (44.15), we have

$$\begin{aligned} \frac{\partial S_0(\varphi, g)}{\partial g} &= i \langle T \int dx \bar{\psi} \Gamma \psi K(\varphi(x)) \exp \left( ig \int \bar{\psi} \Gamma \psi K(\varphi) dz \right) \rangle_{F_0} = \\ &= - \int SP \{ G(x, x | \varphi, g) \Gamma K(\varphi(x)) \} dx \cdot S_0(\varphi, g). \end{aligned}$$

The symbol  $SP$  in this expression represent summation over both spinor and all possible boson indices. Integrating the resulting first-order differential equation subject to the boundary condition  $S_0(\varphi, 0) = 1$ , we obtain

$$S_0(\varphi, g) = \exp \left( - \int_0^g dg' \int dx SP \{ G(x, x | \varphi, g') \Gamma K(\varphi(x)) \} \right). \quad (44.17)$$

The expression given by (44.14) can be transformed with the aid of (44.15) and (44.17) to the form

$$G(x, y) = \int G(x, y | \varphi, g) D\varphi, \quad (44.18)$$

i.e., Green's function for one fermion interacting with the quantized Bose field can be represented by a functional average of ordinary Green's function for the fermion interacting with the classical Bose field. Effects associated with the quantum-mechanical self-action of the Bose field, described by the function  $M(\varphi)$ , and also the vacuum fluctuations, are referred in the above expression to the integration measure

$$D\varphi = \frac{e^{i \int M(\varphi) dx} S_0(\varphi, g) \delta\varphi}{\int e^{i \int M(\varphi) dx} S_0(\varphi, g) \delta\varphi}. \quad (44.19)$$

Thus, for example, the electron Green's function in spinor electrodynamics can be written in the form

$$G(x, y) = \frac{\int G(x, y | A) S_0(A) \delta A}{\int S_0(A) \delta A}, \quad (44.20)$$

where

$$S_0(A) = \langle T \exp \left( i e \int \bar{\psi}(z) \hat{A}(z) \psi(z) dz \right) \rangle_0. \quad (44.21)$$

Formulas analogous to (44.18) can be obtained for other Green's functions. For example, the boson Green's function (44.9) can be written in the form

$$\Delta(x, y) = i \int \varphi(x) \varphi(y) D\varphi \quad (44.22)$$

and so on.

From the formal point of view, the problem of finding the one-particle quantum-field Green's functions  $G$  and  $\Delta$  has been reduced to the determination of Green's function  $G(x, y | \varphi_{cl})$  for one fermion moving in a classical external field  $\varphi_{cl}$ . Here, however, we must emphasize that even if we succeed in obtaining a closed expression for  $G(x, y | \varphi)$  in some particular approximation, which, in general, is a very complicated problem, we still have to cope with the functional quadratures in (44.18), (44.20), and (44.22), and these are not simple. Nevertheless, there is at least one physically important model that is sufficiently simple in mathematical structure, in which the formulas obtained in this subsection can be effectively employed. This is the Bloch-Nordsieck model considered in §46. As already noted, applications of the method of functional integration are seriously restricted by the fact that there are practically no techniques for evaluating functional quadratures. The only form of functional integrals that we can evaluate is the Gaussian quadrature, or other integrals that can be reduced to it by a functional change of variable of integration or variational differentiation with respect to a parameter.

Because of these rather formidable difficulties, functional integrals were practically ignored in modern quantum field theory for a very long time. However, they attracted renewed interest toward the end of the 1960s when it became clear that the method of functional integrals is very convenient in connection with gauge fields (see §8.2) because of the singular character of the corresponding Lagrangians (see §11.4). Faddeev and Popov (1967a, b) used this method of integration in the process of quantization, and Slavnov (1972a, b) developed a perturbation-theory renormalization procedure for Yang-Mills fields. Applications of the functional integral are convenient in this connection because this integral provides a compact method for writing out combinatorial derivations. In particular, transformation to a different basis can be performed by changing the variables of integration.

We shall illustrate these applications by the example of the simplest Abelian gauge field, namely, the electromagnetic field, interacting with fermions. Generalized Ward's identities will be obtained within the framework of spinor electrodynamics (they were obtained previously from perturbation theory in §33), and the structure of transformations of complete Green's functions under the gauge transformation of electromagnetic potentials will be investigated.

## §45. Gauge Transformations In Spinor Electrodynamics

**45.1. Functional Integral in Arbitrary Gauge.** The procedure for introducing the functional integral is simplest in the diagonal Feynman gauge of the electromagnetic potential,

which we used in connection with quantization in §12, because the free Lagrangian in this gauge is the covariant sum of Lagrangians for the individual components

$$\mathcal{L}_0(A_n) = \mathcal{L}_0(A_0, A_1, A_2, A_3) = \sum_{m,n} -\frac{1}{2} \partial_n A_m \partial^n A^m, \quad (45.1)$$

whereas the differential operator in the action is the inverse of chronological pairing (24.3)

$$\mathcal{A} = \int \mathcal{L}_0(A_n) dx = \int dx A^n(x) [D^c(x-y)]_{nm}^{-1} A^m(y) dy. \quad (45.2)$$

Because of (45.1), the functional weight can be factorized

$$\delta A = \delta A_0 \delta A_1 \delta A_2 \delta A_3,$$

and we can define the operation of functional integration with respect to the four-component field  $A_n$  as the product of the corresponding operations with respect to each of the components. By virtue of (45.2), this definition is compatible with the basic formulas such as (43.18) and (43.21) which, in the present case, assume the form

$$\begin{aligned} \exp \left( -\frac{i}{2} \int v^n(k) D_{nm}^c(k) v^m(-k) dk \right) &= \int \exp \left( i \int v^n(k) A_n(k) dk \right) \delta A = \\ &= \int \exp \left( i \int dk \{ A^n(k) [D^c(k)]_{nn}^{-1} A^n(-k) + v^n(k) A_n(k) \} \right) dA. \end{aligned} \quad (45.3)$$

Instead of (43.19) we also have

$$\langle TF(A) \rangle_0 = \int F(A) e^{i\mathcal{A}_0(A)} dA = \int F(A) \delta A. \quad (45.4)$$

Thus, for example, the generating functional for connected Green's functions in spinor electrodynamics can be written in the form

$$Z(\eta, \bar{\eta}, J^m) = -i \ln \left\{ \int e^{i\mathcal{A}(\eta, \bar{\eta}, J^m)} d\psi d\bar{\psi} dA \right\}, \quad (45.5)$$

where  $A$  is the total action corresponding to the Lagrangian with sources

$$\mathcal{L} = \mathcal{L}_0(\psi, \bar{\psi}) + \mathcal{L}_0(A_n) + c\bar{\psi}\hat{A}\psi + \bar{\psi}\eta + \bar{\eta}\psi + A_n J^n, \quad (45.6)$$

and the free Lagrangian for the electromagnetic field,  $\mathcal{L}_0(A_n)$ , has the form given by (45.1).

In order to go over to an arbitrary gauge of the electromagnetic potentials we must use the gauge transformation discussed in §34.1.

Let

$$A'_n(k) = A_n(k) + k_n \beta \frac{(kA(k))}{k^2}, \quad (45.7)$$

where the right-hand side contains the potentials in the diagonal gauge. The chronological pairing of the operators  $A'_n$  is

$$\overline{A'_n(k) A'_m(k')} = \frac{\delta(k+k')}{ik^2} \left\{ \left( g_{nm} - \frac{k_n k_m}{k^2} \right) + d_l \frac{k_n k_m}{k^2} \right\}, \quad (45.8)$$

where

$$d_l = (1 + \beta)^2.$$

At the same time,

$$(kA'(k)) = (1 + \beta)(kA(k)) \quad (45.9)$$

and, consequently,

$$A_n(k) = A'_n(k) - \frac{\beta}{1 + \beta} k_n \frac{(kA'(k))}{k^2}. \quad (45.10)$$

The transformed Lagrangian will therefore be

$$\begin{aligned} \mathcal{L}_0(A) &= -\frac{1}{2} A_m(k) k^2 A^m(-k) = \\ &= -\frac{1}{2} A'_m(k) (g^{mn} k^2 - k^m k^n) A'_n(-k) - \frac{1}{2d_l} (kA'(k)) (kA'(-k)) \equiv \mathcal{L}_0(A', d_l). \end{aligned} \quad (45.11)$$

Correspondingly, in the configuration representation,

$$\mathcal{L}_0(A, d_l) = -\frac{1}{4} H_{mn} H^{mn} - \frac{1}{2d_l} (\partial A)^2 = \mathcal{L}_0^{\text{tr}}(A) - \frac{1}{2d_l} (\partial A)^2. \quad (45.12)$$

The first term of the right-hand side is equal to the gauge-invariant singular Lagrangian (5.13). The second term, which contains  $d_l$ , establishes the gauge. By substituting  $d_l = 1$ , we return to the diagonal expression (45.1).

It is readily verified that the kernel of the operator  $\mathcal{L}_0(A, d_l)$  is the inverse of the chronological pairing (34.5). Transition to arbitrary gauge in the basic formula (45.3) can therefore be achieved by replacing  $\mathcal{L}_0(A)$  by  $\mathcal{L}_0(A, d_l)$  on the right-hand side of the Lagrangian:

$$\begin{aligned} \exp \left( -\frac{i}{2} \int v^n(k) D_{mn}^c(k|d_l) v^m(-k) dk \right) = \\ = \int \exp \left( i \int dk \{ A^n(k) [D^c(k|d_l)]_{nm}^{-1} A^m(-k) + v^n(k) A_n(k) \} \right) dA. \end{aligned} \quad (45.13)$$

This formula gives the functional integral with respect to the electromagnetic fields in arbitrary gauge. Substituting

$$e^i \int \mathcal{L}_0(x|d_t) dx | dA = \delta A(d_t), \quad (45.14)$$

we have

$$\exp \left( -\frac{i}{2} \int v^n(k) D_{nm}^e(k|d_t) v^m(-k) dk \right) = \int e^i \int v^n(k) A_n(k) dk \delta A(d_t). \quad (45.15)$$

Correspondingly, transition to arbitrary gauge in the generating functional (45.5) can be achieved by replacing the action

$$\mathcal{A}(\eta, \bar{\eta}, J) = \mathcal{A}(\eta, \bar{\eta}, J | d_t = 1),$$

corresponding to the Lagrangian in (45.6) by the action

$$\mathcal{A}(\dots | d_t) = \int \mathcal{L}(\dots | d_t) dx, \quad (45.16)$$

in which

$$\mathcal{L}(\dots | d_t) = \mathcal{L}_0(\psi, \bar{\psi}) + \mathcal{L}_0^{\text{tr}}(A) + e\bar{\psi}\hat{A}\psi + \frac{1}{2d_t}(\partial A)^2 + \bar{\psi}\eta + \bar{\eta}\psi + A_n J^n. \quad (45.17)$$

Therefore

$$Z(\eta, \bar{\eta}, J_n | d_t) = -i \ln S_0(\eta, \bar{\eta}, J_n | d_t), \quad (45.18)$$

$$S_0(\eta, \bar{\eta}, J_n | d_t) = \int e^{i\mathcal{A}(\dots | d_t)} d\psi d\bar{\psi} dA. \quad (45.19)$$

Using (43.39) and (45.14), we can also represent  $S_0$  in the form

$$S_0(\eta, \bar{\eta}, J_m | d_t) = \int e^i \int [e\bar{\psi}\hat{A}\psi + \bar{\psi}\eta + \bar{\eta}\psi + JA] dx \delta\psi \delta\bar{\psi} \delta A(d_t). \quad (45.20)$$

Substituting (45.20) in (37.27), and using the reduction formula (38.26), we obtain the expression for the matrix element of the  $S$ -matrix in the form of a variational derivative of  $\ln S_0$ , projected in the appropriate fashion onto momentum space. The differentiation and projection formulas have the following structure.

The operation

$$\lim_{\hat{p}=m} \frac{v^{+-}(p)}{(2\pi)^{3/2}} (m - \hat{p}) \int e^{-ipy} dy \frac{\delta}{\delta \eta(y)}, \quad (45.21)$$

corresponds to an electron in the initial state with momentum  $p$ , and

$$\lim_{\hat{p}=m} \frac{v^{\nu,+}(p)}{(2\pi)^{1/2}} (m - \hat{p}) \int e^{ipx} dx \frac{\delta}{\delta \eta(x)}. \quad (45.21')$$

corresponds to a positron in the final state with momentum  $p$ . The operation

$$\lim_{\hat{p}=m} \frac{\bar{v}^{\nu,\pm}(p)}{(2\pi)^{1/2}} (m - \hat{p}) \int e^{\pm ipx} dx \frac{\delta}{\delta \eta(x)} \quad (45.22)$$

corresponds to an electron in the final (positron in the initial) state with momentum  $p$ .

Finally, the operator

$$\lim_{k^2=0} \frac{e_v^\alpha(k)}{(16\pi^3 k^0)^{1/2}} (-k^2) \int e^{\mp ikx} dx \frac{\delta}{\delta J^\alpha(x)} \quad (43.25)$$

corresponds to a photon in the initial (final) state with momentum  $k$  and polarization  $e_\nu$ .

**45.2. Gauge Invariance of the Scattering Matrix.** We shall now show the gauge invariance of the  $S$ -matrix. We shall take the functional  $S_0$  in the arbitrary gauge (45.19) and change the integration variables with unit Jacobian:

$$\left. \begin{aligned} A_\alpha &\rightarrow A_\alpha + \partial_\alpha \varphi, & \square \varphi(x) &= (1 + \beta)(\partial A), \\ \psi &\rightarrow \psi e^{ie\varphi(x)}, & \bar{\psi} &\rightarrow \bar{\psi} e^{-ie\varphi(x)} \end{aligned} \right\} \quad (45.24)$$

where  $\beta$  is a numerical parameter.

The generating functional (45.19) will assume the form

$$S_0(\dots J) = \int e^{i \int \mathcal{L}(d_i, \beta) dx} d\psi d\bar{\psi} dA, \quad (45.25)$$

where

$$\begin{aligned} \mathcal{L}(d_i, \beta) = & \mathcal{L}_0(\psi, \bar{\psi}) + \mathcal{L}_0^{\text{tr}}(A) + e\bar{\psi}\hat{A}\psi + \frac{\beta^2}{2d_i}(\partial A)^2 + J^\alpha(x)(A_\alpha + \partial_\alpha \varphi) + \\ & + \bar{\psi}(x)\eta(x)e^{-ie\varphi(x)} + \bar{\eta}(x)\psi(x)e^{ie\varphi(x)}. \end{aligned} \quad (45.26)$$

According to the reduction formula, the matrix elements of the  $S$ -matrix are expressed in terms of the variational derivatives of  $S_0$  multiplied by the projection operators (45.21)–(45.23). Variation of the functional  $S_0$ , which depends on the Lagrangian (45.26), will be accompanied by a change in the variational derivatives:

$$\frac{1}{i} \frac{\delta}{\delta J^\alpha(x)} \sim A_\alpha(x) \rightarrow A_\alpha(x) + \partial_\alpha \varphi, \quad (45.27)$$

$$\left. \begin{aligned} \frac{1}{i} \frac{\delta}{\delta \bar{\eta}(x)} &\sim \psi(x) \rightarrow \psi(x) + \psi(x) [e^{ie\varphi(x)} - 1], \\ \frac{1}{i} \frac{\delta}{\delta \eta(x)} &\sim \bar{\psi}(x) \rightarrow \bar{\psi}(x) + \bar{\psi}(x) [e^{-ie\varphi(x)} - 1]. \end{aligned} \right\} \quad (45.28)$$

Substituting the right-hand side of (45.26) in (45.23), we find that since real photons are transverse,  $e_\nu^n(k) \partial_n \sim e_\nu^n(k) k_n = 0$  and the second term in (45.26) does not contribute to the matrix elements. It is readily shown that the additions to the right-hand side of (45.27) after the application of the projection operators (45.21) and (45.22) also provide no contribution to the matrix elements. This occurs because the factors

$$(m - \hat{p}) = [S^c(p)]^{-1} = S_c^{-1}(p),$$

in these operators, when they act on linear objects of the form  $\psi, \bar{\psi}$ , mutually cancel out with the one-particle pairings in accordance with the scheme

$$\begin{aligned} \int S_c^{-1}(y-x) dx \frac{\delta S_0}{\delta \bar{\eta}(x)} &= i \int S_c^{-1}(y-x) dx \langle T(\psi(x) S) \rangle_0 = \\ &= i \int S_c^{-1}(y-x) dx \overline{\psi(x)} \psi(z) dz \frac{\delta S_0}{\delta \bar{\psi}(z)} = \\ &= \int S_c^{-1}(y-x) dx S_c(x-z) dz \frac{\delta S_0}{\delta \psi(z)} = \frac{\delta S_0}{\delta \psi(y)}. \end{aligned}$$

When they act on nonlinear objects such as

$$\bar{\psi}(x) [e^{ie\varphi(x)} - 1] = i \bar{\psi}(x) \varphi(x) + \dots$$

this compensation occurs only for the weakly connected diagrams, so that the contribution of the second terms on the mass surface  $\hat{p} = m$  is in effect reduced to the renormalization of the external lines.

Thus, when the matrix elements are evaluated, the expression given by (45.26) can be effectively replaced by

$$\mathcal{L}(d_i) = \mathcal{L}_0(\psi) + \mathcal{L}_0^{\text{tr}}(A) + c \bar{\psi} \hat{A} \psi + \frac{1}{2d_i'} (\partial A)^2 + \bar{\psi} \eta + \bar{\eta} \psi + J A, \quad (45.29)$$

where

$$\frac{1}{d_i'} = \frac{\beta^2}{d_i}. \quad (45.30)$$

We have thus established that when the matrix elements are evaluated, we can transform from  $d_l$  to  $d_l'$  by changing the integration variables, i.e., by means of an operation which does



not effect  $S_0$  and hence leaves the values of the matrix elements unaltered. We have therefore also shown that the matrix elements are independent of  $d_I$ , and this completes our proof of the gauge invariance of the scattering matrix.

It is, however, important to note that this statement is valid only for the renormalized scattering matrix because (as already noted) transformation from one gauge to another leads to an additional renormalization of the external lines.

**45.3. Generalized Ward's Identities.** We now introduce a linear change of variables in the functional integral on the right-hand side of (45.19)

$$\begin{aligned}\psi(x) &= \psi'(x) e^{ieI(x)}, & \bar{\psi}(x) &= \bar{\psi}'(x) e^{-ieI(x)}, \\ A_n(x) &= A'_n(x) + \partial_n f(x)\end{aligned}\quad (45.31)$$

which is of the form given by (8.17). It was shown in §8.2 that the sum of the first three terms on the right-hand side of (45.17) [which is equal to the Lagrangian (8.16)] is invariant under (45.31). It is also obvious that because the transformations of  $\psi$  and  $\bar{\psi}$  are mutually reciprocal, and the transformation of  $A$  is additive, the product of the differentials is also invariant:

$$d\psi d\bar{\psi} dA = d\psi' d\bar{\psi}' dA'.$$

Variation of the functional (45.19) under the transformation (45.31) is therefore entirely due to the last four terms on the right-hand side of (45.17). Assuming that  $f(x)$  is infinitesimal, we write the variation of the action  $\mathcal{A}(\eta, \bar{\eta}, J_m | d_I)$  in the form

$$\begin{aligned}\delta\mathcal{A}(\eta, \bar{\eta}, J_m | d_I) &= \int dx \left\{ \frac{1}{d_I} \partial^\nu \hat{A}_\nu \partial^\mu f(x) + ie(\bar{\eta}\psi - \bar{\psi}\eta) f(x) + J^\pi(x) \partial_\pi f(x) \right\}, \\ \frac{\delta\mathcal{A}}{\delta f(x)} &= \left\{ -\frac{1}{d_I} \square(\partial A(x)) + ie(\bar{\eta}\psi - \bar{\psi}\eta) - (\partial J(x)) \right\}.\end{aligned}$$

The variation of the functional  $S_0$  will be

$$\frac{\delta S_0}{\delta f(x)} = i \int e^{i\mathcal{A}(\eta, \bar{\eta}, J_m | d_I)} \frac{\delta\mathcal{A}(\eta, \bar{\eta}, J_m | d_I)}{\delta f(x)} d\psi d\bar{\psi} dA.$$

However, the functional  $S_0$  should remain unaltered under the change of the integration variables. The variation  $\delta S_0$  must therefore be zero and, consequently,

$$\begin{aligned}\int e^{i\mathcal{A}(\eta, \bar{\eta}, J_m | d_I)} d\psi d\bar{\psi} dA \left\{ -\frac{\square(\partial A(x))}{d_I} + \right. \\ \left. + ie(\bar{\eta}(x)\psi(x) - \bar{\psi}(x)\eta(x)) - (\partial J(x)) \right\} = 0.\end{aligned}\quad (45.32)$$

This is the generating relationship for generalized Ward's identities (sometimes also referred to as the *Ward-Takahashi identities*).

Thus, differentiating this with respect to  $J^m(y)$  and then substituting  $\eta = \bar{\eta} = J_m = 0$ , we find that

$$\int e^{i\mathcal{A}(0|d)} \left[ \frac{i}{d_l} \square_x \partial_x^n A_n(x) A_m(y) + \partial_m \delta(x-y) \right] d\psi d\bar{\psi} dA = 0,$$

or

$$i \langle T \square_x \partial_x^n A_n(x) A_m(y) S \rangle_0 + d_l \partial_m \delta(x-y) = 0. \quad (45.33)$$

Since the operations of differentiation  $\square_x \partial_x^n$  and functional integration commute in (45.32), we may suppose that  $\square \partial$  commutes with the symbols  $\langle T \dots \rangle_0$  in (45.33) as well. In other words, the symbol  $T$  in (45.33) must be interpreted as the Wick chronological product introduced in §15.3 [see (15.26)]. Therefore

$$\square \partial^n \langle T A_n(x) A_m(0) S \rangle_0 = i d_l \partial_m \delta(x). \quad (45.34)$$

This formula corresponds to the absence of radiative corrections in the longitudinal part of electromagnetic pairing, i.e., the fact that all the radiative corrections are transverse:

$$D_{mn}(x) = \frac{1}{iS_0} \langle T A_m(x) A_n(0) S \rangle_0 = \frac{1}{(2\pi)^4} \int e^{-ikx} \tilde{D}_{mn}(k) dk, \quad (45.35)$$

$$\tilde{D}_{mn}(k) = \frac{d(k^2, e^2)}{-k^2 + ie} \left( g_{mn} - \frac{k_m k_n}{k^2} \right) - \frac{d_l}{k^2} \frac{k_m k_n}{k^2}, \quad (45.36)$$

so that

$$\square \partial^n D_{nm}(x) = -d_l \partial_m \delta(x). \quad (45.37)$$

Differentiating the generating functional relation (45.32) with respect to  $\eta(y)$  and  $\bar{\eta}(z)$ , and going to the limit  $\eta = \bar{\eta} = J = 0$ , we obtain

$$\int e^{i\mathcal{A}(0|d)} \left[ \psi(z) \bar{\psi}(y) \frac{\square (\partial A(x))}{d_l} + \right. \\ \left. + e\psi(z) \bar{\psi}(x) \delta(x-y) - e\psi(z) \bar{\psi}(y) \delta(x-z) \right] d\psi d\bar{\psi} dA = 0,$$

or

$$\square_x \partial_x^m \langle T \psi(z) \bar{\psi}(y) A_m(x) S \rangle_0 = \\ = e d_l \langle T \psi(z) \bar{\psi}(y) S \rangle_0 \delta(x-z) - e d_l \langle T \psi(z) \bar{\psi}(y) S \rangle_0 \delta(x-y). \quad (45.38)$$

This establishes the connection between the longitudinal (in the photon end) component

of the weakly connected three-vertex function and the complete one-particle Green's functions. If we now introduce the strongly connected vertex  $\Gamma$  through a formula analogous to (37.34)

$$\langle T\psi(z)\bar{\psi}(y)A_m(x)S \rangle_0 = ieS_0 \int D_{mn}(x-x') dx' G(z-z') dz' \Gamma^n(z', y' | x') dy' G(y'-y), \quad (45.39)$$

and use (45.37), we obtain

$$\int G(z-z') dz' \frac{\partial}{\partial x^n} \Gamma^n(z', y' | x) dy' G(y'-y) = [\delta(x-z) - \delta(x-y)] G(z-y).$$

Finally, transforming to the momentum representation

$$\Gamma^n(z, y | x) = \frac{1}{(2\pi)^3} \int e^{i(kx + qy - pz)} \delta(p - q + k) \tilde{\Gamma}^n(p, q; k) dp dq dk, \quad (45.40)$$

we obtain

$$G(p) k_n \tilde{\Gamma}^n(p, p+k; k) G(p+k) = G(p+k) - G(p)$$

or

$$k_n \tilde{\Gamma}^n(p, p+k; k) = G^{-1}(p) - G^{-1}(p+k), \quad (45.41)$$

which is identical with (34.28).

Similarly, we can obtain the connection between the higher Green's functions. For example, differentiating the generating functional relation (45.32) with respect to  $\eta(y)$ ,  $\bar{\eta}(z)$ , and  $J_m(t)$ , we obtain the Ward-Takahashi identity, which connects the four-vertex function of the Compton type with the lower vertices, and so on.

This "second" Ward's identity turns out to be important in scalar electrodynamics because of the divergent nature of the Compton-type diagrams.

**45.4. Transition to the Transverse Gauge.** This transition is difficult to perform directly in the expressions given by (45.9), (45.10), (45.13), (45.15), and (45.19) because in the limit as  $d_I = 0$ , the term in the Lagrangian  $\mathcal{L}(\dots | d_I)$  which sets the gauge becomes infinite. We shall therefore introduce a special procedure, taking (45.15) as our starting point.

Consider the specialized gauge transformation of the electromagnetic-field operators, which separates out the transverse part

$$A_n(k) = A_n^{\text{tr}}(k) + k_n f(k). \quad (45.42)$$

Here  $A_n$  is the potential in arbitrary gauge and  $A^{\text{tr}}$  is the transverse potential which, by definition, satisfies the condition

$$(k A^{\text{tr}}(k)) = 0. \quad (45.43)$$

The chronological pairing of  $A^{\text{tr}}$  has the form

$$\overline{A_n^{\text{tr}}(k) A_m^{\text{tr}}(k')} = \frac{\delta(k+k')}{ik^2} \left( g_{nm} - \frac{k_n k_m}{k^2} \right).$$

The second term in (45.42) corresponds to the longitudinal gauge and, in accordance with (45.43), is given by

$$f(k) = \frac{k A(k)}{k^2}.$$

The important point for us is that the cross-pairing of the longitudinal and transverse components in (45.42) is zero:

$$\overline{A_n^{\text{tr}}(k) f(k')} = 0, \quad (45.44)$$

and that the mutual pairing of the operators  $f$  can be written in the form

$$\overline{f(k) f(k')} = \frac{(k A(k)) (k' A(k'))}{k^2 k'^2} = i \delta(k+k') F(k^2), \quad (45.45)$$

where

$$F(k^2) = \frac{d_I}{k^2 k^2}. \quad (45.46)$$

We now introduce a functional change of variables on the left-hand side of (45.15) which correspond to (45.42):

$$A_n(k) = A_n^{\text{tr}}(k) + k_n f(k). \quad (45.42')$$

If we take into account the separation of functional quadratures resulting from (45.44), we obtain

$$\begin{aligned} \int \exp \left( i \int v^n(k) A_n(k) dk \right) \delta A(d_I) &= \int \exp \left( i \int v^n(k) A_n^{\text{tr}}(k) dk \right) \delta A^{\text{tr}} \times \\ &\times \int \exp \left( i \int v^n(k) k_n f(k) dk \right) \delta f. \end{aligned} \quad (45.47)$$

Since

$$\begin{aligned} \int v^n(k) D_{nm}^{\varepsilon}(k | d_I) v^m(-k) dk &= \\ &= \int v^n(k) D_{nm}^{\text{tr}}(k) v^m(-k) dk + \int v^n(k) k_n F(k^2) k_m v^m(-k) dk. \end{aligned}$$

(45.47) yields the following two relationships:

$$\int \exp \left( i \int \mathbf{v}^n(k) A_n^{\text{tr}}(k) dk \right) \delta A^{\text{tr}} = \exp \left( -\frac{i}{2} \int \mathbf{v}^n(k) D_{nm}^{\text{tr}}(k) \mathbf{v}^m(-k) dk \right), \quad (45.48)$$

$$\int \exp \left( i \int \xi(k) f(k) dk \right) \delta f = \exp \left( \frac{i}{2} \int \xi(k) F(k^2) \xi(-k) dk \right), \quad (45.49)$$

where

$$k_n \mathbf{v}^n = \xi.$$

The relation given by (45.48) may be looked upon as the definition of the functional integral with respect to the transverse electromagnetic potential  $A^{\text{tr}}$ , whereas (45.49), in which  $F$  on the right-hand side is defined in accordance with (45.46), can be regarded as the definition of the functional integral with respect to the longitudinal component.

*45.5. Gauge Transformation of Green's Functions.* We now consider expressions of the form

$$\int e^{i \int \mathcal{L}(0|d_i) dx} \prod_1^f \psi(x_k) \bar{\psi}(y_k) \prod_1^b A_{n_j}(z_j) dA d\psi d\bar{\psi}, \quad (45.50)$$

which can be obtained from (45.19) by suitable variational differentiation. It corresponds to the higher weakly connected Green's functions with  $b$  photon and  $2f$  electron external lines. Here

$$\mathcal{L}(0|d_i) = \mathcal{L}_0(\psi, \bar{\psi}) + \mathcal{L}_0^{\text{tr}}(A) + e\bar{\psi}\hat{A}\psi + \frac{1}{2d_i}(\partial A)^2.$$

We now introduce a phase change of fermion variables of integration in (45.42)

$$\psi(x) = \psi'(x) e^{ief(x)}, \quad \bar{\psi}(x) = \bar{\psi}'(x) e^{-ief(x)},$$

such that

$$\square f(x) = -(\partial A). \quad (45.51)$$

The Lagrangian now assumes the form

$$\mathcal{L}(0|d_i) = \mathcal{L}_0(\psi', \bar{\psi}') + \mathcal{L}_0^{\text{tr}}(A) + e\bar{\psi}'(\hat{A} - \hat{\partial}f)\psi' + \frac{1}{2d_i}(\partial A)^2.$$

The next step is to separate the longitudinal and transverse components as in (45.42):

$$A_n(x) = A_n^{\text{tr}}(x) + \partial_n f(x).$$

By virtue of (45.51), the function  $f$  in this expression and in the phase transformation  $\psi \rightarrow \psi'$  is the same. Therefore, using (45.47) and omitting the prime on the fermion variables, we rewrite (45.50) as follows:

$$\int dA^{\text{tr}} d\psi d\bar{\psi} e^{i \int \mathcal{L}^{\text{tr}}(x) dx} \prod_1^f \psi_k(x_k) \bar{\psi}_k(y_k) \times \\ \times \int df e^{\frac{i}{2d_l} \int [\square f(x)]^2 dx + ie \sum_k (f(x_k) - f(y_k))} \prod_1^b (A_{n_j}^{\text{tr}}(z_j) + \partial_{n_j} f(z_j)).$$

It is clear that the integral with respect to  $f$  reduces to (45.49) and its derivatives with respect to  $\xi(k)$ . It can be evaluated explicitly.

As an illustration, consider the simple case of one-electron Green's function ( $f = 1$ ,  $b = 0$ ). The formula for the transformation from an arbitrary gauge to the transverse gauge can be factorized

$$G(x, y | d_l) = J(x, y | d_l) G^{\text{tr}}(x, y), \quad (45.52)$$

where the longitudinal quadrature  $J$  has the form

$$J(x, y | d_l) = \int df \exp \left\{ \frac{i}{2d_l} \int [\square f(x)]^2 dx + ief(x) - ief(y) \right\} = \\ = \int df e^{ie[f(x) - f(y)]} = \int df \exp \left\{ \frac{ie}{(2\pi)^2} \int f(k) [e^{ikx} - e^{iky}] dk \right\}.$$

Evaluating the integrals with the aid of (45.49), we obtain

$$J(x, y | d_l) = \exp \left[ \frac{ie^2}{2(2\pi)^4} \int (e^{ikx} - e^{iky}) (e^{-ikx} - e^{-iky}) F(k) dk \right] = \\ = \exp \{ ie^2 [\tilde{F}(0) - \tilde{F}(x - y)] \},$$

where, according to (45.46),

$$\tilde{F}(x) = \frac{1}{(2\pi)^4} \int e^{ikx} F(k) dk = \frac{1}{(2\pi)^4} \int e^{ikx} \frac{d_l(k) dk}{k^2 \cdot k^2}.$$

The function  $\tilde{F}$  can be obtained in an explicit form in the special case where  $d_l(k^2) = d_l = \text{const}$ . In order to do this, we write it in the form of the convolution in the  $x$ -representation

$$\tilde{F}(x) = \frac{d_l}{(2\pi)^4} \int \frac{e^{ikx} dk}{(k^2 + i\varepsilon)^2} = d_l \int D_0^c(x - y) D_0^c(y) d^4y, \quad (45.53)$$

where  $D_0^c$  is the causal Green's function for the massless field [see (A2B.6)].

The integral in (45.53) is formally identical with the (complex-conjugate) Feynman integral for the simple scalar loop in the self-energy diagram

$$\tilde{F}(x) = \frac{-d_I}{(2\pi)^4} \int \frac{1}{(x-y)^2 - i\epsilon} \frac{d^4 y}{y^2 - i\epsilon}$$

and can be evaluated by the standard methods described at the beginning of Chapter V. This procedure yields

$$\tilde{F}(x) - \tilde{F}(0) = \frac{d_I}{(4\pi)^2 i} [\ln x^2 + C],$$

where  $C$  is a divergent constant. Thus, the result of the gauge transformation is that the singular factor may appear in the fermion Green's function. In accordance with §34, this factor can be removed by suitably modifying the subtraction procedure. All this leads to the following formula for the gauge transformation:

$$G^{\text{tr}}(x) \rightarrow G(x) = \left( \frac{x^2}{x_0^2} \right)^{-\frac{e^2 d^I}{16\pi^2}} G^{\text{tr}}(x). \quad (45.54)$$

#### §46. Investigation of the Bloch-Nordsieck Model\*

*46.1. The Bloch-Nordsieck Model and the Determination of  $G(x, y | A)$ .* In this section we shall examine the application of the method of functional averaging to calculations based on the Bloch-Nordsieck model in spinor electrodynamics. To remove the infrared catastrophe, Bloch and Nordsieck (1937) developed a method for the approximate solution of the problem of the interaction of a fermion with the electromagnetic field in the region of low frequencies, which is not based on perturbation theory. The first approximation in their method is equivalent to replacing the Dirac matrices by constant numbers  $u$ . We shall refer to the equations of electrodynamics in which the above replacement has been made as the Bloch-Nordsieck model.

In addition to its importance for the analysis of the infrared catastrophe, the problem is also of methodologic interest, since in the Bloch-Nordsieck model the functional integration may be carried out to the end without making any approximations, and this allows one to follow in great detail the procedure of obtaining closed expressions for Green's functions and of removing infinities from them. In discussing the problem we shall, as we have done already, employ the auxiliary Pauli-Villars regularization at the intermediate stages.

A characteristic feature of the Bloch-Nordsieck model is the fact that it does not include the vacuum polarization

$$S_0(A) = 1, \quad (46.1)$$

\*We shall follow the account given by Svidzinskii (1956).

i.e., effects connected with the photon self-energy, as may be seen directly from the following simple considerations. Green's function for a free electron,  $G_0$ , is in this case determined by the following first-order equation:

$$\left( i u^n \frac{\partial}{\partial x^n} - m \right) G_0(x - y) = -\delta(x - y), \quad (46.2)$$

where

$$u^n u_n = 1,$$

and, in the momentum representation, instead of having two poles, as in the usual electrodynamics, has only one:

$$G_0(p) = \frac{1}{m - u p - i\varepsilon}. \quad (46.3)$$

As a result,

$$G_0(x - y) = 0 \quad \text{for} \quad x^0 < y^0,$$

i.e.,  $G_0$  is a purely retarded function. Therefore, the matrix element corresponding to a closed electron cycle (for example, such as shown in the diagrams of Fig. 62) will contain at least one function  $G_0$  equal to zero. Physically, this corresponds to the fact that there are no antiparticles in the Bloch-Nordsieck model and that, consequently, no pairs can be created.

Thus, there are no radiative corrections to the photon Green's function in the Bloch-Nordsieck model, and we have to determine only the electron Green's function  $G(x, y)$  which in accordance with (40.12) is determined by taking the functional average of the electron Green's function in a given external field,  $G(x, y | A)$ .

In the present case, the latter function satisfies the equation

$$\left\{ u^n \left( i \frac{\partial}{\partial x^n} + e A_n(x) \right) - m \right\} G(x, y | A) = -\delta(x - y), \quad (46.4)$$

which may be solved by quadrature. For this purpose, we shall use the method of the fifth parameter due to Fock (1937). This method is based on the symbolic representa-

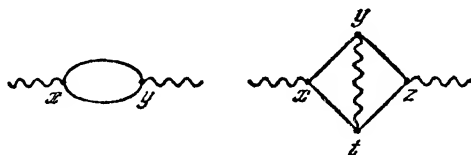


Fig. 62.



tion of the inverse operator in the form of an exponential integral with respect to the fifth parameter:

$$H^{-1} = -i \int_0^{\infty} dv e^{ivH - \varepsilon v}, \quad \varepsilon \rightarrow 0. \quad (46.5)$$

By using this representation and by setting

$$H = \left\{ u^n \left[ i \frac{\partial}{\partial x^n} + e A_n(x) \right] - m \right\}, \quad (46.6)$$

we may write the solution of equation (46.4) in the form

$$G(x, y | A) = i \int_0^{\infty} dv U(v), \quad (46.7)$$

where the function  $U(v)$  which is symbolically defined by the relation

$$U(v) = \exp iv \left\{ u^n \left[ i \frac{\partial}{\partial x^n} + e A_n(x) \right] - m + i\varepsilon \right\} \delta(x - y), \quad (46.8)$$

satisfies the homogeneous first-order differential equation in five variables:

$$-i \frac{\partial U(v)}{\partial v} = \left\{ u^n \left[ i \frac{\partial}{\partial x^n} + e A_n(x) \right] - m + i\varepsilon \right\} U(v) \quad (46.9)$$

with the initial condition

$$U(v)|_{v=0} = \delta(x - y). \quad (46.10)$$

We now go over to the momentum representation for the  $\delta$ -function

$$\delta(x - y) = \frac{1}{(2\pi)^4} \int e^{ip(x-y)} dp, \quad (46.11)$$

and seek  $U(v)$  in the form

$$U(v) = \frac{1}{(2\pi)^4} \int e^{i[K(x, v) - p(x-y) - (m - up - i\varepsilon)v]} dp. \quad (46.12)$$

The function  $K$  in the foregoing is determined by the equation

$$\frac{\partial K}{\partial v} = -u^n \frac{\partial K}{\partial x^n} + eu^n A_n(x) \quad (46.13)$$

with the boundary condition

$$K(v)|_{v=0} = 0.$$

Equation (46.13) is a linear equation with constant coefficients. Solving it by means of a Fourier transformation we obtain:

$$K = \frac{e}{(2\pi)^2} \int dk (u \cdot A(k)) e^{-ikx} \int_0^v dv' e^{i(uk)v'} \quad (46.14)$$

and, consequently, after taking (46.7) and (46.12) into account,

$$G(x, y | A) = i \frac{1}{(2\pi)^4} \int_0^\infty dv \int dp \exp[-ip(x-y) - iv(m-up - i\epsilon) + iK(v | A)]. \quad (46.15)$$

The formula given by (46.15) is a closed explicit expression for the electron Green's function in a given external field in the Bloch-Nordsieck model. Having obtained it, we can now turn to the determination of the complete electron Green's function.

46.2. *Evaluation of  $G(x, y)$ .* We shall use (44.20), taking into account the fact that  $S_0(A)$  is now given by (46.1) and not (44.21), i.e., it is equal to unity. This gives

$$G(x-y) = \frac{1}{(2\pi)^4} \int e^{-ip(x-y)} G(p) dp, \quad (46.16)$$

$$G(p) = i \int_0^\infty \exp[-iv(m-up - i\epsilon)] dv \int \exp[iK(v | A)] dA. \quad (46.17)$$

Thus, in order to find  $G$ , it is necessary to evaluate the functional integral of the following form:

$$\int \exp[iK(v | A)] dA = \int \exp\left[\frac{i}{(2\pi)^2} \int dk \sum_n R^n A_n(k)\right] dA, \quad (46.18)$$

where

$$R^n(k) = eu^n e^{-ikx} \int_0^v dv' e^{i(uk)v'}.$$

The integral in (46.18) is of the Gaussian type (with respect to  $A_n$ ) and may be evaluated by means of (45.12), yielding

$$\int \exp[iK(v | A)] dA = \exp\left\{-i \frac{1}{2(2\pi)^4} \int R^n(k) D_{nm}^0(k) R^m(-k) dk\right\} = \exp[f(v)],$$

where

$$f(v) = -\frac{ie^2}{2(2\pi)^4} \int (uD^0(k)u) \int_0^v dv' e^{i(uk)v'} \int_0^v dv'' e^{-i(uk)v''} dk. \quad (46.19)$$

We thus obtain

$$G(p) = i \int_0^\infty dv \exp[-iv(m - up - i\epsilon) + f(v)]. \quad (46.20)$$

Evaluating the integrals with respect to  $v'$  and  $v''$  in (46.19), we obtain

$$f(v) = -\frac{ie^2}{(2\pi)^4} \int [uD^0(k)u] \frac{1 - \cos(uk)v}{(uk)^2} dk.$$

From now on we shall use an arbitrary gauge, so that  $D_{nm}^0$  will be given by (34.5).

It is then obvious that, since  $u^2 = 1$ ,

$$f(v) = \frac{ie^2}{(2\pi)^4} \int \frac{1 - \cos(uk)v}{(uk)^2 k^2} \left[1 + (d_l - 1) \frac{(uk)^2}{k^2}\right] dk. \quad (46.21)$$

This integral diverges logarithmically in the ultraviolet region. However, the important point is that it does not contain any divergences for small  $k$ . It follows that the *infrared catastrophe is absent from the complete solution* (46.20).

To evaluate the integral (46.21) in an explicit form, it is convenient to use the auxiliary Pauli-Villars regularization.

Omitting all details, we reproduce only the final result:

$$f(v) = \frac{e^2}{8\pi^2} (3 - d_l) \ln Mv - \frac{ie^2}{8\pi} Mv,$$

where  $M$  is the Pauli-Villars mass. Comparison with (46.20) will show that the second term on the right-hand side is the mass renormalization term:

$$m \rightarrow m' = m + \frac{e^2}{8\pi} M = m + \frac{\alpha}{2} M; \quad \alpha = \frac{e^2}{4\pi}. \quad (46.22)$$

Equation (46.22) is therefore conveniently rewritten in the form

$$f(v) = -i \frac{\alpha}{2} Mv + \frac{\alpha}{2\pi} (3 - d_l) \ln \frac{M}{m'} + \frac{\alpha}{2\pi} (3 - d_l) \ln m'v. \quad (46.23)$$

The first term in this expression is the mass renormalization term and the second is the

wave-function renormalization term, i.e., the term corresponding to the multiplicative renormalization of Green's function:

$$Z = \exp \left[ \frac{\alpha}{2\pi} (3 - d_l) \ln \frac{M}{m'} \right] = \left( \frac{M}{m'} \right)^{\frac{\alpha}{2\pi} (3 - d_l)}. \quad (46.24)$$

Both terms can be eliminated by the corresponding computational procedure which, according to the general theory of the removal of divergences, reduces to the redefinition of chronological products. Henceforth, instead of  $G$  we shall therefore consider the function

$$G'(m', p) = Z^{-1} G(m, p),$$

which does not contain ultraviolet divergences in the limit as  $M \rightarrow \infty$ . We obtain

$$\begin{aligned} G'(m, p) &= i \int_0^\infty dv \exp \left[ -iv(m - up - i\varepsilon) + \frac{\alpha}{2\pi} (3 - d_l) \ln mv \right] = \\ &= im^{\frac{\alpha(3-d_l)}{2\pi}} \int_0^\infty dv \exp \left[ -i(m - up - i\varepsilon)v \right] v^{\frac{\alpha(3-d_l)}{2\pi}}. \end{aligned} \quad (46.25)$$

We now turn to the case where  $m - (up) > 0$ . Changing the variable in (46.25) so that

$$[m - (up)]v = x,$$

we obtain

$$G'(p) = \frac{\left[ 1 - \frac{(up)}{m} \right]^{\frac{\alpha(d_l-3)}{2\pi}}}{m - (up)} I, \quad (46.26)$$

where

$$I = i \int_0^\infty dx x^{\frac{\alpha(3-d_l)}{2\pi}} e^{-ix - \varepsilon x}.$$

The integral  $I$  can be estimated approximately, using the fact that  $e^2(3 - d_l)8\pi^2 \ll 1$ . This yields

$$I = i \int_0^\infty dx e^{-ix - \varepsilon x} + O(e^2) = 1 + O(e^2).$$

Similarly, when

$$(up) - m > 0,$$

the change of variable

$$[(up) - m]v = y$$

yields

$$G'(p) = \frac{\left[ \frac{(up)}{m} - 1 \right]^{\frac{\alpha(d_l - 3)}{2\pi}}}{(up) - m} I', \quad (46.27)$$

where

$$I' = i \int_0^\infty dy y^{\frac{\alpha(3-d_l)}{2\pi}} e^{iy - \varepsilon y} = -1 + O'(e^2).$$

Combining (46.26) and (46.27), we obtain the final expression for the electron Green's function in the Bloch-Nordsieck model:

$$G'(p) = \frac{1}{m - (up)} \left| 1 - \frac{(up)}{m} \right|^\beta, \quad (46.28)$$

where

$$\beta = -\frac{e^2(3-d_l)}{8\pi^2} = -\frac{\alpha}{2\pi}(3-d_l). \quad (46.29)$$

Comparison of (46.28) with (46.3) will show that the complete Green's function differs from the free-field Green's function by the factor

$$\left| 1 - \frac{(up)}{m} \right|^\beta.$$

If we expand the complete Green's function  $G'(p)$  into a series in powers of  $\alpha$ , we obtain the logarithmic terms in the lowest-order approximation

$$\left| 1 - \frac{(up)}{m} \right|^\beta = e^{\beta \ln \left| 1 - \frac{up}{m} \right|} = 1 - \frac{\alpha}{2\pi}(3-d_l) \ln \left| 1 - \frac{(up)}{m} \right| + \dots,$$

which are characteristic for the infrared catastrophe.

It will be shown in §50.3 that the infrared singularity in the electron Green's function in ordinary spinor electrodynamics is also of the form given by (46.28).

# Chapter 9

## THE RENORMALIZATION GROUP

### §47. The Group of Multiplicative Renormalizations in Quantum Field Theory

*47.1. Introduction.* We shall now consider a method of investigating the asymptotic properties of Green's functions, based on the group of renormalizing transformations which is available in quantum field theory and which was mentioned in Chapter 6. The essence of the method is as follows. The system of interacting fields is described by the complete Lagrangian  $\mathcal{L}$  written as the sum of the free Lagrangian  $\mathcal{L}_0$  and the interaction Lagrangian  $\mathcal{L}_{\text{int}}$ :

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{\text{int}}. \quad (47.1)$$

Fields corresponding to  $\mathcal{L}_0$  are quantized, and one then proceeds to the interaction representation and uses  $\mathcal{L}_{\text{int}}$  to construct a perturbation theory. In renormalizable theories (after infinities have been removed) this results in a finite arbitrariness which can be removed by adding counterterms to  $\mathcal{L}_{\text{int}}$  which in their operator structure are identical with the individual terms in  $\mathcal{L}_0$  (finite renormalization of mass and field operators) and in  $\mathcal{L}_{\text{int}}$  renormalization of coupling constants).

As was shown in §§34 and 36, transformations of Green's functions and coupling constants that are connected with these finite renormalizations have the group property and form the renormalization group. The existence of the renormalization group imposes special restrictions on the structure of Green's functions and their dependence on coupling constants. The nature of these restrictions can be illustrated as follows. If we write the complete Lagrangian  $\mathcal{L}$  in the form

$$\mathcal{L} = \mathcal{L}'_0 + \mathcal{L}'_{\text{int}} \quad \mathcal{L}'_0 = z\mathcal{L}_0, \quad \mathcal{L}'_{\text{int}} = (1-z)\mathcal{L}_0 + \mathcal{L}_{\text{int}}$$

where  $z$  is a real constant, the expressions for Green's functions (and matrix elements) constructed with the aid of the "new" interaction Lagrangian will differ in the individual orders of perturbation theory from the expressions obtained from (47.1). However, the complete expressions for physical variables should be the same.

It thus turns out that finite sums of perturbation-theory terms do not exhibit renormalization invariance. The imposition of the invariance conditions enables us, starting with a finite number of perturbation-theory terms for Green's functions, to obtain for these functions expressions which, on the one hand, do exhibit the renormalization invariance and, on the other hand, give terms that are identical with the original terms when expanded into a series in terms of the coupling constants. In other words, in many important cases, it is possible to use the first few perturbation-theory terms to establish the leading terms in all the higher orders of perturbation theory, and obtain expressions for their sums.

This procedure of partial summation of the infinite perturbation-theory series is of interest not only in the case of strong interactions. We have already seen in the example of Green's functions in quantum electrodynamics that the expansion parameter in the ultraviolet region is in fact, not the square of the charge  $e^2$ , but its product with a large logarithm,  $e^2 \ln k^2$ . An analogous situation arises in the infrared region.

The existence of the group of renormalizing transformations in quantum field theory was first noted by Stueckelberg and Peterman (1953), who also showed that it is possible to introduce the corresponding infinitesimal operators and thus construct the differential Lie equations. Soon after, Gell-Mann and Low (1954) actually used the group structure of the renormalization transformations in spinor electrodynamics to obtain information on the ultraviolet asymptotics of Green's functions. Their treatment was based on the perturbation-theory scheme, using a cut-off in momentum space. The group transformations were formulated in the language of momentum cut-off.

Bogolyubov and Shirkov (1955b, c; 1956a, b) expressed the functional and, later, the differential equations of the renormalizing transformations in terms of normalization momenta (i.e., the parameters of the  $R$ -operation). This idea, which is not organically connected with the ultraviolet divergences, was useful in analyzing not only the ultraviolet but also the infrared properties of electrodynamic Green's functions. This logical scheme was subsequently used to examine the ultraviolet asymptotic behavior of the two-charge meson-nucleon model [Shirkov (1955), Ginzburg (1956)] and also in a number of other models not containing the ultraviolet divergence [see, for example, Shirkov (1959), Maier and Shirkov (1958)].

This approach, which is founded on the  $R$ -operation, forms the basis of the present chapter. A discussion of its connection with other possible formulations, and a brief review of the most interesting results obtained during the last decade, are given in §§49.1, 49.2, and 51.6.

**47.2. Group Character of Multiplicative Renormalizations.** It was established in Chapter 5 that the introduction of finite counterterms into the interaction Lagrangian, whose structure is identical with the components of the Lagrangian, is equivalent to a finite renormalization (Dyson transformation) of the basic Green's functions and the coupling

constants. This is subject to the reservation that counterterms of the type of proper mass do not lead to multiplicative renormalizations, and that only the transverse part of the photon function is subjected to the multiplicative transformation in electrodynamics. We begin our account with the simplest (pseudo) scalar field  $\varphi$  with the interaction

$$\mathcal{L}_{\text{int}}(x) = h\varphi^4(x). \quad (47.2)$$

In accordance with the results of §36, the introduction into the Lagrangian (47.2) of the counterterms

$$\mathcal{L}_{\text{int}} \rightarrow \mathcal{L}_{\text{int}} + \Delta\mathcal{L}, \quad \Delta\mathcal{L} = \frac{z_3 - 1}{2} \varphi(p)(p^2 - \mu^2)\varphi(p) + (z_4 - 1)h\varphi^4 \quad (47.3)$$

is equivalent to the following Dyson transformations:

$$\begin{aligned} \Delta(\dots, h) &\rightarrow z_3^{-1} \Delta(\dots, h'), \\ \square(\dots, h) &\rightarrow z_4 \square(\dots, h') \end{aligned} \quad (47.4)$$

where

$$h \rightarrow h' = z_4 z_3^{-2} h. \quad (47.5)$$

The repeated dots in (47.4) represent the appropriate configurational or momentum variables.

In other words, the introduction of the counterterms (47.3), which reduce to the renormalization of the elementary pairing and the elementary vertex

$$\overline{\varphi}\varphi \rightarrow z_3^{-1} \overline{\varphi}\varphi, \quad h \rightarrow z_4 h,$$

induces the corresponding renormalization (47.4) of the complete Green's function  $\Delta$  and the four-vertex  $\Gamma_4 = \square$ , and in the final analysis amounts to the renormalization of the coupling constant (47.5).

This essentially very simple property can also be illustrated as follows. The complete Lagrangian

$$\mathcal{L}_{\text{tot}} = \mathcal{L}_0 + \mathcal{L}_{\text{int}} + \Delta\mathcal{L},$$

which consists of the sum of the expressions given by (3.1), (47.2), and (47.3), can be written in the form

$$\mathcal{L}_{\text{tot}} = \frac{z_3}{2} \varphi(p^2 - \mu^2)\varphi + z_4 h \varphi^4 = \mathcal{L}_0(\varphi') + h'(\varphi')^4; \quad \varphi' = \sqrt{z_3} \varphi. \quad (47.6)$$

We now look upon the first term in (47.6) as the free Lagrangian, quantize it, and



proceed to the interaction representation. The role of the boson field, renormalized by the commutation relations, will be played by  $\varphi'$ , and we obtain a theory with the new coupling constant (47.5).

We emphasize that this discussion is not based on perturbation theory at all.

If *simultaneously* with the introduction of the counterterms (47.3) we perform the renormalization of the coupling constant, which is inverse of (47.5),

$$h \rightarrow z_3^{-1} z_3^2 h, \quad (47.7)$$

then the result of the operations defined by (47.2) and (47.7) will be a theory with an unaltered value of the coupling constant  $h$ .

It will be more convenient in the ensuing discussion to use transformations that are inverse to (47.4) and (47.7):

$$\begin{aligned} \Delta_1 \rightarrow \Delta_2 = z_3 \Delta_1, \quad \square_1 \rightarrow \square_2 = z_4^{-1} \square_1, \\ h_1 \rightarrow h_2 = z_4 z_3^{-2} h_1, \end{aligned} \quad (47.8)$$

where

$$\Delta_i \equiv \Delta(\dots, h_i), \quad \square_i \equiv \square(\dots, h_i). \quad (47.9)$$

The significance of (47.8) is that the set  $\Delta_1, \square_1, h_1$  leads to the same physical results as  $\Delta_2, \square_2, h_2$ . It is clear that the number of such equivalent sets is unlimited. It is also clear that the transformation (47.8) exhibits the group property. The corresponding group will therefore be referred to as the group of multiplicative renormalizing transformations, or the *renormalization group*. The validity of the transformations in the renormalization group can be established for a broad class of models in quantum field theory. The above is a representative example of a class of renormalizable models with one dimensionless coupling constant.

Spinor electrodynamics is another important example from this class. We shall introduce the following notation for the square of the coupling constant:

$$\alpha = \frac{e^2}{4\pi},$$

and repeat the above discussion, using the “multiplicative” counterterms from (34.10)

$$\Delta \mathcal{L} = (z_1 - 1) \bar{\psi} \hat{A} \psi + (z_2 - 1) \bar{\psi} (\hat{p} - m) \psi + \frac{z_3 - 1}{2} A_n (g^{nm} p^2 - p^n p^m) A_m. \quad (47.10)$$

Recalling Ward's identity,  $z_1 = z_2 = z$ , we obtain

$$G_1 \rightarrow G_2 = z G_1, \quad D_1 \rightarrow D_2 = z_3 D_1, \quad \Gamma_1 \rightarrow \Gamma_2 = z^{-1} \Gamma_1,$$

$$\alpha_1 \rightarrow \alpha_2 = z_3^{-1} \alpha_1, \quad (47.11)$$

instead of (47.8), where

$$\left. \begin{aligned} G_i &\equiv G(\dots, \alpha_i, \alpha_i), & D_i &\equiv D_{tr}(\dots, \alpha_i), \\ \Gamma_i &\equiv \Gamma(\dots, \alpha_i, \alpha_i), & \alpha_i &\equiv \alpha d_i. \end{aligned} \right\} \quad (47.12)$$

Here we have used the fact that according to (44.42), the product  $\alpha d_i$  is an invariant of the renormalizing transformations.

Finally, consider the model (36.44) of the meson-nucleon interaction with two coupling constants. Using the “multiplicative” counterterms in (36.45) and the Dyson transformations (36.46), (36.47), we obtain the group transformations

$$\begin{aligned} G_1 \rightarrow G_2 &= z_2 G_1, & \Delta_1 \rightarrow \Delta_2 &= z_3 \Delta_1, & \Gamma_1 \rightarrow \Gamma_2 &= z_1^{-1} \Gamma_1, \\ \square_1 \rightarrow \square_2 &= z_1^{-1} \square_1, & g_1^2 \rightarrow g_2^2 &= z_1^2 z_3^{-2} z_3^{-1} g_1^2, & h_1 \rightarrow h_2 &= z_4 z_3^{-2} h_1. \end{aligned} \quad (47.13)$$

The corresponding group contains transformations of the two coupling constants. They will be referred to as the *two-charge groups*.

Renormalizable models in quantum field theory usually contain dimensionless coupling constants corresponding to zero values of the maximum indices of the corresponding vertices (see §32). It is readily verified that the dimensionality of the coupling constants is not an essential feature in the formulation of the group renormalizing transformations. The consequence of this is that the renormalizing transformations and the entire subsequent apparatus can be formulated for quantum fields with dimensional coupling constants [see, for example, Shirkov (1959), Blokhintsev, Yefremov, and Shirkov (1974)].

Group transformations of the form given by (47.8), (47.11), and (47.13) can be used to obtain relatively simple functional equations for the Green's functions involved in them (we shall call them the basic functions), and also the corresponding differential equations (Lie equations).

**47.3. Momentum Representation.** The functional equations are usually written for the “dimensionless” basic Green's functions, considered in the momentum representation.

For the meson propagator in the model given by (47.2)

$$\Delta(k^2, h) = \frac{d(k^2, h)}{\mu^2 - k^2} \quad (47.14)$$

the dimensionless quantity of this kind is  $d(k^2, h)$ . In addition to  $d$ , we must also introduce the strongly connected four-vertex function

$$\Gamma_4 \sim \square(\{k^2\}, h). \quad (47.15)$$

This function must be taken in the form of a dimensionless Lorentz-invariant quantity that is a function of six linearly independent invariant variables:

$$\{k^2\} = (k_1^2, k_2^2, k_3^2, k_4^2, (k_1 + k_2)^2, (k_1 + k_3)^2). \quad (47.16)$$

It is normalized so that as  $h \rightarrow 0$ , the function  $\square$  is similar to  $d$  in that it becomes equal to unity:

$$d(k^2, h=0) = \square(\{k^2\}, h=0) = 1. \quad (47.17)$$

It follows that the expansion of the function

$$\square = \frac{1}{h} \Gamma_4 \quad (47.18)$$

in powers of  $h$  is

$$\square(\{k^2\}, h) = 1 + h[I(s) + I(t) + I(u)] + \dots, \quad (47.19)$$

where

$$\begin{aligned} s &= (k_1 + k_2)^2, \quad t = (k_1 + k_3)^2, \\ u &= (k_1 + k_4)^2 = k_1^2 + k_2^2 + k_3^2 + k_4^2 - s - t. \end{aligned}$$

The functions  $I$  correspond to simple second-order diagrams and were evaluated explicitly in §36:

$$I(z) = \int_0^1 dx \ln \left[ \frac{zx(1-x) - \mu^2 + i\varepsilon}{\lambda x(1-x) - \mu^2} \right]. \quad (47.20)$$

Here  $\lambda$  is a subtraction point such that

$$I(z = \lambda) = 0.$$

In accordance with (47.8), the functions  $d$  and  $\square$  transform as follows:

$$d_1 \rightarrow d_2 = z_3 d_1, \quad \square_1 \rightarrow \square_2 = z_4^{-1} \square_1. \quad (47.21)$$

They are therefore defined to within arbitrary multiplicative constants. This arbitrariness corresponds to the arbitrariness in the  $R$  operation and can be tackled by specifying the subtraction point in the  $R$ -operation.

We shall suppose that

$$d_i(k^2) = 1 \quad \text{for} \quad k^2 = \lambda_i, \quad (47.22)$$

where  $\lambda_i$  plays the role of the square of the normalization momentum.

Normalization to unity is possible only for such  $\lambda$  for which the meson propagator  $d$  is real. This is connected with the fact that the renormalization of the propagator  $\Delta$  is performed by the constant  $z_3$  which is present in the counterterm. The requirement that the counterterm be Hermitian ensures that  $z_3$  is real. The condition for  $\Delta$  to be real, leads, in accordance with (46.29), to a restriction on the possible values of  $\lambda$ :

$$\lambda \leq 9\mu^2. \quad (47.23)$$

Accordingly, the normalization condition for  $\square$  will be written in the form

$$\square_j(\{k^2\}) = 1 \quad \text{for} \quad \{k^2\} = \{\lambda_j\}, \quad (47.24)$$

i.e., for

$$k_1^2 = \lambda_{1j}, \dots, k_t^2 = \lambda_{tj}, \quad s = \lambda_{5j}, \quad t = \lambda_{6j}. \quad (47.25)$$

The condition that  $z_3$  be real imposes the corresponding restrictions on  $\lambda_{aj}$ .

From the condition of homogeneity in momentum space it now follows that  $d$  and  $\square$  can be represented by functions of dimensionless arguments:

$$d_i = d\left(\frac{k^2}{\lambda_i}, \frac{m^2}{\lambda_i}, h\right),$$

$$\square_j = \square\left(\frac{k_1^2}{\lambda_{1j}}, \frac{k_2^2}{\lambda_{2j}}, \frac{k_3^2}{\lambda_{3j}}, \frac{k_4^2}{\lambda_{4j}}, \frac{k_5^2}{\lambda_{5j}}, \frac{k_6^2}{\lambda_{6j}}, \frac{\mu^2}{\lambda_{1j}}, h\right) = \square\left(\left\{\frac{k^2}{\lambda_j}\right\}, \frac{\mu^2}{\lambda_{1j}}, h\right),$$

where

$$k_s^2 = (k_1 + k_2)^2 = s, \quad k_t^2 = (k_1 + k_3)^2 = t.$$

*47.4. Derivation of Functional Equations.* Equations (47.21) can now be written in the form

$$d\left(\frac{k^2}{\lambda_2}, \frac{\mu^2}{\lambda_2}, h_2\right) = z_3 d\left(\frac{k^2}{\lambda_1}, \frac{\mu^2}{\lambda_1}, h_1\right), \quad (47.26)$$

$$\square\left(\left\{\frac{k^2}{\lambda_2}\right\}, \frac{\mu^2}{\lambda_{12}}, h_2\right) = z_4^{-1} \square\left(\left\{\frac{k^2}{\lambda_1}\right\}, \frac{\mu^2}{\lambda_{11}}, h_1\right), \quad (47.27)$$

where, in accordance with (47.8),

$$h_2 = z_4 z_3^{-2} h_1. \quad (47.28)$$

The normalization conditions (47.22) and (47.24) have the following form in the new notation:

$$d(1, y, h) = 1, \quad \square(\{1\}, y, h) = 1. \quad (47.29)$$

The symbol  $\{1\}$  represents the "six-component unit vector"

$$\{1\} = (1, 1, 1, 1, 1, 1),$$

and the letter  $y$  represents the mass variable. The set of relations given by (47.26) and (47.29) is the set of functional equations for the functions  $d$  and  $\square$ . To give it a more compact form, we first eliminate the constants  $z_3$  and  $z_4$ . Assuming in (47.26) that  $k^2 = \lambda_1$ , and using the first equation (47.29), we obtain

$$z_3 = d\left(\frac{\lambda_1}{\lambda_2}, \frac{\mu^2}{\lambda_2}, h_2\right). \quad (47.30)$$

On the other hand, assuming in (47.27) that  $k_\alpha^2 = \lambda_{\alpha 1}$ , and using the second equation in (47.29), we obtain

$$z_4^{-1} = \square\left(\left\{\frac{\lambda_1}{\lambda_2}\right\}, \frac{\mu^2}{\lambda_{12}}, h_2\right). \quad (47.31)$$

Here the symbol  $\{\lambda_1/\lambda_2\}$  represents the "six-vector" with components

$$\frac{\lambda_{\alpha 1}}{\lambda_{\alpha 2}} = \tau_\alpha, \quad \alpha = 1, \dots, 6. \quad (47.32)$$

Equations (47.26)–(47.27) describe multiplicative transformations of the basic Green's functions under a change in the normalization points (i.e., points of subtraction in the  $R$ -operation). Accordingly, transition from the normalization points  $\lambda_1$  and  $\{\lambda_1\}$  to  $\lambda_2$  and  $\{\lambda_2\}$  reduces to the multiplication of  $d$  and  $\square$  by  $z$  and  $z_4^{-1}$ , and to a change in the coupling constant  $h$  in accordance with (47.28). The constants  $z_3$  and  $z_4^{-1}$  which normalize  $d$ ,  $\square$ , and  $h$  are then expressed in terms of  $d$  and  $\square$  in accordance with (47.30) and (47.31).

We note that the product  $hd^2\square$  is an invariant of the transformation

$$h_2 d^3\left(\frac{k^2}{\lambda_2}, \frac{\mu^2}{\lambda_2}, h_2\right) \square\left(\left\{\frac{k^2}{\lambda_2}\right\}, \frac{\mu^2}{\lambda_{12}}, h_2\right) = h_1 d^3\left(\frac{k^2}{\lambda_1}, \frac{\mu^2}{\lambda_1}, h_1\right) \square\left(\left\{\frac{k^2}{\lambda_1}\right\}, \frac{\mu^2}{\lambda_{11}}, h_1\right). \quad (47.33)$$

Substituting (47.30) and (47.31) in (47.26), (47.27), and (47.28), we find that

$$d(x, y, h_2) = d(t, y, h_2) d\left(\frac{x}{t}, \frac{y}{t}, h_1\right), \quad (47.34)$$

$$\square(\{x\}, \bar{y}, h_2) = \square(\{\tau\}, \bar{y}, h_2) \square\left(\left\{\frac{x}{\tau}\right\}, \frac{\bar{y}}{\tau}, h_1\right), \quad (47.35)$$

where we have introduced an obvious notation for the dimensionless arguments, and also

$$h_1 = h_2 d^2(t, y, h_2) \square(\{\tau\}, \bar{y}, h_2). \quad (47.36)$$

Finally, we make the technically simplifying assumption that

$$\tau_1 = \tau_2 = \dots = \tau_6 = t, \quad \bar{y} = y \quad (\lambda_{12} = \lambda_2), \quad (47.37)$$

and substitute

$$\square(t, t, t, t, t, t, y, h) \equiv \square(t, y, h), \quad (47.38)$$

$$h d^2(t, y, h) \square(t, y, h) \equiv \bar{h}(t, y, h). \quad (47.39)$$

Equations (47.34) and (47.35) will now look as follows:

$$d(x, y, h) = d(t, y, h) d\left(\frac{x}{t}, \frac{y}{t}, \bar{h}(t, y, h)\right), \quad (47.40)$$

$$\square(\{x\}, y, h) = \square(t, y, h) \square\left(\left\{\frac{x}{t}\right\}, \frac{y}{t}, \bar{h}(t, y, h)\right). \quad (47.41)$$

Equating in (47.41) all the components of the “six-vector”  $\{x\}$ , we obtain the equation for the “symmetric” vertex function:

$$\square(x, y, h) = \square(t, y, h) \square\left(\frac{x}{t}, \frac{y}{t}, \bar{h}(t, y, h)\right). \quad (47.42)$$

It is clear that the functional equations for  $d$ ,  $\square$  and the “symmetric”  $\square$  involve the function  $\bar{h}$  given by (47.39). This function satisfies the normalization condition

$$\bar{h}(1, y, h) = h, \quad (47.43)$$

and is a transformation invariant of the renormalization group. We shall call it the *invariant coupling constant* (or the *invariant charge*).

The equation for the invariant coupling constant can be obtained from (47.39), (47.40), and (47.42):

$$\bar{h}(x, y, h) = \bar{h}\left(\frac{x}{t}, \frac{y}{t}, \bar{h}(t, y, h)\right). \quad (47.44)$$

We must now relate the coupling constant  $h$  in (47.40)–(47.44) to the “observed” low-energy coupling constant  $h_0$  defined as the magnitude of the scattering amplitude, i.e., the four-tail function  $\square$  taken on the mass surface:

$$M(s, t) \equiv h \square (k_1^2 = k_2^2 = k_3^2 = k_4^2 = \mu^2, s, t; h),$$

at some "low-energy" point.

$$h_0 = M(s_0, t_0), \quad |s_0| \sim |t_0| \sim \mu^2.$$

If we now use the invariance (47.33) of the product  $h d^2 \square = \bar{h}$ , we may write

$$\begin{aligned} h_\lambda d^2 \left( \frac{k^2}{\lambda}, \frac{\mu^2}{\lambda}, h_\lambda \right) \square \left( \left\{ \frac{k^2}{\lambda} \right\}, \frac{\mu^2}{\lambda}, h_\lambda \right) = \\ = h_0 d_0^2 \left( \frac{k^2}{\mu^2}, h_0 \right) \square_0 \left( \frac{k_1^2}{\mu^2}, \dots, \frac{k_4^2}{\mu^2}, \frac{k_5^2}{s_0}, \frac{k_6^2}{t_0}, h_0 \right). \end{aligned} \quad (47.45)$$

Here

$$d_0(1, h_0) = 1, \quad \square_0(1, 1, 1, 1, 1, 1; h_0) = 1. \quad (47.46)$$

Substituting

$$k^2 = \lambda, \quad k_1^2 = \lambda_1, \dots, k_6^2 = \lambda_6,$$

in (47.45) and using the normalization conditions (47.29), we finally obtain the required relation:

$$h_\lambda = h_0 d_0^2 \left( \frac{\lambda}{\mu^2}, h_0 \right) \square \left( \frac{\lambda_1}{\mu^2}, \dots, \frac{\lambda_4}{\mu^2}, \frac{\lambda_5}{s_0}, \frac{\lambda_6}{t_0}; h_0 \right). \quad (47.47)$$

## § 48. General Analysis of Group Equations

*48.1. Structure of Functional Equations.* Equations (47.40), (47.41), (47.42), and (47.44) form a complete set of functional equations of the renormalization group for the model of a scalar field with fourfold interaction. However, the structure of these equations is governed only by the general algebraic structure of the Lagrangian, and is therefore very general and universal. Thus, for example, it is identical with the set of functional group equations for the model of the spinor field with fourfold interaction

$$\mathcal{L}_{\text{int}} \sim \bar{\psi} 0_i \psi \bar{\psi} 0_i \psi.$$

This model is renormalizable in two-dimensional space-time. It can be solved in the massless case and is known as the Thirring model (1958). Maier and Shirkov (1968) have carried out a renormalization-group analysis of this model.

It will be shown below that the set of equations given by (47.40), (47.41), (47.42), and (47.44) has much in common with the set of equations for spinor electrodynamics. We shall now consider certain features of this system of equations that are of very general

character. The system has a definite hierarchic structure, and, consequently, contains equations of three different classes. The first class contains the equation for the invariant charge  $\bar{h}$ . In this case, this class consists of the single equation (47.44). In the general case, for a model with  $k$  coupling constants, the equations in the first class contain  $k$  equations for  $k$  invariant charges. The two-charge model of the meson-nucleon interaction will be discussed below [see (48.12) and (48.13)].

Equations in the first class form a closed system. They must be solved first. The procedure for finding the general solution will be considered below.

If we suppose that equations in the first class have been solved, we can proceed to the investigation of equations in the second class. In the present case, this class incorporates the equation for the one-particle Green's function (47.40) and for the "symmetric" vertex (47.42). In the general case, this class contains equations for all the one-particle functions and all the "symmetric" vertices.

The last, third, class contains equations such as (47.41) for the multiple-argument vertex functions.

Let us now consider functional equations of the renormalization group for the other quantum-field models in §47.2.

In spinor electrodynamics, there are three basic Green's functions. They are: the transverse part of the photon propagator

$$D_{mn}^{\text{tr}}(k) = -\frac{1}{k^2} \left( g_{mn} - \frac{k_m k_n}{k^2} \right) d(k^2, \alpha), \quad (35.7)$$

the electron Green's function

$$G(p) = \frac{s_1(p^2, \dots) \hat{p} + s_2(p^2, \dots) m}{m^2 - p^2} \quad (48.1)$$

and the three-vertex function

$$\Gamma^n(p, q; p - q) = \gamma^n \Gamma(p^2, q^2, (p - q)^2) + \dots \quad (48.2)$$

These three formulas express  $D$ ,  $G$ , and  $\Gamma^n$  in terms of the dimensionless scalar Lorentz-invariant functions  $d$ ,  $s$ ,  $\Gamma$ ,  $\dots$ .

The Dyson multiplicative transformations (47.11) can be written down for the scalar functions  $d$ ,  $s$ , and  $\Gamma$  (it is precisely these functions that are renormalized by the multiplicative counterterms):

$$\left. \begin{aligned} d\left(\frac{k^2}{\lambda_2}, \frac{m^2}{\lambda_2}, \alpha_2\right) &= z_3 d\left(\frac{k^2}{\lambda_1}, \frac{m^2}{\lambda_1}, \alpha_1\right), \\ s\left(\frac{k^2}{\lambda_2}, \frac{m^2}{\lambda_2}, \alpha_2, \alpha_l\right) &= z^{-1} s\left(\frac{k^2}{\lambda_1}, \frac{m^2}{\lambda_1}, \alpha_1, \alpha_l\right), \\ \Gamma\left(\frac{k^2}{\lambda_2}, \frac{m^2}{\lambda_{12}}, \alpha_2, \alpha_l\right) &= z \Gamma\left(\frac{k^2}{\lambda_1}, \frac{m^2}{\lambda_{11}}, \alpha_1, \alpha_l\right), \\ \alpha_2 &= z_3^{-1} \alpha_1. \end{aligned} \right\} \quad (48.3)$$



Ward's identity ( $z_1 = z_2 = z$ ) then ensures that the electron propagator and the vertex function are absent from the invariant charge

$$[\bar{\alpha}(x, y, \alpha) = \alpha d(x, y, \alpha), \quad (48.4)$$

the functional equation for which is

$$\bar{\alpha}(x, y, \alpha) = \bar{\alpha}\left(\frac{x}{t}, \frac{y}{t}, \bar{\alpha}(t, y, \alpha)\right). \quad (48.5)$$

The equations for the one-electron function can be written in the form

$$s(x, y, \alpha, \alpha_t) = s(t, y, \alpha, \alpha_t) s\left(\frac{x}{t}, \frac{y}{t}, \bar{\alpha}(t, y, \alpha), \alpha_t\right). \quad (48.6)$$

The equation for the "symmetric" vertex  $\Gamma(x, y, \alpha, \alpha_t) \equiv \Gamma(\{x, x, x\}, y, \alpha, \alpha_t)$  has the same form:

$$\Gamma(x, y, \alpha, \alpha_t) = \Gamma(t, y, \alpha, \alpha_t) \Gamma\left(\frac{x}{t}, \frac{y}{t}, \bar{\alpha}(t, y, \alpha), \alpha_t\right). \quad (48.7)$$

Finally, the equation for  $\Gamma$  is

$$\Gamma(\{x\}, y, \alpha, \alpha_t) = \Gamma(t, y, \alpha, \alpha_t) \Gamma\left(\left\{\frac{x}{t}\right\}, \frac{y}{t}, \bar{\alpha}(t, y, \alpha, \alpha_t)\right). \quad (48.8)$$

The coupling constant  $\alpha$  in these equations is related to the observed low-energy constant (fine structure constant)

$$\alpha_0 = \frac{1}{137}$$

through the formula

$$\alpha_\lambda = \alpha_0 d_0\left(\frac{\lambda}{m^2}, \alpha_0\right), \quad (48.9)$$

where  $d_0$  is the photon propagator normalized on the mass surface of the real photon:

$$d_0\left(\frac{k^2}{m^2}, \alpha_0\right)\Big|_{k^2=0} = d_0(0, \alpha_0) = 1.$$

In conclusion, we shall write down, without giving a detailed derivation, the functional equations for the two-charge model of the meson-nucleon interaction (36.44).

We introduce dimensionless functions of dimensionless arguments: the meson and nucleon propagators

$$d\left(\frac{k^2}{\lambda_d}, \frac{\mu^2}{\lambda_d}, \frac{\mu^2}{M^2}; g^2, h\right), \quad s\left(\frac{p^2}{\lambda_s}, \frac{\mu^2}{\lambda_s}, \frac{\mu^2}{M^2}; g^2, h\right),$$

the three-vertex and four-vertex functions

$$\Gamma\left(\left\{\frac{p^2}{\lambda}\right\}, \frac{\mu^2}{\lambda_1}; g^2, h\right), \quad \square\left(\left\{\frac{p^2}{\lambda}\right\}, \frac{\mu^2}{\lambda_1}; g^2, h\right)$$

(from this point onward we omit the argument  $\mu^2/M^2$  for the sake of brevity), which satisfy the normalization conditions

$$d(1, \dots) = s(1, \dots) = \Gamma(\{1\}, \dots) = \square(\{1\}, \dots) = 1,$$

we define the "symmetric vertices

$$\left. \begin{aligned} \Gamma_s(x, y; g^2, h) &= \Gamma(x, x, x, y; g^2, h), \\ \square_s(x, y; g^2, h) &= \square(x, x, x, x, x, y; g^2, h) \end{aligned} \right\} \quad (48.10)$$

and we introduce the invariant coupling constants

$$\left. \begin{aligned} \bar{g}^2(x, y; g^2, h) &= g^2 s^2(x, y; g^2, h) \Gamma^2(x, y; g^2, h) d(x, y; g^2, h), \\ \bar{h}(x, y; g^2, h) &= h d^2(x, y; g^2, h) \square(x, y; g^2, h). \end{aligned} \right\} \quad (48.11)$$

The functional equations for  $\bar{g}^2$  and  $\bar{h}$  are

$$\bar{g}^2(x, y; g^2, h) = \bar{g}^2\left(\frac{x}{t}, \frac{y}{t}; \bar{g}^2(t, y; g^2, h), \bar{h}(t, y; g^2, h)\right), \quad (48.12)$$

$$\bar{h}(x, y; g^2, h) = \bar{h}\left(\frac{x}{t}, \frac{y}{t}; \bar{g}^2(t, y; g^2, h), \bar{h}(t, y; g^2, h)\right). \quad (48.13)$$

This closed set of equations forms the first class. The second class includes the four equations for  $d$ ,  $s$  and the symmetric vertices  $\Gamma_s$  and  $\square_s$ :

$$\begin{aligned} F(x, y; g^2, h) &= F(t, y; g^2, h) F\left(\frac{x}{t}, \frac{y}{t}; \bar{g}^2(t, y; g^2, h), \bar{h}(t, y; g^2, h)\right), \\ F &= (d, s, \Gamma_s, \square_s). \end{aligned} \quad (48.14)$$

Finally, the equations in the third class are:

$$\Gamma(\{x\}, y; g^2, h) = \Gamma_s(t, y; g^2, h) \Gamma\left(\left\{\frac{x}{t}\right\}, \frac{y}{t}, \bar{g}^2, \bar{h}\right), \quad (48.15)$$

$$\square(\{x\}, y; g^2, h) = \square_s(t, y; g^2, h) \square\left(\left\{\frac{x}{t}\right\}, \frac{y}{t}, \bar{g}^2, \bar{h}\right). \quad (48.16)$$

We note that the functional equations of the renormalization group are *universal*. The equation for the invariant coupling constant has the same form for all models with one coupling constant [see (47.44) and (48.5)]. Equations in the second class are also the same for such models [for example, (47.40), (47.42), (48.6), and (48.7)], and this is also the case for the equations for the vertex functions in the third class.

This universal property is valid for two-charge and other models. For example, the set of functional equations for the two invariant charges in scalar electrodynamics is identical with (48.2), (48.13), and so on.

**48.2. Differential Equations.** Since the renormalization group is a group of continuous transformations, it can be characterized by an infinitesimal element, i.e., the corresponding Lie differential equations.

These differential equations can be obtained by differentiating the functional equations. They are very useful both for general analysis and for particular applications.

We begin by considering the equations for the invariant charge in the single-charge model. Differentiating (47.44) with respect to  $x$  and then substituting  $t = x$ , we obtain

$$x \frac{\partial \bar{h}(x, y, h)}{\partial x} = \varphi\left(\frac{y}{x}, \bar{h}(x, y, h)\right), \quad (48.17)$$

where

$$\varphi(y, h) = \left. \frac{\partial \bar{h}(t, y, h)}{\partial t} \right|_{t=1} \quad (48.18)$$

On the other hand, differentiating (47.44) with respect to  $t$  and then substituting  $t = 1$ , we obtain

$$\left\{ x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} - \varphi(y, h) \frac{\partial}{\partial h} \right\} \bar{h}(x, y, h) = 0, \quad (48.19)$$

where the function  $\varphi$  is defined in (48.18).

Equations (48.17) and (48.19) are different forms of the group differential Lie equations. Equations such as (48.17) were derived in the fundamental papers of 1954–1955. Equations such as (48.19) were first introduced by Ovsyannikov (1956) in the course of his derivation of the general solution of the functional equations (see below, §48.3). Equations such as (48.17) will be referred to as Lie equations. Equations such as (48.19) will be called Lie equations in the Ovsyannikov form, or, briefly, the Ovsyannikov equations.

The Ovsyannikov equations are, of course, completely equivalent to the Lie equations.

To some extent, they are more impressive because they describe in an explicit form the change in the function for an infinitesimal change in the normalization point. It is important to emphasize that the differential equations given by (48.17) and (48.19) are somewhat less informative than the functional equation (47.44) because they do not take into account the normalization condition (47.43). This condition must be imposed on solutions of the differential equations as a boundary condition.

We must now obtain the differential equations for the functional equations in the second class. Differentiating the functional equation (47.40) in the appropriate fashion, we obtain the Lie equation

$$x \frac{\partial \ln d(x, y, h)}{\partial x} = \psi_d \left( \frac{y}{x}, \bar{h}(x, y, h) \right) \quad (48.20)$$

and the Ovsyannikov equation

$$\left\{ x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} - \varphi(y, h) \frac{\partial}{\partial h} \right\} \ln d(x, y, \bar{h}) = \psi_d(y, h). \quad (48.21)$$

Here

$$\psi_d(y, d) = \left. \frac{\partial \ln d(t, y, h)}{\partial t} \right|_{t=1}, \quad (48.22)$$

and the function  $\varphi$  is defined by (48.18).

Finally, we must consider the equations in the third class. They contain a number of independent momentum arguments. The Lie equations for them will therefore form the system

$$\begin{aligned} \frac{\partial \square(x_1, \dots, x_6, y, h)}{\partial \ln x_i} &= \square(x_i, y, h) \Psi_i \left( \frac{x_1}{x_i}, \dots, \frac{x_6}{x_i}, \frac{y}{x_i}, \bar{h}(x_i, y, h) \right), \\ \Psi_i(\{x\}, y, h) &= \left. \frac{\partial \square(x_1, \dots, x_6, y, h)}{\partial x_i} \right|_{x_i=1}, \quad i = 1, \dots, 6, \end{aligned} \quad (48.23)$$

and the Ovsyannikov equation will contain all the partial derivatives:

$$\left\{ \sum_i x_i \frac{\partial}{\partial x_i} + y \frac{\partial}{\partial y} - \varphi(y, h) \frac{\partial}{\partial h} \right\} \ln \square(\{x\}, y, h) = \psi_{\square}(y, h). \quad (48.24)$$

The above equations can be generalized in a natural way to the case of two or more coupling constants. Thus, the Lie equations for two invariant charges (48.11) have the form

$$\left. \begin{aligned} x \frac{\partial \bar{g}^2(x, y; g^2, h)}{\partial x} &= \Phi_g \left( \frac{y}{x}, \bar{g}^2, \bar{h} \right), \\ x \frac{\partial \bar{h}(x, y; g^2, h)}{\partial x} &= \Phi_h \left( \frac{y}{x}, \bar{g}^2, \bar{h} \right). \end{aligned} \right\} \quad (48.25)$$

where

$$\varphi_g(y, g^2, h) = \frac{\partial \bar{g}^2(t, y, g^2, h)}{\partial t} \Big|_{t=1}, \quad \varphi_h(y, g^2, h) = \frac{\partial \bar{h}(t, y, g^2, h)}{\partial t} \Big|_{t=1}, \quad (48.26)$$

and the functions  $\bar{g}^2$  and  $\bar{h}$  in the arguments on the right-hand side depend on the same arguments as the left-hand side.

The Ovsyannikov equations corresponding to (48.25) have the form

$$\begin{aligned} \left\{ x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} - \varphi_g(y, g^2, h) \frac{\partial}{\partial g^2} - \varphi_h(y, g^2, h) \frac{\partial}{\partial h} \right\} \bar{g}^2(x, y, g^2, h) &= 0, \\ \left\{ x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} - \varphi_g(y, g^2, h) \frac{\partial}{\partial g^2} - \varphi_h(y, g^2, h) \frac{\partial}{\partial h} \right\} \bar{h}(x, y, g^2, h) &= 0. \end{aligned} \quad (48.27)$$

The differential equations corresponding to the equations in the second and third classes can also be written out without difficulty. The differential Lie equations turn out to be of the same form as the Lie equations for the one-charge case, and the operators in the Ovsyannikov equations contain the combination

$$\varphi_g \frac{\partial}{\partial g^2} + \varphi_h \frac{\partial}{\partial h}$$

instead of  $\varphi \cdot \frac{\partial}{\partial h}$ . Generalization to the many-charge case is obvious.

**48.3. General Solution of the Equations.** The functional equations of the renormalization group can be solved in the general form. This was done by Ovsyannikov (1956).

Consider, to begin with, the invariant charge in the one-charge theory [equations (47.44), (48.17), and (48.19)].

Assuming for the moment that  $\varphi(x, y)$  is a known function, we may look upon (48.19) as a linear homogeneous partial differential equation. The first two integrals of the differential equations for the characteristics can be taken in the form

$$\frac{y}{x} = C_1, \quad \Phi_1(y, h) = C_2,$$

where  $\Phi_1$  is a certain function. To obtain the explicit form of the latter function, we must use the explicit form of  $\varphi$  and solve the equation

$$\frac{dh}{dy} + \frac{\varphi(y, h)}{y} = 0.$$

The required function  $\bar{h}$  can therefore be written in the form

$$\bar{h}(x, y, h) = \Phi_2\left(\frac{y}{x}, \Phi_1(y, h)\right), \quad (48.28)$$

where  $\Phi_2$  is an arbitrary function of its arguments. We now use the normalization condition for the invariant charge (47.43). In order to do this, we solve (48.28) for the second argument, and obtain

$$\Phi_1(y, h) = \Phi_3\left(\frac{y}{x}, \bar{h}(x, y, h)\right), \quad (48.29)$$

where  $\Phi_3$  is the inverse of  $\Phi_2$ . We now substitute  $x = 1$  and, using (47.43), we obtain  $\Phi_1 = \Phi_3$ . Equation (48.29) then assumes the form

$$\Phi_1(y, h) = \Phi_1\left(\frac{y}{x}, \bar{h}(x, y, h)\right). \quad (48.30)$$

This functional relation is the general solution of (47.44). It shows that to any solution  $\bar{h}$  of the original functional equation there corresponds a certain function  $\Phi_1(y, h)$  of the two arguments, which is such that this solution is implicitly given by (48.30).

The converse is also valid: if the function  $\Phi_1(y, h)$  has an inverse with respect to the second argument, then (48.30) defines the function  $\bar{h}(x, y, h)$  satisfying the functional equation (47.44). A proof of this can be reduced to a simple manipulation of the arguments of (48.30).

As already noted, to determine the explicit form of  $\Phi_1$  and, consequently, the explicit form of  $\bar{h}$ , it is sufficient to specify the function  $\varphi(y, h)$ . This fact follows directly from the Lie equation given by (48.17).

We now consider the solution of the equation in the second class. The general solution of (48.21) can be written in the form

$$\ln d(x, y, h) = Q_1\left(\frac{y}{x}, \bar{h}(x, y, h)\right) + Q_2(y, h).$$

The first term in this expression is the general solution of the corresponding homogeneous equation, and the second term satisfies the equation

$$\left\{y \frac{\partial}{\partial y} - \varphi(y, h) \frac{\partial}{\partial h}\right\} Q_2(y, h) = \psi_d(y, h).$$

From the normalization condition, we have

$$Q_1(y, h) = -Q_2(y, h) = \ln Q(y, h)$$

and, consequently,

$$d(x, y, h) = \frac{Q\left(\frac{y}{x}, \bar{h}(x, y, h)\right)}{Q(y, h)}, \quad (48.31)$$

which satisfies the functional equation given by (47.40) for any arbitrary function  $Q(y, h)$  of the two arguments. The expression given by (48.31) will also satisfy the differential equation given by (48.21) with the right-hand side defined by

$$\psi_d(y, h) = \varphi(y, h) \frac{\partial \ln Q(y, h)}{\partial h} - \frac{\partial \ln Q(y, h)}{\partial \ln y}. \quad (48.32)$$

We have thus found the most general solution of the functional equation (47.40) in the second class. The general solution of the differential equation in the second class, which is given by (48.21), for given functions  $\bar{h}(x, y, h)$  and  $\psi_d(y, h)$  has been reduced to the solution of (48.32). Finally, the general solution of the functional equation in the third class, which is given by (47.41), can be written in the form

$$\square(x_1, \dots, x_n, y, h) = \frac{K\left(\frac{x_2}{x_1}, \dots, \frac{x_n}{x_1}, \frac{y}{x_1}, \bar{h}(x_1, y, h)\right)}{K(y, h)}, \quad (48.33)$$

where the function  $K(y, h)$  satisfies the normalization condition

$$K(y, h) = K(1, \dots, 1, y, h), \quad (48.34)$$

and also a differential equation such as (48.32)

$$\varphi(y, h) \frac{\partial \ln K(y, h)}{\partial h} - \frac{\partial \ln K(y, h)}{\partial \ln y} = \psi_\square(y, h). \quad (48.35)$$

It is readily verified that the expression given by (48.33) will satisfy the functional equation given by (47.41) for an arbitrary function of the  $n + 1$  arguments

$$K(\xi_2, \dots, \xi_n, y; h).$$

It will also satisfy the Ovsyannikov differential equation in the third class, (48.24), subject to the condition that the function  $\psi_\square(y, h)$  is related to  $\varphi$  and  $K(y, h)$  by (48.35).

This completes our analysis of the one-charge case. We must now consider the two-charge case. We shall confine our attention to the general solution of the equations in the first class, i.e., the set of two differential equations given by (48.12) and (48.13). The corresponding Ovsyannikov equations form a set of two differential equations given by (48.27). There is no difficulty in showing that the general solution of (48.27) can be written in the form

$$\begin{aligned} \bar{g}^2(x, y, g^2, h) &= \Phi_1\left(\frac{y}{x}, F_1(y, g^2, h), F_2(y, g^2, h)\right), \\ \bar{h}(x, y, g^2, h) &= \Phi_2\left(\frac{y}{x}, F_1(y, g^2, h), F_2(y, g^2, h)\right), \end{aligned} \quad (48.36)$$

where  $\Phi_1$  and  $\Phi_2$  are arbitrary functions of the three arguments, and the functions  $F_1$  and  $F_2$  satisfy the equations

$$\left\{ y \frac{\partial}{\partial y} - \Phi_g(y, g^2, h) \frac{\partial}{\partial g^2} - \Phi_h(y, g^2, h) \frac{\partial}{\partial h} \right\} F_i(y, g^2, h) = 0, \quad i = 1, 2.$$

The normalization conditions for the invariant charges impose relationships between the functions  $F_1$ ,  $F_2$  and  $\Phi_1$ ,  $\Phi_2$ . Thus, solving the right-hand side of (48.36) for  $F_1$  and  $F_2$ , we write

$$F_i(y, g^2, h) = \Psi_i\left(\frac{y}{x}, \bar{g}^2(x, y, g^2, h), \bar{h}(x, y, g^2, h)\right), \quad i = 1, 2.$$

The normalization condition leads to the functional equations

$$F_i(y, g^2, h) = \Psi_i(y, g^2, h), \quad i = 1, 2,$$

so that we finally have

$$F_i(y, g^2, h) = F_i\left(\frac{y}{x}, \bar{g}^2(x, y, g^2, h), \bar{h}(x, y, g^2, h)\right), \quad i = 1, 2. \quad (48.37)$$

These relationships are the analog of (48.30) and give the most general solution of the set of functional equations given by (48.12) and (48.13).

#### §49. Asymptotic Analysis in the Ultraviolet Region

The method based on the renormalization group has found its most fruitful applications in the study of the ultraviolet asymptotic behavior of renormalizable quantum-field models. We shall therefore now give a general analysis of the solutions of the renormalization group equations in the ultraviolet region.

*49.1. Asymptotic Solutions for the Invariant Charge.* Let us take a normalization point in the ultraviolet region

$$|k^2| \gg m^2 \quad (49.1)$$

which satisfies

$$|\lambda| \gg m^2, \quad (49.2)$$

and let us consider a range of variation for the variable  $k^2$  such that the ratio  $k^2/\lambda$  lies in the interval



$$1 \lesssim \frac{k^2}{\lambda} = x \lesssim \infty. \quad (49.3)$$

In renormalizable models with dimensionless coupling constants, the limiting transition

$$y = \frac{m^2}{\lambda} \rightarrow 0$$

turns out to be regular, and the functional equations of the renormalization group are found to be much simpler and do not contain the mass variable. The equation for the invariant charge is

$$\bar{h}(x, h) = \bar{h}\left(\frac{x}{t}, \bar{h}(t, h)\right). \quad (49.4)$$

where

$$\bar{h}(x, h) = \bar{h}(x, 0, h) = \lim_{y \rightarrow 0} \bar{h}(x, y, h). \quad (49.5)$$

Henceforth, all quantities that do not contain the mass variable must be regarded as related to the corresponding quantities introduced in §§37 and 48 by a similar limiting transition.

We must now introduce a remark relating to (49.5). The asymptotically invariant charge introduced in (49.5) is normalized so that

$$\bar{h}(1, h) = h \quad (49.6)$$

at the point  $k^2 = \lambda$  satisfying (49.2). The coupling constant  $h = h_\lambda$  encountered in this context is therefore also an asymptotic constant. By definition, it is equal to the invariant charge at the ultraviolet renormalization point satisfying (49.2). On the other hand, in realistic models of quantum field theory, the parameter connected with observable quantities is the coupling constant determined in terms of low-energy values of propagators and vertex Green's functions [see (34.54) and (36.34)]. It is therefore necessary to have the connection between the asymptotic constant  $h_\lambda$  and the low-energy constant  $h_0$ .

The invariance condition for this invariant charge yields

$$\bar{h}_{\text{l.e.}}(k^2, m^2, h_0) = \bar{h}_{\text{h.e.}}(k^2, \lambda, m^2, h_\lambda) \simeq \bar{h}\left(\frac{k^2}{\lambda}, h_\lambda\right).$$

The low-energy invariant charge  $\bar{h}_{\text{l.e.}}$  which we have introduced in this expression is normalized at the point  $k^2 = \xi m^2$  where  $\xi$  is of the order of unity. We may therefore write

$$\bar{h}_{\text{l.e.}}(k^2, m^2, h_0) = \bar{h}_0\left(\frac{k^2}{m^2}, h_0\right),$$

where

$$\bar{h}_0(1, h_0) = h_0. \quad (49.7)$$

Therefore

$$\bar{h}_0\left(\frac{k^2}{m^2}, h_0\right) \simeq \bar{h}\left(\frac{k^2}{\lambda}, h_\lambda\right). \quad (49.8)$$

When the right-hand side of this relation is expanded into a series in the coupling constant  $h_\lambda$ , it becomes a polynomial of degree  $\ln k^2/\lambda$ . The exact equation sign can be used in (49.8) only when the left-hand side is taken to the asymptotically limiting form

$$\bar{h}_0\left(\frac{k^2}{m^2}, h_0\right) \rightarrow \bar{h}_0^{\text{as}}\left(\frac{k^2}{m^2}, h_0\right),$$

which is a polynomial in  $\ln k^2/m^2$ . Therefore

$$\bar{h}\left(\frac{k^2}{\lambda}, h_\lambda\right) = \bar{h}_0^{\text{as}}\left(\frac{k^2}{m^2}, h_0\right). \quad (49.9)$$

the function  $\bar{h}_0^{\text{as}}$  in (49.9) is obtained from  $\bar{h}_0$  by rejecting all terms that tend to zero as  $|k^2/m^2| \rightarrow 0$ . We can now formally substitute  $\lambda = m^2$ , so that (49.9) with  $k^2 = \lambda = m$  yields

$$h_{\text{l.e.}} = h_{m^2} = \bar{h}_{\text{l.e.}}(m^2, m^2, 0, h_{m^2}) = \bar{h}_0^{\text{as}}(1, h_0)$$

which is the required relation between the low-energy coupling constant  $h_\lambda$  and the low-energy physical coupling constant  $h_0$ .

We now turn to the solution of (49.4). The Ovsyannikov differential equation for  $\bar{h}(x, h)$  is

$$\left[ x \frac{\partial}{\partial x} - \varphi(h) \frac{\partial}{\partial h} \right] \bar{h}(x, h) = 0, \quad (49.10)$$

where

$$\varphi(h) = \left. \frac{\partial \bar{h}(x, h)}{\partial x} \right|_{x=1}. \quad (49.11)$$

The general nonnormalized solution of (49.10) can be written in the form of an arbitrary function  $\Phi$  of the first integral of the equation for the characteristic:

$$h(x, h) = \Phi(\ln x + \Psi(h)),$$

where

$$\Psi(h) = \int^h \frac{dh'}{\varphi(h')}. \quad (49.12)$$

Using the normalization condition (49.6), we find that the functions  $\Phi$  and  $\Psi$  are mutually inverse. Therefore

$$\Psi(\bar{h}(x, h)) = \ln x + \Psi(h). \quad (49.13)$$

Using (49.12), we obtain

$$\int_{\bar{h}}^{\bar{h}(x, h)} \frac{da}{\varphi(a)} = \ln x \quad (49.14)$$

which is the so called Gell-Mann–Low equation (1954).

We note that the solution given by (49.13) and (49.14) is a special case of the general solution given by (48.30). The connection between  $\Phi_1$  in (48.30) and  $\Psi$  in (49.13) is

$$\Psi(h) = \lim_{y \rightarrow 0} \ln \frac{\Phi_1(y, h)}{y}. \quad (49.15)$$

We also note that the solution given by (49.14) can be obtained directly from the asymptotic form of the Lie equation (48.17)

$$x \frac{\partial \bar{h}(x, h)}{\partial x} = \varphi(\bar{h}(x, h)). \quad (49.16)$$

The Gell-Mann–Low equation is very convenient for investigating the possible asymptotic behavior of the invariant charge.

Occasionally (for example, in calculations using dimensional regularization and subtraction of divergences according to 't Hooft (1973)), it is convenient to work with non-normalized Green's functions and invariant coupling constants. The basic functional and differential group equations are then readily modified.

Suppose, for example, that the normalization condition for the invariant charge is

$$\bar{h}_q(1, h) = q(h). \quad (49.17)$$

Here we have introduced the new symbol  $\bar{h}_q$  for the invariant charge normalized with the aid of the function  $q(h)$ . The functional equation for  $\bar{h}_q$  in this case assumes the form

$$\bar{h}_q(x, h) = \bar{h}_q\left(\frac{x}{t}, Q(\bar{h}_q(t, h))\right), \quad (49.18)$$

where  $Q$  is a function that is the inverse of  $q$ :

$$Q(q(x)) = q(Q(x)) = x.$$

The general solution of (49.18) can be obtained by the Ovsiannikov method. The corresponding differential equation is

$$\left[ x \frac{\partial}{\partial x} - \beta(h) \frac{\partial}{\partial h} \right] \bar{h}_q(x, h) = 0, \quad (49.19)$$

where

$$\beta(h) = \varphi_q(h) Q'(q(h)), \quad (49.20)$$

$$\varphi_q(h) = \frac{\partial \bar{h}_q(t, h)}{\partial t} \Big|_{t=1}. \quad (49.21)$$

We shall not solve (49.19). To obtain the general solution of (49.18), we note that the substitution

$$Q(\bar{h}_q) = \bar{h} \quad (49.22)$$

reduces it to the standard form given by (49.4). The general form of  $\bar{h}_q$  can therefore be obtained from (49.14) with the aid of the relation

$$\bar{h}_q = Q^{-1}(\bar{h}) = q(\bar{h}). \quad (49.23)$$

All that remains is to note the following point. Differentiating (49.18) with respect to  $x$  and then substituting  $t = x$ , we obtain the Lie equation

$$x \frac{\partial \bar{h}_q(x, h)}{\partial x} = \tilde{\varphi}(h_q(x, h)), \quad (49.24)$$

in which

$$\tilde{\varphi}(h) = \varphi_q(Q(h)). \quad (49.25)$$

It is thus clear that in the above case, the functions  $\beta$  and  $\tilde{\varphi}$ , which are present in both the Lie and the Ovsiannikov equations, are different. They are related to the function  $\varphi$  in (49.11) by the formula

$$\varphi_q(h) = \varphi(h) q'(h), \quad (49.26)$$

which follows from (49.21).

Finally, we note that the equations for the invariant charge (49.10), (49.19), and also equation (49.38) given below for the one-particle Green's functions, are the asymptotic forms of the corresponding Ovsyannikov equations given by (48.19) and (48.21). These asymptotic Ovsyannikov equations are identical with the ultraviolet limits of the so-called Callan (1970) and Symanzik (1970) equations. It is therefore reasonable to refer to them as the asymptotic of Ovsyannikov-Callan-Symanzik equations.

*49.2. Asymptotic Behavior of Invariant Charge.* It follows from (49.14) and (49.11) that, to determine the ultraviolet asymptotic behavior of the invariant charge  $\bar{h}(x, h)$ , it is sufficient to know its derivative in the neighborhood of the normalization point  $x = 1$ . To find it, we can use the asymptotic perturbation theory

$$\bar{h}(x, h) = h + h^2 a_1 L + h^3 (b_2 L^2 + b_1 L) + \dots \quad (49.27)$$

where  $L = \ln x$ . The series given by (49.27) can be written in the form of a series in powers of  $L$ :

$$\bar{h}(x, h) = h + f_1(h) L + f_2(h) L^2 + \dots \quad (49.28)$$

Substituting (49.28) in (49.11), we obtain

$$\varphi(h) = f_1(h) = a_1 h^2 + b_1 h^3 + \dots \quad (49.29)$$

Thus, to determine the asymptotic function  $\bar{h}(x, h)$ , we must evaluate, in each order of perturbation theory, the coefficient in front of the linear logarithmic term (the "lowest" logarithm).

The arithmetic sign of the function  $\varphi(h)$  is exceedingly important for further analysis. It is clear from (49.29) that, for sufficiently small  $h$ , this sign is determined by the sign of  $a_1$ , i.e., the sign of the first logarithmic radiative correction in (49.27).

Consider, to begin with, the case where  $\varphi(h)$  is positive in the range of values of  $h$  in which we are interested. It then follows from the Lie equation (49.16) that  $\bar{h}(x, g)$  increases with increasing  $L = \ln x$ . The limit of infinitely large  $L$  corresponds to the divergence on the integral

$$I(z) = \int_h^z \frac{da}{\varphi(a)}, \quad (49.30)$$

on the left-hand side of (49.14) at the upper limit.

Three cases are possible (Fig. 63):

(a) The integral  $I(z)$  diverges for finite  $z = H$ . This is possible if the function  $1/\varphi$  has a nonintegrable singularity at the point  $H$ . For example,

$$\varphi(h) \simeq a(H - h)^{1+m}, \quad m \geq 0.$$

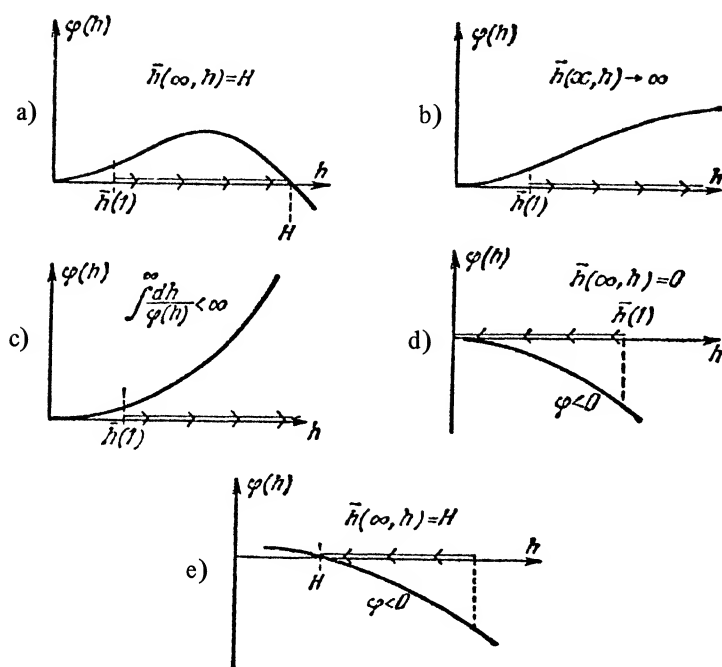


Fig. 63.

In this case,  $\bar{h}$  tends from below to a finite  $H$  in the limit of large  $L$ :

$$\bar{h}(\infty, h) = H, \quad (49.31)$$

and, in the example we have considered, this has the form

$$\bar{h}(x, h) \simeq H - \frac{1}{(amL)^{1/m}}.$$

(b) The integral  $I(z)$  diverges for infinite  $z$ . The invariant charge then increases without limit:

$$\lim_{x \rightarrow \infty} \bar{h}(x, h) = \infty. \quad (49.32)$$

For example,

$$\alpha) \quad \varphi(h) \sim h^{1-n}, \quad n > 0$$

or

$$\beta) \quad \varphi(h) \sim bh(\ln h)^{1-q}, \quad 0 \leq q \leq 1.$$

Substituting these expressions in (49.30), we obtain:

$$\left. \begin{array}{l} \text{(a)} \quad \bar{h}(x, h) \sim (\ln x)^{1/\alpha} \\ \text{and} \\ \text{(b)} \quad \bar{h}(x, h) \sim \exp(bq \ln x)^{1/q}. \end{array} \right\} \quad (49.33)$$

(c) The integral in (49.30) remains finite for  $z \rightarrow \infty$ . In this case, equation (49.14) does not admit of a self-consistent asymptotic behavior for  $\bar{h}$  as  $x \rightarrow \infty$ . We thus arrive at an internal *contradiction*.

We now consider the situation in which  $\varphi(h)$  is negative. According to (49.16), the invariant charge  $\bar{h}$  then decreases with increasing  $L = \ln x$ . Equation (49.14) is conveniently written in the form

$$\bar{h}(x, h) \int_0^h \frac{da}{|\varphi(a)|} = L. \quad (49.30')$$

Here we have two possible cases:

(d) the integral (49.30') diverges for  $\bar{h} = 0$ ; then

$$\lim_{x \rightarrow \infty} \bar{h}(x, h) = 0; \quad (49.34)$$

(e) the integral on the right-hand side of (49.30') diverges for certain finite  $h' = H$ ; this case is close to (a) and differs from it only by the fact that  $\bar{h}$  tends to its asymptotic value *from above*.

We now consider the physical significance of the above possibilities. Since, according to the quantum-mechanical reciprocity relation (see also §41.1), the ultraviolet momentum limit  $L \rightarrow \infty$  corresponds to small separations, our discussion can be conveniently given in terms of the renormalization of the coupling constant. If we define the renormalization constant  $Z_h$  as the ratio of the coupling constant at large distances (physical constant  $h$ ) to its value at small distances (residual constant  $h_0 \sim \bar{h}(\infty, h)$ ), we have

$$Z_h^{-1} = \frac{\bar{h}(\infty, h)}{h}. \quad (49.35)$$

Case (a) corresponds to finite renormalization of the coupling constant

$$Z_h^{-1} = \frac{H}{h}. \quad (49.35a)$$

Case (b) corresponds to infinite renormalization

$$Z_h^{-1} = \infty, \quad Z_h = 0. \quad (49.35b)$$

The effective coupling constant increases without limit at short distances. Conversely, if we fix any finite value for the residual charge, we find that the physical charge becomes zero. Vacuum fluctuations completely screen the charge.

Finally, in case (d), we have

$$Z_h^{-1} = 0. \quad (49.35c)$$

A finite value of the physical charge  $h$  then corresponds to zero value of the asymptotic (i.e., residual) charge. The vacuum polarization effect is opposite to the case of complete screening and reduces to an effectively infinite increase in the residual coupling constant. This behavior of the invariant charge in the ultraviolet region is often referred to as *asymptotically free* in modern literature. Asymptotic freedom in the ultraviolet region is attractive from the general point of view because, even for large numerical values of the physical coupling constants, it enables us to reach the weak-coupling region in the ultraviolet asymptotics, where perturbation theory can be employed.

A class of quantum field theory models containing non-Abelian gauge fields (Yang-Mills fields) was discovered in the last few years. Such fields lead to asymptotically free ultraviolet behavior.

We shall not discuss case (c) because it involves an internal contradiction. However, analysis of many quantum-field models leads to this case when it is based on information obtained from lowest orders of perturbation theory. Spinor electrodynamics, which we shall consider in the next subsection, is among these models. A more detailed discussion of variant (c) will also be given in the next subsection.

We have thus concluded our examination of possible ultraviolet asymptotic behavior of the invariant charge in singly charged renormalizable models. In the case of models containing several coupling constants, it is necessary to perform the corresponding analysis of the system of asymptotic equations for the invariant charges.

The Lie differential equations for the asymptotically invariant charges in the case of  $k + 1$  charges has the form

$$\frac{\partial \bar{g}_i}{\partial L} = \varphi_i(\bar{g}_1, \dots, \bar{g}_{k+1}), \quad i = 1, \dots, k+1. \quad (49.36)$$

where

$$L = \ln x, \quad \bar{g}_i = \bar{g}_i(L, g_1, \dots, g_{k+1}).$$

An important property of (49.36) is the absence of an explicit dependence on the right-hand sides on the argument  $L$  which can therefore be eliminated, for example, by dividing the first  $k$  equations by the last. Substituting

$$\begin{aligned} \bar{g}_{k+1} &= \sigma, \quad \bar{g}_i(L, \dots) = y_i(\sigma), \quad i = 1, \dots, k, \\ \frac{\varphi_i(\bar{g}_1, \dots, \bar{g}_{k+1})}{\varphi_{k+1}(\bar{g}_1, \dots, \bar{g}_{k+1})} &= \psi_i(\sigma, y_1(\sigma) \dots y_k(\sigma)), \end{aligned}$$



we obtain

$$\frac{dy_i(\sigma)}{d\sigma} = \psi_i(\sigma, y_1, \dots, y_k), \quad i = 1, \dots, k \quad (49.37)$$

i.e., a set of  $k$  first-order nonlinear differential equations. A qualitative analysis of this set of equations can be performed by standard methods, but we shall not do this here. We merely note that equation (49.37) will be used in §51 in connection with the two-charge meson-nucleon model.

*49.3. Asymptotic Green's Functions.* We now consider the ultraviolet analysis of equations in the second class, i.e., equations for the single-particle Green's functions and symmetric higher Green's functions. The corresponding differential equations, i.e., (48.20) and (48.21), will be written in the asymptotic form

$$\left\{ \frac{\partial}{\partial L} - \varphi(h) \frac{\partial}{\partial h} \right\} D(L, h) = \psi(h), \quad (49.38)$$

$$\frac{\partial D(L, h)}{\partial L} = \psi(\bar{h}(x, h)). \quad (49.39)$$

where

$$D(L, h) = \ln d(x, h), \quad \psi(h) = \psi_d(y=0, h).$$

We shall now solve equation (49.38). Its general solution can be written in the form

$$D(L, h) = \tilde{\Psi} \left( L + \int_c^h \frac{da}{\varphi(a)} \right) - \int_c^h \frac{\psi(a)}{\varphi(a)} da,$$

where  $\tilde{\Psi}$  is an arbitrary function. The first term on the right-hand side is the general solution of the homogeneous equation (48.10) corresponding to (49.38). It can be written in the form

$$\tilde{\Psi} \left( L + \int_c^h \frac{da}{\varphi(a)} \right) = \ln Q(\bar{h}(L, h)).$$

and therefore

$$d(L, h) = Q(\bar{h}(x, h)) \exp \left( - \int_c^h \frac{\psi(a)}{\varphi(a)} da \right). \quad (49.40)$$

Bearing in mind the normalization condition

$$d(0, h) = 1,$$

we obtain

$$d(L, h) = \exp \left( \int_h^{\bar{h}(x, h)} \frac{\psi(a)}{\varphi(a)} da \right). \quad (49.41)$$

The last equation is the analog of the Gell-Mann—Low equation (49.14). It is very convenient in the analysis of possible ultraviolet asymptotics of single-particle Green's functions and symmetric many-particle functions.

We shall outline this analysis in connection with the possible asymptotics of the invariant charge, considered in §49.2:

(a) *Finite renormalization of the charge* (49.31). There are three possibilities:

(a1) The integral

$$I_0 = \int_h^H \frac{\psi(a)}{\varphi(a)} da$$

converges at the upper limit. We then have

$$d(\infty, h) = e^{I_0} \quad (49.42)$$

i.e., Green's function has a finite limit. The renormalization of the wave function is also finite.

(a2) The integral  $I_0$  diverges at the upper limit, but the integral

$$I_1 = \int_h^H \frac{\psi(a) - \psi(H)}{\varphi(a)} da$$

converges. We then have

$$d(L, h) \rightarrow x^\nu e^{I_1}, \quad (49.43)$$

i.e., the single-particle Green's function increases (or decreases) asymptotically in a power-law fashion. The exponent

$$\nu = \psi(H)$$

is called the *anomalous dimension*.

(a3) The integral  $I_0$  diverges and the “single subtraction,” introduced in the transition to  $I_1$ , is insufficient, i.e.,  $I_1$  also diverges. In this case, the power-law asymptotic behavior is, in general, subject to the weaker logarithmic behavior

$$d(L, h) \sim x^\nu (\ln x)^\mu, \quad (49.44)$$

and so on.

Since the invariant charge  $\bar{h}$  is the product of the single-particles and symmetric Green's functions raised to different powers

$$\bar{h}(L, h) = \prod_i (d_i(L, h))^{N_i}, \quad (49.45)$$

the asymptotic behavior of the individual factors in (49.45) in case (a1) correspond to the asymptotic behavior of the product. This may be referred to as the *normal* case. In cases (a2) and (a3), the asymptotic behavior of the factors differs from the asymptotic behavior of the product. The anomalous dimensions must then, obviously, satisfy the relation

$$\sum_i \nu_i N_i = 0. \quad (49.46)$$

Such cases will be referred to as anomalous.

(b) *Infinite renormalization of the coupling constant* (49.32). The number of possible subvariants in this case is greater than for finite renormalization of  $h$ . These correspond to different asymptotics  $\bar{h}$ . These asymptotics may, in particular, have a (quasi) logarithmic character (49.33a) or (quasi) power-law character (49.33b). In normal cases, the asymptotic behavior of the individual Green's functions will be given by (49.33), whereas in anomalous cases it may deviate toward either side. We note that power-law asymptotic behavior of the form of (49.43) may be obtained in the anomalous case corresponding to (49.33a) and also in the normal case corresponding to (49.33b). In the latter variant, the sum rule (49.46) will not be valid.

(c) *The case of asymptotic freedom* (49.34). In this case, we do not leave the region of weak coupling for small values of the constant  $h$ . The explicit evaluation of the functions  $\varphi$ ,  $\psi_i$  can therefore be performed with the aid of perturbation theory. In the lowest non-vanishing order of perturbation theory, we find that the ratio

$$\frac{\psi_i(h)}{\varphi(h)} = c_i$$

is independent of  $h$ . Therefore

$$d_i(L, h) \simeq \exp c_i (\bar{h}(L, h) - h), \quad (49.47)$$

and, as  $L \rightarrow \infty$  ( $\bar{h} \rightarrow 0$ ), the function  $d_i$  tends to a constant, i.e., the renormalization of the wave function turns out to be finite.

**49.4. Higher Green's Functions.** The renormalization-group technique enables us to consider not only the basic but also the higher Green's functions and vertices (i.e., strongly connected vertex functions, introduced in §37). Consider, for example, the multiparticle strongly connected Green's function with  $b$  boson and  $2f$  fermion external lines, defined by (37.37). It follows from the general analysis given in §36 that, when  $b + 3f > 4$ , such objects contain divergences corresponding only to divergent subdiagrams, but do not contain specific divergences connected with the corresponding diagrams "as a whole." Accordingly, under transformations of multiplicative renormalization, these functions transform as follows:

$$\Gamma_{b, 2f} \rightarrow z_2^{-f} z_3^{-b/2} \Gamma_{b, 2f} \quad (49.48)$$

Using (47.30)–(47.31), we have for the symmetric asymptotic behavior

$$\Gamma_{b, 2f}(x, g^2, h) = s^{-f}(x, g^2, h) d^{-b/2}(x, g^2, h) \Gamma_{b, 2f}(1, g^2, \bar{h}). \quad (49.49)$$

We note that the symmetric asymptotic behavior (49.49) corresponds to the case where all the momentum arguments of the function  $\Gamma$  tend to a single (large) value of  $k^2$ :

$$|k_i k_j| \sim |k^2|, \quad |k^2| \gg m^2. \quad (49.50)$$

This case has no direct relation to real physical processes (in which  $k_i^2 = m_i^2$ ). However, by investigating it, we obtain some idea about the nature of the interaction at "short distances."

## §50. Analysis of Green's Functions in Spinor Electrodynamics

**50.1. Formulation of the Problem.** It was established above that the equations of the renormalization group imposed definite restrictions on Green's functions and vertex functions but, nevertheless, involve an arbitrariness of the functional type. This property is completely natural because renormalization invariance is a reflection of the peculiar self-similar property of the basic Green's functions corresponding to the Dyson transformations so that, for example, as noted in §38.1, the equations of the renormalization group have the property of universality. In other words, dynamics is not reflected in the equations of the renormalization group and their solutions.

The introduction of dynamic information is therefore an essential step in the particularization of the above general results. Unfortunately, perturbation-theory calculations are the only source of this information. Here we draw attention to the fact that expansion into a series (even an asymptotic series) is a *hypothesis* that is adequately justified only in electrodynamics (see the discussion in §20.1). Our task now is to place the general solutions in correspondence with perturbation theory, remembering that this procedure is undoubtedly meaningful in spinor electrodynamics.

The reduction of perturbation theory to the renormalizationally invariant form enables us to improve the quality of approximations based on the expansions of the electrodynamic

perturbation theory when the terms of these expansions do not decrease sufficiently rapidly. This situation occurs in the ultraviolet and infrared ranges of the momentum variables in which the effective expansion parameter is the product of the fine structure constant and a large logarithm.

50.2. *Asymptotic Behavior in the Ultraviolet.* In accordance with the general prescription given in §49, we begin by considering the asymptotic behavior of the invariant charge (48.4) which, by virtue of Ward's identity, is identical with the photon Green's function. Substituting  $y = 0$ , we have instead of (48.5),

$$\bar{\alpha}(x, \alpha) = \bar{\alpha}\left(\frac{x}{t}, \bar{\alpha}(t, \alpha)\right). \quad (50.1)$$

The integral of the corresponding differential equation is, according to (49.14),

$$\bar{\alpha}(x, \alpha) \int_{\alpha}^{\alpha} \frac{da}{\varphi(a)} = \ln x, \quad (50.2)$$

where

$$\varphi(\alpha) = \left. \frac{\partial \bar{\alpha}(x, \alpha)}{\partial x} \right|_{x=1}. \quad (50.3)$$

We now use perturbation theory to determine the function  $\varphi(\alpha)$ . According to (35.15), we have in the second order

$$\bar{\alpha}_2(x, \alpha) = \alpha d_2(x, \alpha) = \alpha + \frac{\alpha^2}{3\pi} L, \quad L = \ln x. \quad (50.4)$$

Consequently

$$\varphi(\alpha) \sim \varphi_2(\alpha) = \frac{\alpha^2}{3\pi}. \quad (50.5)$$

Substituting (50.4) in (50.2), we find that

$$\bar{\alpha}(x, \alpha) = \frac{\alpha}{1 - \frac{\alpha}{3\pi} L}. \quad (50.6)$$

This formula was first obtained by Landau, Abrikosov, and Khalatnikov (1954b) by direct summation of the leading logarithmic terms.

The expression given by (50.6) is remarkable in two respects. First, it is a function of the product  $\alpha L$  and, when expanded into a series, contains all powers  $(\alpha L)^n$ . Thus, by

starting with the first logarithmic term (50.2) of second-order perturbation theory, and using renormalization invariance, we have obtained (50.6) which contains all the higher logarithmic terms in any order of perturbation theory.

Second, the expression given by (50.6) increases without limit for  $\alpha L \rightarrow 3\pi$ , i.e., it contains the so-called "ghost" pole, the possible existence of which leads to serious contradictions with a number of general principles of the theory [see Landau and Pomeranchuk (1955) and Fradkin (1955a)]. According to the classification of §49.2, this second property corresponds to case (c). The expression given by (50.6) cannot be used for

$$\ln x = L \gtrsim \frac{3\pi}{\alpha}. \quad (50.7)$$

The internal contradiction in (50.6) leads us to an analysis of the reasons for it. It is readily verified that any approximation to  $\varphi(\alpha)$  obtained from perturbation theory cannot be used for large values of the argument. This means that the upper limit on the left-hand side of (50.2) cannot be much greater than unity. Thus, any consequences of (50.2) based on perturbation-theory approximations for  $\varphi(\alpha)$  and leading to

$$\bar{\alpha}(x, \alpha) \gg 1 \quad (50.8)$$

are unjustified.

When the higher-order terms of perturbation theory are taken into account, we obtain [Baker and Jensen (1969)]

$$\varphi_6(\alpha) = \frac{\alpha^2}{3\pi} + \frac{\alpha^3}{4\pi^2} + \frac{\alpha^4}{8\pi^3} \left( \frac{8}{3} \xi(3) - \frac{101}{36} \right); \quad \xi(3) \simeq 1.19. \quad (50.9)$$

If we directly substitute this expression in (50.2), we again arrive at (49.2c), i.e., a pole-type expression and an internal contradiction. It is, however, possible to construct an expansion for the function  $\varphi^{-1}$ :

$$\frac{1}{\varphi(\alpha)} \simeq \frac{3\pi}{\alpha^2} \left[ 1 - \frac{3\alpha}{4\pi} + \frac{\alpha^2}{\pi^2} \left( \frac{155}{96} - \xi(3) \right) + \dots \right]. \quad (50.10)$$

The use of the above three terms leads us to (49.2c) because of the positive sign in front of the last term. Substituting (50.10) in (50.2), and integrating, we obtain

$$\frac{1}{\alpha} - \frac{1}{\bar{\alpha}(x, \alpha)} - \frac{3}{4\pi} \ln \frac{\bar{\alpha}(x, \alpha)}{\alpha} + \frac{1}{\pi^2} \left[ \frac{155}{96} - \xi(3) \right] (\bar{\alpha}(x, \alpha) - \alpha) + \dots = \frac{1}{3\pi} L. \quad (50.11)$$

In the limit as  $x \rightarrow \infty$ , we find, formally, that

$$\bar{\alpha}(x, \alpha) \rightarrow \text{const} \cdot \ln x. \quad (50.12)$$

In this connection, it is important to note that (50.2) has one interesting property. Let us suppose that the function  $\varphi(z)$  is defined and positive in the entire range of its argument  $0 < z < \infty$ , and that

$$\int_{\alpha}^{\infty} \frac{dz}{\varphi(z)} = \infty.$$

We thus assume that the true function  $d(x, \alpha)$  has a unique singularity for  $x \rightarrow \infty$ , and the theory does not contain any undesirable surprises such as the "logarithmic pole" (50.6).

Next, suppose that we wish to evaluate the true function  $\varphi(z)$  with the aid of some limiting transition for the function  $\varphi_{\Lambda}(z)$  which has the property

$$\lim_{\Lambda \rightarrow \infty} \varphi_{\Lambda}(z) = \varphi(z)$$

for any  $z$  where, however, for any fixed  $\Lambda$ ,

$$\varphi_{\Lambda}(z) > z^2 \tau_{\Lambda} > 0. \quad (50.13)$$

If we now introduce  $\varphi_{\Lambda}$  into (50.2), we have

$$\int_{\alpha_{\Lambda}}^{\alpha_{\Lambda} d_{\Lambda}} \frac{dz}{\varphi_{\Lambda}(z)} = \ln x, \quad (50.14)$$

and hence, using (50.13), we obtain

$$\ln x < \int_{\alpha_{\Lambda}}^{\infty} \frac{dz}{z^2 \tau_{\Lambda}} = \frac{1}{\tau_{\Lambda} \alpha_{\Lambda}}.$$

It is now clear that for (50.14) to be valid throughout the entire region of variation of  $\ln x$  (between zero and infinity), we must put

$$\alpha_{\Lambda} = 0.$$

It is not difficult to see that it is precisely this type of approximation to  $\varphi(z)$  that is obtained in the case of the auxiliary regularization performed by introducing the nonlocal interactions, as was done by Landau and Pomeranchuk (1955). The above analysis shows that it is dangerous to draw any conclusions with regard to the exact problem on the basis of approximations.

We now consider the electron Green's function. Starting with the perturbation theory expansion for the structure functions

$$\begin{aligned} a(x, y, \alpha, \alpha d^l) &= 1 + \alpha a_1(x, y, d^l) + \alpha^2 a_2(x, y, d^l) + \dots, \\ b(x, y, \alpha, \alpha d^l) &= 1 + \alpha b_1(x, y, d^l) + \alpha^2 b_2(x, y, d^l) + \dots \end{aligned} \quad (50.15)$$

in second-order perturbation theory, we find that in the ultraviolet region,

$$a_1(x, d^l) = \frac{d^l}{4\pi} \ln x, \quad b_1(x, d^l) = \frac{d^l - 3}{4\pi} \ln x. \quad (50.16)$$

Substituting these expressions into (49.39) and (48.21), we obtain

$$\left. \begin{aligned} \ln \frac{a(x, \alpha, \alpha^l)}{a(1, \alpha, \alpha^l)} &= \int_1^x \frac{dz}{z} \frac{\alpha^l}{4\pi} = \frac{\alpha^l}{4\pi} \ln x, \\ \ln \frac{b(x, \alpha, \alpha^l)}{b(1, \alpha, \alpha^l)} &= \frac{\alpha^l}{4\pi} \ln x - \frac{3\alpha}{4\pi} \int_1^x \frac{dz}{z} d(z, \alpha). \end{aligned} \right\} \quad (50.17)$$

Using the explicit expression for  $d$ , given by (50.6), in the second equation, we obtain, neglecting the higher orders of  $\alpha$ .\*

$$\frac{a(x, \alpha, \alpha^l)}{a(1, \alpha, \alpha^l)} = x^{\frac{\alpha^l}{4\pi}}, \quad (50.18)$$

$$\frac{b(x, \alpha, \alpha^l)}{b(1, \alpha, \alpha^l)} = x^{\frac{\alpha^l}{4\pi}} [d(x, \alpha)]^{-9/4}. \quad (50.19)$$

A similar procedure can be used to determine the *symmetric* ultraviolet asymptotic behavior of the vertex function when  $p^2 \sim q^2 \sim k^2 \gg m^2$ . Assuming that  $x_1 = x_2 = x_3 = x$ , and following a procedure similar to that used above, we obtain

$$\frac{\Gamma(x, \alpha, \alpha d^l)}{\Gamma(1, \alpha, \alpha d^l)} = x^{-\frac{\alpha d^l}{4\pi}}. \quad (50.20)$$

It is clear from (43.12) and (43.17), that expansions of the following type have been obtained for  $d, a, b$ :

$$f_0(\alpha \ln x) + \alpha f_1(\alpha \ln x) + \dots$$

\*We note that these formulas were first obtained by Landau, Abrikosov, and Khalatnikov (1954a, c) by solving the approximate integral equations for the Green's functions.



Comparison of these expressions with the original expansions given by (50.9) and (50.15) shows that in order to reconstruct the form of the function  $f_i(\alpha \ln x)$  (with the aid of the differential equations of the renormalization group), it is sufficient to know the first term in its expansion into a Maclaurin series. Thus, for example, the term  $\alpha^2/3\pi \ln x$  in  $\bar{\alpha}_3$  gives the sum of the leading logarithmic terms of the form  $\alpha(\alpha \ln x)^n$ , the term  $\alpha^3/4\pi^2 \ln x$  gives the sum of all terms of the form  $\alpha^2(\alpha \ln x)^m$ , and so on.

**50.3. Asymptotic Behavior of the Electron Green's Function in the Infrared.** We now turn to the region  $p^2 \sim m^2$  where the electron Green's function has a singularity. We shall write the functions  $s = a, b$  in the form

$$s\left(\frac{p^2}{\lambda}, \frac{m^2}{\lambda}, \alpha, \alpha^l\right) = S\left(\frac{p^2 - m^2}{\lambda - m^2}, \frac{\lambda - m^2}{m^2}, \alpha, \alpha^l\right). \quad (50.21)$$

Substituting (50.21) into the right-hand side of (48.20), and using (48.22), we obtain

$$\ln \frac{s(x, y, \alpha, \alpha^l)}{s(x_0, y, \alpha, \alpha^l)} = \int_{x_0/y}^{x/y} \frac{dz}{z-1} \left[ \frac{\partial}{\partial \tau} \ln S(\tau, z-1, \alpha, \alpha^l) \right]_{\tau=1}. \quad (50.22)$$

To determine the integrand, we use second-order of perturbation theory:

$$\left. \begin{aligned} A(\tau, z-1, \alpha, \alpha^l) |_{z \rightarrow 1} &= 1 + \frac{\alpha}{2\pi} \{ (d^l - 3) \ln[\tau(z-1)] + c \} + \dots, \\ B(\tau, z-1, \alpha, \alpha^l) |_{z \rightarrow 1} &= 1 + \frac{\alpha}{2\pi} \{ (d^l - 3) \ln[\tau(z-1)] + c \} + \dots \end{aligned} \right\} \quad (50.23)$$

The final result is

$$\frac{a(x, y, \alpha, \alpha^l)}{a(x_0, y, \alpha, \alpha^l)} \Big|_{x \sim y} = \frac{b(x, y, \alpha, \alpha^l)}{b(x_0, y, \alpha, \alpha^l)} \Big|_{x \sim y} = \frac{\alpha(d^l - 3)}{2\pi} \ln \left| \frac{x-y}{x_0-y} \right|.$$

We have thus found that in the neighborhood of the point  $p^2 \sim m^2$ , instead of the single pole, the complete Green's function has the singularity

$$S(p) \sim \frac{1}{(m^2 - p^2)^{1+\beta}}, \quad \beta = \frac{\alpha(d^l - 3)}{2\pi}. \quad (50.24)$$

This is in complete agreement with the results of §45.5 and 46.2.

**50.4. Matrix Elements and Transition Probabilities.** As noted in §49.4, the multiplicative Dyson transformations [see (49.48)] can be written for the higher strongly connected vertices. The reduction formulas of §38.2 then enable us to go over to the transformation formulas for the matrix elements

$$M_{b,2f} \rightarrow z_2^f z_3^{b/2} M_{b,2f}, \quad (50.25)$$

introduced in accordance with (25.17). We note that the matrix elements transformed inversely with respect to the strongly connected vertices  $\Gamma$  and similarly to the weakly connected functions  $G$ .

Following Blank (1957a), we can now take the next step and consider the squares of the matrix elements, i.e., the transition probabilities which are proportional to the squares of the matrix elements:

$$W \sim |M|^2.$$

Transformation formulas similar to (50.25), and also the corresponding functional and differential equations, can be written for the transition probabilities.

When these equations are used, it must be remembered that in the matrix elements and the transition probabilities, some of the invariant momentum arguments lie on the mass surface and are therefore fixed. As noted in §49.4, certain difficulties will therefore arise when the asymptotic behavior of the matrix elements is examined in the ultraviolet region (i.e., the nonsymmetric asymptotic behavior of Green's functions).

This, however, does not apply to the case of infrared singularities where the equations of the renormalization group can be effectively employed. The main instrument used here is the set of differential equations such as (48.20) and their integrals such as (50.22), written for the squares of the matrix elements.

A consequence of integrals such as (50.22) is that the normalized radiative correction to  $W$  must be "placed in the exponential"

$$W = W_0 (1 + \alpha W_1 + \dots) \rightarrow W_0 e^{\alpha W_1}, \quad (50.26)$$

similarly to the situation encountered in the transition from the perturbation-theory formulas (50.23) for the electron Green's function to the result of integration of the group differential equation given by (50.24).

It is well known (see also §35.4) that in ordinary perturbation theory, the removal of the infrared catastrophe involves consideration of both the basic process and the emission of one additional soft photon whose energy does not exceed  $E_{\max}$ . The dependence of the resultant probability on  $E_{\max}$  then turns out to be singular ( $\sim \ln E_{\max}$ ). To obtain a physically correct result, we must sum the infinite series of diagrams describing the emission of different numbers of long-wave photons. This problem is substantially simplified when the renormalization-group technique is employed. The contribution  $W_1$  due to the process with one soft photon is proportional to  $\ln E_{\max}$  and, as a result of (50.26), turns up in the argument of the exponential. The resulting expression is equivalent to an infinite sum over the soft photons, and has the correct limit (vanishes) for  $E_{\max} \rightarrow 0$ .

Blank (1957b) has reported detailed calculations for the process involving the scattering of an electron in an external field.

## §51. Strong-Interaction Models

We must now consider the ultraviolet asymptotic behavior of strong-interaction models. We begin with a detailed examination of the one-charge model of the pseudoscalar field

introduced in §36.2. Despite its simplicity, this model provides an exhaustive illustration of the one-charge renormalization group.

*51.1. Main Logarithmic Approximation of the  $\phi^4$  Model.* The Lagrangian for this model will be taken in the form (57.2), with somewhat modified normalization of the coupling constant:

$$\mathcal{L} = \frac{4\pi^2}{3} h \varphi^4. \quad (51.1)$$

The invariant charge  $\bar{h}$  contains the product of the symmetric four-vertex  $\square$  and the square of the meson propagator

$$\bar{h}(x, h) = h \square(x, h) d^2(x, h). \quad (51.2)$$

Since the first correction to  $d$  is of the second order in  $h$ , the first logarithmic contributions to  $\bar{h}$  originate in  $\square$ .

According to (36.30), we have

$$\square(x, h) \simeq 1 - 3hL; \quad L = \ln x \quad (51.3)$$

and, consequently,

$$\bar{h}(x, h) \simeq h - 3h^2L, \quad \varphi(h) = -3h^2.$$

The differential equation (49.16) now assumes the form

$$\frac{\partial \bar{h}(x, h)}{\partial L} = -3\bar{h}^2(x, h).$$

Therefore, the invariant charge  $\bar{h}$  has a negative derivative and decreases from initial positive values  $h > 0$ :

$$\bar{h}(x, h) = \frac{h}{1 + 3hL} \rightarrow 0. \quad (51.4)$$

We thus arrive at case (d) in the classification of §49.2. The model defined by (51.1) is thus asymptotically free for  $h > 0$ .

It is interesting to determine the asymptotic behavior of the propagator in this case. If we evaluate the asymptotic behavior of (62.29), corresponding to the “walnut” type diagram shown in Fig. 50, we obtain

$$d(x, h) = 1 + \frac{h^2}{3} L. \quad (51.5)$$

The asymptotic behavior of the function  $d$  can now be determined on the basis of the general prescription given by (49.31). Using (51.5) to determine the corresponding function

$$\psi_d(h) = \frac{h^2}{3}, \quad (51.6)$$

we obtain

$$\ln d(x, h) = \int_h^{\bar{h}} \frac{\psi_d(h')}{\varphi(h')} dh' = \frac{\bar{h} - h}{9},$$

i.e.,

$$d(x, h) = \exp\left(\frac{h^2 L/3}{1 + 3hL}\right) = x^{\frac{h^2}{3(1 + 3h \ln x)}}. \quad (51.7)$$

Thus the propagator  $d$  tends asymptotically to the free expression  $d(\infty, h) = \text{const.}$  This property is characteristic for asymptotic freedom: the asymptotic behavior of the individual Green's functions tends to the free expression with logarithmic precision.

The properties of the above results, given by (51.4) and (51.6), are essentially based on the fact that the coupling constant  $h$  is positive.

The picture changes radically when  $h < 0$ . To discuss this case, it is convenient to substitute  $k = -h$ .

Thus, instead of (51.1) we consider the model

$$\mathcal{L} = -\frac{4\pi^2}{3} k\varphi^4, \quad k > 0 \quad (51.1')$$

(this sign of the coupling constant corresponds to a positive Hamiltonian, i.e., to repulsion in the boson plus boson system).

For the redefined invariant charge

$$\bar{k}(x, k) = k \square d^2 = -\bar{h}(x, -k) \quad (51.8)$$

we have, instead of (51.4),

$$\bar{k}(x, k) = \frac{k}{1 - 3kL} \quad (51.9)$$

i.e., case (c) in the classification of §49.2. Because of this, on the right-hand side of the Lie differential equation

$$\frac{\partial \bar{k}}{\partial L} = \varphi(\bar{k})$$

we must take into account the next terms in the power expansion, and the approximation of leading logarithms

$$\varphi(k) \simeq \varphi_2(k) = 3k^2$$

turns out to be inadequate.

*51.2. Second Logarithmic Approximations.* To determine the next terms in  $\varphi(k)$ , proportional to  $k^3$ , we must evaluate the logarithmic ultraviolet contributions in (51.8) of order  $k^3$  (i.e., terms  $\sim k^2$  in  $\square$  and  $d$ ). For the scalar propagator, we have from (51.5)

$$d(x, -k) \simeq 1 + \frac{k^2}{3} L, \quad (51.10)$$

and, to determine the corresponding contributions to the symmetric four-vertex, we must evaluate the ultraviolet asymptotic behavior of the third-order two-loop diagrams shown in Fig. 64.

The diagrams of Fig. 64a, which represent “fish” type interactions, provide contributions to  $\square$  that are proportional to the squares of the “fish” contributions

$$3k^2 L^2$$

(i.e., do not contain the leading logarithms), whereas the diagrams of Fig. 64b give the contributions

$$6k^2 (L^2 - 2L).$$

Therefore

$$\square \simeq 1 + 3kL + 9k^2 L^2 - 12k^2 L. \quad (51.11)$$

Using (51.10), we obtain

$$\bar{k}(x, k) = k + 3k^2 L + 9k^3 L^2 - \frac{34}{3} k^3 L. \quad (51.12)$$

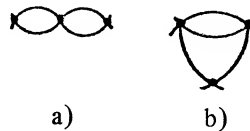


Fig. 64. Third-order contributions to the vertex function.

Therefore, the third approximation is

$$\varphi_3(k) = 3k^2 - \frac{34}{3}k^3. \quad (51.13)$$

This expression is very remarkable. The function  $\varphi_3$  vanishes linearly at the point

$$k^* = \frac{9}{34}, \quad (51.14)$$

so that the integral on the left-hand side of the equation

$$\int_k^{\bar{k}} \frac{dk'}{\varphi_3(k')} = L \quad (51.15)$$

increases without limit as  $\bar{k}$  tends to  $k^*$ . The quantity  $k^*$  is the ultraviolet limit of the invariant charge  $\bar{k}$ :

$$\bar{k}(\infty, k) = k^*. \quad (51.16)$$

We have thus obtained case (a) in the classification of §49.2. The Lie equation can be integrated in terms of quadratures which is equivalent to the explicit evaluation of the integral on the left-hand side of (51.15). We thus obtain

$$\frac{1}{k} - \frac{1}{\bar{k}} + \frac{1}{k^*} \ln \left( \frac{\bar{k}}{k} \cdot \frac{k^* - k}{k^* - \bar{k}} \right) = 3L \quad (51.17)$$

which is a transcendental equation for  $\bar{k} = \bar{k}(x, k)$ . It can be rewritten in the form

$$\bar{k}(x, k) = k \left[ 1 - 3kL + \frac{k}{k^*} \ln \left( \frac{\bar{k}}{k} \cdot \frac{k^* - k}{k^* - \bar{k}} \right) \right]^{-1},$$

which is convenient for expanding in powers of  $k$  in the denominator. The approximate result is

$$\bar{k}(x, k) = k \left[ 1 - 3kL + \frac{3k^2}{k^*} L + \frac{9k^3}{2k^*} L^2 + \dots \right]^{-1}, \quad (51.18)$$

which is in complete agreement with (51.12). To analyze the behavior of as  $L \rightarrow \infty$ , it is convenient to start directly with (51.15) in which the following approximation must be made:

$$\varphi_3(k') \simeq \frac{34}{3} (k^*)^2 (k^* - k).$$

This yields

$$\bar{k}(x, k) \simeq k^* - cx^{-27/34}. \quad (51.19)$$

Thus, the invariant charge  $\bar{k}$  tends to its limiting value  $k^*$  in a power-law fashion, and the exponent is not very different from unity. The limiting value is approached rapidly.

We now consider the asymptotic behavior of the single-particle Green's function. Substituting (51.6) and (51.13) in (49.13), and rearranging, we obtain

$$d(x, -k) \simeq x^\nu, \quad \nu = \frac{27}{(34)^2} \simeq 0.023. \quad (51.20)$$

The propagator  $d$  thus turns out to have a power-law asymptotic behavior, and the anomalous-dimension exponent is small in comparison with unity.

*51.3. Reliability of Results.* In conclusion of our analysis of the two-loop approximation, we note the following. We have verified that the inclusion of the second term in the function  $\varphi(k)$  produces a radical change in the qualitative picture. Instead of the internal contradiction in (49.2c) we obtain (49.2a) which corresponds to finite renormalization of the coupling constant. The result of the first logarithmic approximation thus turns out to be unstable with respect to the next corrections. We thus have the problem of the stability of the second approximation which we have just considered. The general answer to this question can be obtained within the framework of the second approximation. With this in mind, and recalling that the expression for  $\varphi_3$  given by (51.13) was obtained with the aid of perturbation theory, we use the expansion

$$\frac{1}{\varphi_3(k)} \simeq \frac{1}{3k^2} + \frac{34}{27} \frac{1}{k}. \quad (51.21)$$

when substituting (51.13) into the quadrature (51.15).

Substitution of this expression in (51.15) leads us to case (b) of §49.2. Since the approximation given by (51.21) is justified to the same extent as the approximation given by (51.13), we may conclude that the second-approximation results are generally unreliable.

This conclusion can be confirmed quantitatively by direct third-approximation calculations. The expression for the third approximation obtained by Belokurov et al. (1974) has the form

$$\varphi_4(k) = 3k^2 - \frac{34}{3}k^3 + 153.6k^4. \quad (51.22)$$

The function  $\varphi_4(k)$  has no zeros for real value of  $k$ , and we thus return to the internal contradiction of (49.2c).

We may summarize our discussion as follows. Perturbation theory gives a reliable result only when the function  $\varphi$  on the right-hand side of the group differential equation is

negative in the first logarithmic approximation. The invariant charge  $\bar{g}$  then decreases with increasing  $L = \ln x$ , and, in the limit as  $L \rightarrow \infty$ ,

$$\bar{g}(\infty, g) = 0 \quad (51.23)$$

and we obtain the asymptotically free theory. The higher-order approximation cannot change the sign of the function  $\varphi$  for sufficiently small values of the original coupling constant

$$\bar{g}(1, g) = g,$$

so that the asymptotically free case is stable with respect to the higher approximations.

In the opposite case (positive  $\varphi_2$ ),  $\bar{g}$  leaves the region of small values of the order of  $g \ll 1$  and no stable results based on a finite number of perturbation-theory terms for the function  $\varphi$  can in general be obtained. A reliable choice between (a), (b), and (c) cannot be made.

*51.4. Two-Charge Model.* Finally, we considered the asymptotic behavior of perturbation theory in the two-charge model

$$\mathcal{L} = ig\bar{\psi}\gamma^5\psi\varphi + h\varphi^4. \quad (36.44)$$

The functional and differential equations for this model were written out above in a general form [see (47.13), (48.12), (48.13), and (48.25)]. Using the lower-order perturbation-theory terms to evaluate the right-hand sides of the Lie equations (48.25), we obtain

$$\frac{\partial \bar{g}^2}{\partial L} = a_1 \bar{g}^4 + a_2 \bar{g}^2 \bar{h}^2 + \dots, \quad \frac{\partial \bar{h}}{\partial L} = b_1 \bar{g}^4 + b_2 \bar{h} \bar{g}^2 - b_3 \bar{h}^2 + \dots \quad (51.24)$$

If we now introduce the (arbitrary) assumption that  $g^2$  and  $h$  are small quantities of the same order, i.e.,

$$g^2 \sim h \ll 1, \quad (51.25)$$

we obtain

$$\frac{\partial \bar{g}^2}{\partial L} = a_1 \bar{g}^4, \quad (51.26a)$$

$$\frac{\partial \bar{h}}{\partial L} = b_1 \bar{g}^4 + b_2 \bar{g}^2 \bar{h} - b_3 \bar{h}^2. \quad (51.26b)$$

The coefficients  $a_i$ ,  $b_k$  are positive. Their numerical values are given in Table III.

As noted in §49.2, the set of equations given by (51.2) can be investigated by standard methods of the theory of differential equations.



Table III

Theory	$16\pi^2 a_1$	$16\pi^2 a_2$	$16\pi^2 b_1$	$16\pi^2 b_2$	$16\pi^2 b_3$	$\alpha_s$	$\alpha_d$	$\alpha_\Gamma$	$\alpha_\square$	$c_-$	$c_+$
Neutral	5	3	4	4	9	-0.1	-0.4	-0.2	-0.2	$\frac{\sqrt{145}-1}{18}$	$-\frac{1+\sqrt{145}}{18}$
Symmetric, charged	5	5	8	8	11	-0.3	-0.8	+0.2	+0.6	1	-8/11

The results of this analysis [Ginzburg (1956)] can conveniently be presented on the phase plane of the variables  $\bar{g}^2$  and  $\bar{h}$  (Fig. 65). This figure shows the integral curves  $\bar{h}(\bar{g}^2)$  on the half-plane  $\bar{g}^2 \geq 0$ . The momentum variable  $L = \ln x$  is the parameter increasing along the curves in the directions indicated by the arrows. Each pair of values of the coupling constant  $h$  and  $g^2 \geq 0$  corresponds to one integral curve such that  $\bar{h}(g^2) = h$ . The only exception is the origin  $h = 0, g^2 = 0$  through which an infinite set of integral curves is found to pass. This point is a singular point. Transition from a point  $\bar{g}_1^2, \bar{h}_1$  on a given integral curve to another  $\bar{g}_2^2, \bar{h}_2$  is equivalent to the transition from one value of the parameter  $L$  to another, or the transition from one pair of coupling constants  $g_1^2, h_1$  to another  $g_2^2, h_2$  under the transformations given by (47.13).

The above system has three singular solutions:

$$\bar{h} = c_+ \bar{g}^2, \quad \bar{h} = c_- \bar{g}^2, \quad \bar{g}^2 = 0, \quad (51.27)$$

where

$$c_{\pm} = \frac{a_1 - b_2 \pm \sqrt{(a_1 - b_2)^2 + 4b_1 b_3}}{-2b_3}.$$

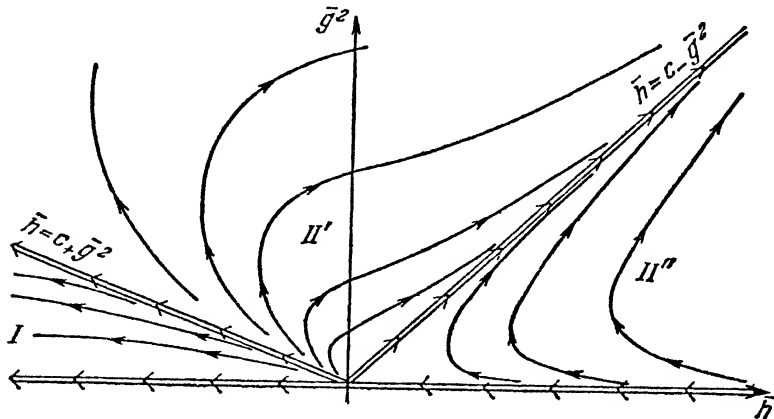


Fig. 65.

The solutions  $\bar{h} = c_+ \bar{g}^2$  and  $\bar{g}^2 = 0$ ,  $\bar{h} > 0$  are unstable, whereas  $\bar{h} = c_- \bar{g}^2$  and  $\bar{g}^2 = 0$ ,  $\bar{h} < 0$  are stable singular solutions. The line corresponding to the unstable singular solution  $\bar{h} = c_+ \bar{g}^2$  is the separation boundary between regions *I* and *II* = *II'* + *II''* on the phase plane. All solutions in region *II* ( $\bar{h} > c_+ \bar{g}^2$ ) tend to the stable singular solution  $\bar{h} = c_- \bar{g}^2$  which is therefore the asymptotic solution for this region. To the left of the separation boundary, for  $\bar{h} < c_+ \bar{g}^2$ , the solutions asymptotically tend to the neighborhood of the singular solution  $\bar{g}^2 = 0$ , i.e., the region  $\bar{g}^2 \ll |\bar{h}|$ .

Thus, to establish the ultraviolet asymptotic behavior, it is sufficient to analyze two stable solutions.

(a) *Singular solution*  $\bar{h} = c_- \bar{g}^2$ . By solving (51.26a), we obtain

$$\bar{g}^2(L) = \frac{g^2}{1 - a_1 g^2 L}; \quad \bar{h} = c_- \bar{g}^2. \quad (51.28)$$

The determining charge in this case is the invariant charge  $\bar{g}^2$  which governs the emergence from the framework of weak coupling. In asymptotic calculations, an effective device is to substitute

$$\bar{h}_{\text{eff}} = c_- \bar{g}^2. \quad (51.29)$$

By solving (49.29) for Green's functions and vertex functions, we obtain the corresponding symmetric asymptotic behavior:

$$\begin{aligned} s(x, g^2, h) &= \left(\frac{g^2}{\bar{g}^2}\right)^{\alpha_s}, & \Gamma(x, g^2, h) &= \left(\frac{g^2}{\bar{g}^2}\right)^{\alpha_\Gamma}, \\ d(x, g^2, h) &= \left(\frac{g^2}{\bar{g}^2}\right)^{\alpha_d}, & \square(x, g^2, h) &= \left(\frac{g^2}{\bar{g}^2}\right)^{\alpha_\square}. \end{aligned} \quad (51.30)$$

Numerical values of the parameters  $\alpha_i$ ,  $a_k$ ,  $b_l$ ,  $c_\pm$  for the neutral and isospin symmetric versions of the pion-nucleon interaction listed in Table III.

(b) *Singular solution*  $\bar{g}^2 = 0$ ,  $\bar{h} < 0$ . In this case, we must solve (51.26b) with  $\bar{g}^2 = 0$ . This equation was investigated in §51.1. The solution is

$$\bar{h}(L) = \frac{h}{1 + b_3 h L} \quad (h < 0), \quad \bar{g}^2 = 0. \quad (51.31)$$

Here we have the framework of weak coupling because of the increase in  $\bar{h}$ . To investigate the neighborhood of the singular solution, we must take into account the term containing  $g^2 \bar{h}^2$  in the equation for  $\bar{g}^2$  [this was not taken into account in (51.26a)] and omit the term  $\bar{g}^4$ :

$$\frac{\partial \bar{g}^2}{\partial L} = a_2 \bar{g}^2 \bar{h}^2, \quad a_2 > 0.$$

The solution of this equation, with  $\bar{h}$  given by (51.31), is

$$\bar{g}^2(L) = g^2 \exp \left\{ \frac{a_2}{b_3} (h - \bar{h}) \right\} = g^2 x^{\frac{a_2 h^2}{1 + b_3 h \ln x}} \simeq g^2. \quad (51.32)$$

Moreover, in this case,

$$s(x) \sim \Gamma(x) \sim 1, \quad d(x) \sim \bar{g}^2/g^2, \quad \square(x) \sim \bar{h}/h. \quad (51.33)$$

Thus, both in region *I* and in region *II*, sufficiently large values of the argument  $L = \ln x$  correspond to emergence out of the framework of weak coupling, independently of the degree of smallness of the initial values of  $g^2$  and  $h$ .

We note that for  $h > 0$ , the emergence out of the framework of weak coupling does not occur for the singular solution  $\bar{g}^2 = 0$ . However, this solution corresponds to the complete absence of the Yukawa interaction in the Lagrangian (36.44), and is unstable. The corresponding integral curves approach  $\bar{h} = c - \bar{g}^2$  asymptotically for  $g^2 > 0$  as small as desired.

We have thus established that the two-charge model (36.44) of the pion-nucleon interaction (like spinor electrodynamics) takes us out of the framework of weak coupling, i.e., we have a situation equivalent to positive  $\varphi$  in one-charge models. In the lowest approximation, this yields formal expressions for  $\bar{g}^2$  and  $\bar{h}$ , containing the "ghost" poles. However, experience gained as a result of analyses of the  $\varphi^4$  model enables us to conclude that we cannot as yet make a realistic choice between (a), (b), and (c) (§49.2). Reliable enough arguments of this kind cannot be based on perturbation-theory data.

An important qualitative result is the fact that the "zero-charge" properties of the Yukawa-type interaction (which appear in electrodynamics as well) cannot be corrected by the addition of a four-scalar interaction independently of its sign. This situation is typical for a broad class of Yukawa and four-scalar models used in phenomenologic descriptions of hadronic interactions.

Thus, the general and relatively uninspiring conclusion relating to all such models is that departure from the framework of weak coupling is unavoidable, and that perturbation theory is fundamentally inadequate in the qualitative analysis of ultraviolet asymptotic behavior.

A new class of quantum field models, based on the non-Abelian Yang-Mills gauge fields, was discovered quite recently.

Models belonging to this class contain only one coupling constant,  $g_{YM}$ , and perturbation-theory expansions involve powers of its square. The corresponding renormalization group turns out to be a one-charge group, and the function  $\varphi$  turns out to be negative in the lowest-order approximation of perturbation theory:

$$\varphi_2(g^2) = -c(g^2)^2, \quad c > 0.$$

In view of this, we arrive at case (d) in the classification of §49.2, and models in this class turn out to be asymptotically free.

51.5. *Symmetric Asymptotic Behavior of the Higher Green's Functions.* It was shown in §37.1 that the weakly connected multiparticle Green's functions with  $b$  meson and two  $f$  fermion external lines can be determined on the basis of the chronological products such as (37.1)

$$G_{b,2f}(x_1, \dots, x_b; y_1, \dots, y_f, z_1, \dots, z_f) \sim \langle T \varphi_1(x_1) \dots \varphi_b(x_b) \psi_1(y_1) \dots \psi_f(y_f) \bar{\psi}_1(z_1) \dots \bar{\psi}_f(z_f) S \rangle_0.$$

Correspondingly, the strongly connected (i.e., vertex) multiparticle functions were introduced there with the aid of the corresponding variational derivatives (37.37).

The Green's functions  $G_{b,2f}$  contain self-energy inserts in external lines. We shall now consider symmetric ultraviolet asymptotic behavior when all the invariant momentum arguments of the dimensionless function  $G$  are large quantities of the same order, i.e.,

$$|k_i k_j| \sim k^2, \quad k^2 \gg m^2. \quad (51.35)$$

This case is not directly related to real physical processes (in which  $k_i^2 = m_i^2$ ). However, by investigating it, we can throw light on the nature of the interaction at "short distances."

The relative simplicity of the analysis of the higher Green's functions is due to the fact that these functions do not contain intrinsic divergences and are renormalized by the factors  $z_2$  and  $z_3$  that renormalize the single-particle Green's functions:

$$G_{b,2f} \rightarrow z_2^f z_3^{\frac{b}{2}} G_{b,2f}. \quad (51.36)$$

Using (47.30) to determine  $z_2$  and  $z_3$ , we obtain

$$G_{b,2f}(x, g^2, h) = s^{-f}(x, g^2, h) d^{-\frac{b}{2}}(x, g^2, h) G_{b,2f}(1, \bar{g}^2(x), \bar{h}(x)). \quad (51.37)$$

we note that the symmetric asymptotic behavior of the basic functions  $\Gamma$  and  $\square$  can also be written in this form:

$$\Gamma(x) = s^{-1}(x) d^{-1/2}(x) \left[ \frac{\bar{g}^2(x)}{g^2} \right]^{1/2}, \quad \square(x) = d^2(x) \frac{\bar{h}(x)}{h}. \quad (51.38)$$

These relationships are a direct consequence of the definitions of the invariant coupling constants  $\bar{g}^2$  and  $\bar{h}$ .

In the region of weak coupling, (51.37) yields the following expression for (51.20):

$$G_{b,2f}(x) = \left( \frac{\bar{g}(x)}{g} \right)^{2\alpha_s f + \alpha_d b}, \quad (51.39)$$

and also

$$\Gamma(x) = \left( \frac{\bar{g}(x)}{g} \right)^{1 + \alpha_d + 2\alpha_s}, \quad \square(x) = \left( \frac{\bar{g}(x)}{g} \right)^{1 + 2\alpha_d}. \quad (51.40)$$

*51.6. Conclusion. Hypothesis of Finite Renormalization of Coupling Constants.* Thus, both in spinor electrodynamics and in the two-charge meson-nucleon theory, the improvement in the convergence of perturbation-theory series in the ultraviolet region, which is achieved by solving the group differential equations (or some other way), leads us to the conclusion that the effective parameters of the improved expansions (i.e., the invariant charges) unavoidably cease to be small in the ultraviolet region, and we leave the framework of weak coupling. Of course, expressions such as (50.6) and (51.28), obtained in the “first renormalization-group approximation” (equivalent to the summation of the leading powers of the ultraviolet logarithms) and containing the logarithmic poles, provide only information on the “rate” of emergence from the weak-coupling region and cannot serve as a basis for any reasonable conclusions relating to the invariant charges or Green’s functions for sufficiently large values of the ultraviolet logarithms. As noted in §51.3, such an analysis cannot be carried out within the framework of the weak-coupling model.

It follows that one can only speculate about the behavior of the invariant charges for sufficiently large values of  $L = \ln x$ . The assumption of finite invariant charge for  $x \rightarrow \infty$  is physically interesting. This behavior, which is consistent with a finite value of the renormalizing constant  $z_3$ , can be obtained in spinor electrodynamics within the framework of certain models [see, for example, Redmond (1958) and Bogolyubov, Logunov, and Shirkov (1959)].

This hypothesis arose in the theory of strong interactions in connection with attempts to obtain a reasonable description of the high-energy experimental situation in the diffraction region [Shirkov (1963), Ginzburg and Shirkov (1965)] and in the region of deep inelastic scattering [Wilson (1971)].

The hypothesis of finite renormalization of the coupling constant corresponds to variant (a) of §49.2. It was shown in §49.3 that, in this case, the single-particle and the higher Green’s functions may have power-law asymptotic behavior of the form given by (49.43). Such a behavior in the ultraviolet region would physically correspond to *scale invariance* (or *scaling*) which satisfactorily describes many of the qualitative features of high-energy hadronic and electromagnetic interactions. It is precisely for this reason that the hypothesis of finite renormalization has attracted attention.

We note in this connection that as was pointed out in §49.3, the power- and quasipower-law asymptotic behavior of the Green’s functions and the higher vertices may arise for (49.2b) as well, i.e., infinite renormalization of the coupling constant [this case is analyzed in detail by Shirkov (1973)].

In the multicharge case, one must consider the possible properties of the functions  $\varphi_i$  on the right-hand sides of (49.36). The case of finite renormalization of the coupling constants corresponds to the simultaneous vanishing of all functions  $\varphi_i$  at a certain point

$$g_1^*, \dots, g_{k+1}^* \quad (51.41)$$

in the phase space of the variables  $\bar{g}_1, \dots, \bar{g}_{k+1}$ . If this point is stable, then the quantities

(51.41) are the asymptotic values of the invariant coupling constants

$$\bar{g}_i(x = \infty) = g_i^*.$$

The other favorable possibility, which does not have an analog in the one-charge case, is connected with the existence of a *limit cycle*. The invariant coupling constants oscillate around certain mean values in the limit as  $x \rightarrow \infty$ .



# Chapter 10

## DISPERSION RELATIONS

### §52. Basic Properties of the $S$ -Matrix in Local Field Theory

*52.1. Introduction.* In this chapter, we shall present the fundamentals of a further method that is not based on an expansion in powers of the interaction strength, i.e., on perturbation theory.

We have already noted that the perturbation-theory method is totally invalid for the so-called strong interactions (i.e., interactions of mesons and baryons) because the corresponding coupling constants are large. It is now known that, for example, the renormalized coupling constant for the pseudoscalar interaction between pions and nucleons (8.8) is greater than unity. The corresponding dimensionless expansion parameter that is analogous to the fine structure constant  $\alpha$  turns out to be equal to (see below, §57.5)

$$\frac{g^2}{4\pi} \simeq 14.7.$$

This means that the simplest Feynman diagrams corresponding to the lowest powers in expansions in terms of the parameter  $g^2$  do not provide even a qualitative agreement with experiment.

The method of dispersion relations, to which the present chapter is devoted, was put forward by Gell-Mann, Goldberger, and Thirring (1954) and was rigorously justified in quantum field theory by Bogolyubov (1956). It has played an exceedingly important role in the development of the theory of strong interactions. In recent years, it has become the basic rigorous method for theoretical investigations in the physics of fundamental interactions. In addition to providing a large number of rigorous results, it has become



an intimate part of various heuristic schemes of semiphenomenologic character and is also the starting point for many approximate methods of investigating the dynamics of strong interactions, based on rigorously established or postulated [for example, the double spectral representation of Mandelstam (1958)] analytic properties of matrix elements.

In what follows, we give a compressed but relatively complete and rigorous presentation of the fundamentals of the method and of its principal applications. Our account is based on general physical principles drawn from quantum field theory, such as, for example, invariance and unitarity, i.e., principles lying at the basis of *modern axiomatic formulation* of quantum field theory.

The mathematical basis of the dispersion relations is the Cauchy integral formula. It is well known that this formula represents an analytic function  $f$  of the complex variable  $z$  as an integral over a closed contour  $\Gamma$  bounding the region  $G$  within which  $f(z)$  is analytic:

$$\frac{1}{2\pi i} \int_{\Gamma} \frac{f(z')}{z' - z} dz' = \begin{cases} f(z), & \text{if } z \text{ lies inside } \Gamma, \\ 0, & \text{if } z \text{ lies outside } \Gamma. \end{cases} \quad (52.1a)$$

$$(52.1b)$$

In the intermediate case, when the point  $z$  lies on the contour of integration, the interpretation of the Cauchy integral in the sense of the principal value leads to the formula\*

$$f(z) = \frac{1}{\pi i} \oint_{\Gamma} \frac{f(z')}{z' - z} dz'. \quad (52.2)$$

Taking separately the real and the imaginary parts of (52.2), we obtain relations between  $\text{Re } f$  and  $\text{Im } f$ . The dispersion relations are relations of just this type between the real and the imaginary parts of the matrix elements of the scattering matrix.

It is therefore clear that in order to obtain the dispersion relations, it is very important to carry out an investigation of the analytic properties of the matrix elements. It is customary to investigate the analytic nature of the elements of the scattering matrix considered as functions of energy, and the possibility of analytic continuation of these matrix elements into the upper half-plane.

We now turn to the case where as a result of such analytic continuation we obtain a function  $g(E)$  which falls not slower than  $\text{const}/|E|$  as  $|E| \rightarrow \infty$  in the upper half-plane, i.e.,

$$|g(E)| \leq \frac{A}{|E|} \quad \text{for} \quad |E| \rightarrow \infty. \quad (52.3)$$

Then, by choosing for the contour of integration in (52.2) the real axis and the upper semicircle of infinitely large radius, we can neglect the integral over the semicircle and obtain the following formula:

$$g(E) = \frac{1}{\pi i} \oint \frac{g(E')}{E' - E} dE'.$$

\*This formula may be obtained from (52.1a) with the aid of the symbolic identity in (A2A.6).

Taking the real part of this expression, we obtain a relation between the real and the imaginary parts of the function  $g$ :

$$\operatorname{Re} g(E) = \frac{1}{\pi} \mathcal{P} \int_{-\infty}^{\infty} \frac{\operatorname{Im} g(E')}{E' - E} dE'. \quad (52.4)$$

It should be noted that condition (52.3) turns out to be somewhat restrictive for physical applications. However, it is not difficult to extend the argument given above to the case of functions  $g(E)$  which increase polynomially for large  $|E|$ . For example, let

$$g(E) \leq A |E|^n \quad \text{for} \quad |E| \rightarrow \infty.$$

The quantity

$$\tilde{g}(E) = \frac{g(E)}{(E - E_0)^{n+1}}$$

will then be an analytic function in the upper half-plane and will satisfy (52.3) for all  $E_0$  that have a negative imaginary part. Substituting for  $\tilde{g}$  in (52.2), where the contour  $\Gamma$  has been chosen in the same way as in (52.4), we obtain, after going to the limit\*  $\operatorname{Im} E_0 \rightarrow 0$ , a relation whose real part has the form

$$\begin{aligned} \operatorname{Re} g(E) = & \frac{(E - E_0)^{n+1}}{\pi} \mathcal{P} \int_{-\infty}^{\infty} \frac{\operatorname{Im} g(E') dE'}{(E' - E)(E' - E_0)^{n+1}} + \\ & + \operatorname{Re} g(E_0) + \dots + \frac{\operatorname{Re} g^{(n)}(E_0)}{n!} (E - E_0)^n \quad (-\infty < E, E_0 < \infty). \end{aligned} \quad (52.5)$$

A relation of this type is, for example, the dispersion relation of classical electrodynamics between the real and the imaginary parts of the index of refraction\*\*, which was obtained in the 1920s by Kronig (1926) and by Kramers (1927)

$$\operatorname{Re} [n(\omega) - n(0)] = \frac{2}{\pi} \mathcal{P} \int_0^{\infty} \frac{\omega'^2 \operatorname{Im} n(\omega')}{\omega'(\omega'^2 - \omega^2)} d\omega'.$$

\*This transition to the limit may be carried out with the aid of the formula

$$\frac{1}{(E' - E + i\epsilon)^{n+1}} = \mathcal{P} \frac{1}{(E' - E)^{n+1}} - \frac{i\pi(-1)^n}{n!} \delta^{(n)}(E' - E),$$

which is obtained from (46.3) by differentiating  $n$  times.

\*\*The Kramers-Kronig formula may be obtained from (52.5) by setting

$$n=0, \quad E_0=0 \quad \text{and} \quad \operatorname{Im} g(E') = -\operatorname{Im} g(-E').$$

Relations such as (52.5) are called dispersion relations in order to indicate their link with the Kramers-Kronig formula.

We emphasize that the possibility of analytic continuation into the upper half-plane of the energy variable and, consequently, the possibility of obtaining formulas such as (52.4), rests on *considerations of causality* which form the *physical basis of the dispersion relations*.

In order to clarify the connection just indicated, let us suppose, simplifying somewhat the actual state of affairs, that

$$f(E) = \int_{-\infty}^{\infty} F(t) e^{iEt} dt, \quad (52.6)$$

where as a result of the "condition of causality" the function  $F$ , which depends on the time  $t$ , has the property

$$F(t) = 0 \quad \text{for} \quad t < 0.$$

In passing into the upper half-plane, when  $+i|\text{Im } E|$  is added to  $E$ , and the integral (52.6) acquires the factor  $\exp(-t|\text{Im } E|)$ , this factor will play the role of a cutoff factor guaranteeing the convergence of the integral, since for  $t < 0$ , when  $\exp(-t|\text{Im } E|)$  increases, the function  $F(t)$  is zero because of the condition of causality.

It may also be shown that even if  $F(t)$  is a singular function, as long as it is integrable in the sense of the definition given in §19, the integral (52.6) converges and defines a function without essential singularities at infinity.

A different situation occurs when  $F(t)$  vanishes only for  $t < -a$ , where  $a$  is some "elementary length." Then, replacing  $t$  by  $t - a$  in the integrand of (52.6), we find that

$$f(E) = e^{-iaE} f_1(E),$$

where the function  $f_1(E)$  has no essential singularity at infinity (the factor  $\exp(-iaE)$  does have such a singularity). Therefore, to obtain in this case a function for which the dispersion relation holds, it is necessary to multiply  $F(E)$  by  $\exp(i\alpha E)$  with  $\alpha \geq a$ .

In actual fact, the situation is much more complicated if only because the expressions replacing (52.6) contain a large number of integration variables. Nevertheless, despite the substantial technical improvement in the above discussion, the foundations have remained unaltered.

**52.2. General Properties of the Scattering Matrix.** We now proceed to the explicit formulation of the basic physical principles of quantum field theory that are necessary for establishing the dispersion relations.

We have already partially carried out a program of this kind in connection with the development of expansions for the scattering matrix (Chapter 4). This was done by taking as a starting point explicitly formulated conditions of unitarity, causality, and covariance for the  $S$ -matrix. However, the development of the theory was confined to the framework of perturbation theory.

Moreover, this treatment had one further serious defect, connected with the use of the auxiliary function  $g(x)$ . This function was used to formulate the generally accepted "pseudo-physical" principle of adiabatic switching on and off of the interaction, and it also enabled us to investigate the local features of the theory. This possibility was technically connected with the operation of functional differentiation of the  $S$ -matrix with respect to  $g(x)$ .

It therefore seems desirable to review our system of basic principles in order to free ourselves from the function  $g(x)$  which is, essentially, necessary only for power-type expansions.

As indicated in §20.6, instead of using variational derivatives with respect to  $g(x)$  in the description of the local structure of the theory, we can use the derivatives with respect to the field functions. Such variational derivatives with respect to the Bose and Fermi operator arguments were introduced in §§37 and 38. We shall now use them.

We shall also abandon the idea of introducing the  $S$ -matrix with the aid of the generally accepted adiabatic switching on and off of the interaction, and will return to the original definition given by Heisenberg, in which it is defined as the matrix whose elements are the transition probability amplitudes between a state at  $t = -\infty$  and another state at  $t = \infty$ . Each of these states can be occupied both by the system of "infinitely" distant individual elementary particles and their complexes in bound states. This way of introducing the  $S$ -matrix is more realistic than that employed earlier because it does not assume the absence of interaction with virtual fields for  $t = \pm\infty$  (see §20.1).

This presents us with the problem of describing the initial and final states of spatially separated real particles and bound states. We shall not, however, concern ourselves with these complicated questions but proceed directly to the formulation of the necessary basic physical principles of modern field theory.

These principles can be conveniently divided into two groups: (a) *general properties* characteristic for a very broad class of possible theories, and (b) *special local properties*, connected, in particular, with the condition of microscopic causality.

#### *General properties*

A\*) The asymptotic states of a system contain infinitely distant real particles and their bound complexes. There is no interaction between such particles and complexes, so that the fundamental characteristics of the system (such as energy, momentum, angular momentum, and so on) are additive. Such states are described by amplitudes  $|\dots\rangle$  that are the elements of a linear space.

We emphasize that in contrast to our previous presentation, where the asymptotic state corresponded to free noninteracting particles (and fields), here we are concerned with states with real observable (in the adiabatic approach, renormalized) characteristics.

B) There is a group  $G$  of transformations  $L$  that includes the Poincaré group of transformations of space-time ( $G$  may include other elements, for example, isospin transformations). The state amplitudes  $|\dots\rangle$  transform under the transformations  $L$  in  $G$  with the aid of a certain unitary representation  $U_L$  of the group.

C) If, in the state  $|p\rangle$  there is a definite energy-momentum four-vector, then

\*These points will be cited in the following as (52.2A), (52.2B), and so on.

$$U_{L_a} |p\rangle = e^{-ip a} |p\rangle, \quad (52.7)$$

where  $L_a$  is the translation  $x \rightarrow x + a$ . There exists a vacuum state  $|0\rangle$  for which

$$U_{L_a} |0\rangle = |0\rangle. \quad (52.8)$$

We note that in accordance with (9.23) and (52.7), it follows, in particular, that

$$\langle p' | A(x) | p \rangle = e^{ix(p' - p)} \langle p' | A(0) | p \rangle. \quad (52.9)$$

D) There exists a system of amplitudes  $|n, k\rangle$  which, together with the vacuum amplitudes  $|0\rangle$ , is closed, so that

$$\langle \alpha | AB | \beta \rangle = \langle \alpha | A | 0 \rangle \langle 0 | B | \beta \rangle + \sum_n \int dk \langle \alpha | A | n, k \rangle \langle n, k | B | \beta \rangle, \quad (52.10)$$

and is such that definite momentum  $k$  and energy  $E_n(k) > 0$  can be associated with the state  $|n, k\rangle$ . The index  $n$  represents the set of quantum numbers, both discrete and continuous, which, together with  $k$ , completely characterizes the state of a given closed system. Analogous properties can be formulated for irreducible representations of other subgroups of  $G$ , in particular, representations corresponding to the angular momentum.

E) The transition probability amplitude between state  $|\alpha\rangle$  and state  $|\beta\rangle$  is given by the matrix element  $\langle \beta | S | \alpha \rangle$  of the operator  $S$  (the scattering matrix) satisfying the unitarity condition

$$S S^\dagger = 1.$$

F) The scattering matrix  $S$  transforms under a transformation  $L$  in  $G$  with the aid of the unitary representation  $U_L$ .

G) If  $|\alpha\rangle$  is the vacuum amplitude, or the amplitude of a state containing one real particle or one stable complex, the stability conditions for such states is

$$S |\alpha\rangle = |\alpha\rangle. \quad (52.11)$$

Condition (52.11) determines the unitary phase factor to within which the scattering matrix is usually defined. Thus, from the point of view of the usual theory given in Chapters VI and V, condition (52.11) reduces to the stability condition for the vacuum state and the one-particle state.

This condition could have been obtained in our original presentation by replacing the usual scattering matrix  $S$  by an operator  $S'$  obtained from  $S$  by subtracting the vacuum and one-particle counterterms from the interaction Lagrangian. For example, for spinor electrodynamics

$$S' = T \left[ \exp i \int d\rho \left\{ \mathcal{L}(\rho) - (Z_2 - 1) \bar{\psi}(\rho - m) \psi - \delta m \bar{\psi} \psi \right\} - \right. \\ \left. - (Z_3 - 1) A_\mu (g^{\mu\nu} \rho^2 - \rho^\mu \rho^\nu) A_\mu - R_{\text{vac}} \right],$$

so that

$$\langle 0 | S' | 0 \rangle = 1,$$

and

$$\begin{aligned} \langle 1 \text{ photon} | S' | 1 \text{ photon} \rangle &= 1, \\ \langle 1 \text{ electron} | S' | 1 \text{ electron} \rangle &= 1, \end{aligned}$$

which is equivalent to (52.11).

Propositions (52.2A)–(52.2G) are so general that they will probably remain unaltered in possible future modifications of the theory of elementary particles.

**52.3. Local Properties.** Before we begin the formulation of the local properties of the theory, which we require, we recall the corresponding points of the usual formulation. We begin with the fact that the representation of the  $S$ -matrix in terms of the  $T$ -exponential

$$S = T \left( \exp i \int \mathcal{L}(z) dz \right)$$

allows us to obtain at once its variational derivatives. For example,

$$\frac{\delta S}{\delta u_\alpha(x)} = iT \left( \frac{\partial \mathcal{L}(x)}{\partial u_\alpha(x)} \exp i \int \mathcal{L}(z) dz \right).$$

Expressions of this type will also be obtained for the higher variational derivatives.

We may now write the condition of causality in a form analogous to (20.31):

$$\frac{\delta}{\delta u_\alpha(x)} \left[ \frac{\delta S}{\delta u_\beta(y)} \overset{\dagger}{S} \right] = 0 \quad \text{for} \quad x \lesssim y. \quad (52.12)$$

We note further that the probability amplitude for the scattering of free particles (i.e., for such processes in which bound complexes are not present initially nor appear in the course of the process) may be expressed in terms of the vacuum expectation values of the variational derivatives of the  $S$ -matrix with respect to the free fields. Indeed, consider the matrix element

$$\frac{(2\pi)^{\frac{3(r+s)}{2}}}{S_0} \langle 0 | a_{\beta_1}^-(p'_1) \dots a_{\beta_s}^-(p'_s) S a_{\alpha_1}^+(p_1) \dots a_{\alpha_r}^+(p_r) | 0 \rangle \quad (52.13)$$

for a process at the beginning of which we have particles with momenta  $p_1, \dots, p_r$  and other quantum numbers  $\alpha_1, \dots, \alpha_r$ , and at the end of which we have particles with momenta  $p'_1, \dots, p'_s$  and other quantum numbers  $\beta_1, \dots, \beta_s$  (we assume, as usual, that there are no identical pairs among the pairs  $p_i \alpha_i, p'_j \beta_j$ ). We now use the usual commutation relations (24.11), (24.12):

$$\left. \begin{aligned} \{a_{\beta}^{-}(\mathbf{p}'), u_{\gamma}(x)\}_{\pm} &= \frac{1}{(2\pi)^{3/2}} e^{i\mathbf{p}'x} v_{\gamma}^{\beta,+}(\mathbf{p}'), \\ \{u_{\gamma}(x), a_{\alpha}^{+}(\mathbf{p})\}_{\pm} &= \frac{1}{(2\pi)^{3/2}} e^{-i\mathbf{p}x} v_{\gamma}^{\alpha,-}(\mathbf{p}) \end{aligned} \right\}$$

and take into account the fact that  $\mathcal{L}(x)$  must be an even function of the Fermi fields. This gives [cf. (38.11) and (38.12)]

$$\left. \begin{aligned} [a_{\beta}^{-}(\mathbf{p}'), S]_{-} &= \frac{1}{(2\pi)^{3/2}} \sum_{\sigma} v_{\sigma}^{\beta,+}(\mathbf{p}') \int dx e^{i\mathbf{p}'x} \frac{\delta S}{\delta u_{\sigma}(x)}, \\ [S, a_{\alpha}^{+}(\mathbf{p})]_{-} &= \frac{1}{(2\pi)^{3/2}} \sum_{\sigma} v_{\sigma}^{\alpha,-}(\mathbf{p}) \int dx e^{-i\mathbf{p}x} \frac{\delta S}{\delta u_{\sigma}(x)}. \end{aligned} \right\} \quad (52.14)$$

We now transfer all the annihilation operators in (52.13) to the right until they yield zero by acting on  $|0\rangle$ , and we move all the creation operators to the left. We then use (52.14) to express the matrix element as the sum of terms proportional to the integrals

$$\int \langle 0 | \frac{\delta^{r+s} S}{\delta u_{\beta_1}(x'_1) \dots \delta u_{\alpha_r}(x_r)} | 0 \rangle \exp i \left( \sum_{\beta} \mathbf{p}' x' - \sum_{\alpha} \mathbf{p} x \right) dx'_1 \dots dx_r. \quad (52.15)$$

After eliminating the vacuum loops, we obtain the following matrix elements in the integrands:

$$\frac{1}{S_0} \langle 0 | \frac{\delta^n S}{\delta u_1(x_1) \dots \delta u_n(x_n)} | 0 \rangle = \langle 0 | \frac{\delta^n S}{\delta u_1(x_1) \dots \delta u_n(x_n)} \hat{S}^{\dagger} | 0 \rangle. \quad (52.16)$$

We can easily verify this relation if we take into account the fact that the amplitude  $S^{\dagger} | 0 \rangle$  may differ from  $| 0 \rangle$  only by a phase factor equal to  $(S_0)^{-1}$ .

We have thus arrived at the important concept of the *radiation operator of order  $n$* :

$$H_n(x_1, \dots, x_n) = \frac{\delta^n S}{\delta u_1(x_1) \dots \delta u_n(x_n)} \hat{S}^{\dagger}. \quad (52.17)$$

It is seen that the order of the above operator is taken to mean the resultant order of the variational derivatives. The matrix elements may be reduced to the vacuum expectation values of such radiation operators (52.13).

Doubts may arise as to the admissibility of the transition to the variational derivatives with respect to the quantized fields such as (38.9), since the property of the quantized functions of satisfying the field equations is not in any way apparent from the definition (38.9). However, it may be shown that there are no foundations whatever for doubts of this nature. Indeed, in the  $S$ -matrix, the foregoing property is in fact related to the procedure of calculating the matrix elements and becomes apparent only for those field

operators which from the point of view of Feynman diagrams correspond to external lines. The above property then turns out to be a trivial consequence of the commutation relations (52.4) between the operator  $u(x)$  in the matrix  $S$  and the creation (or annihilation) operator for a free particle  $a^\pm(k)$  in the expression for the state amplitude. Therefore, in the investigation of the  $S$ -matrix and of its variational derivatives, we may completely ignore the above property (and consider that we are dealing with a formal extension of the functional  $S$  to a class of functions which do not have to satisfy any equations). Of course, at the final stage of going over to the matrix elements, we have to consider the projection of  $S$  on the set of operators that satisfy the free-field equations (here we have in mind only operators corresponding to the *external* lines of Feynman diagrams!).

In subsequent presentations we shall need neither the representation of the  $S$ -matrix in terms of the  $T$ -exponential, nor even the concept of the interaction Lagrangian. It will be sufficient to retain only the possibility of variational differentiation of the  $S$ -matrix, the condition of causality in the form of (52.12), and the possibility of expressing the transition amplitude (52.13) in terms of integrals such as (52.15). Therefore, we arrive at the possibility of formulating the following local properties.

#### *Local properties*

A.\* Real elementary particles are characterized by boson and fermion fields  $u(x)$  with the same transformation and commutation properties as in the theory of free fields. The operator  $S$  has variational derivatives of arbitrary order with respect to these fields. These variational derivatives have all the usual properties. Thus, their transformation properties are determined by the transformation properties of the fields. The derivatives with respect to the boson fields commute, and those with respect to the fermion fields anticommute among themselves.

The radiation operators (52.7) and products of such operators with independent arguments are integrable (in the sense of the definition given in §18), i.e., all the matrix elements

$$\langle \beta | H(x_1, \dots, x_n) \dots H(z_1, \dots, z_k) | \alpha \rangle$$

are generalized functions integrable over one of the classes  $C(q, r, n)$ .

B. The condition of causality holds in the form of (52.22).

C. Let

$$|\alpha_1, p_1, \dots, \alpha_r, p_r\rangle = |\omega\rangle$$

denote the state amplitude for a system of infinitely separated elementary particles with momenta  $p_1, \dots, p_r$  and other quantum numbers  $\alpha_1, \dots, \alpha_r$ . The matrix element

$$S_{\omega'\omega} = \langle \omega' | S | \omega \rangle$$

may then be expressed in terms of vacuum expectation values of radiation operators (52.12)

\*These properties are referred to later as 52.3A, 52.3B, and so on.



by means of the following formal procedure. We write

$$S_{\omega'\omega} = \langle 0 | a_{\alpha_1'}^-(p_1') \dots a_{\alpha_s'}^-(p_s') S a_{\alpha_1}^+(p_1) \dots a_{\alpha_r}^+(p_r) | 0 \rangle$$

and transfer the annihilation operators  $a^{(-)}$  to the right, and  $a^{(+)}$  to the left until we obtain terms in which  $a^{(-)}$  acts on  $|0\rangle$  and  $a^{(+)}$  acts on  $\langle 0|$  and which therefore vanish. In doing this, we use the usual commutation relations for  $a^{(+)}$ ,  $a^{(-)}$  from the theory of noninteracting fields, and relations such as (52.14) for the commutation of  $a^{(+)}$ ,  $a^{(-)}$ , and  $S$ .

Having thus expressed  $S_{\omega'\omega}$  in terms of the vacuum expectation values of the variational derivatives of the  $S$ -matrix, and having then made use of condition 52.2G, we can reduce them to the vacuum expectation values of the radiation operators.

The general and local properties formulated above may be looked upon as axioms at the basis of our quantum theory of interacting fields. We shall see below that these axioms lead to a large number of rigorous consequences of a general character such as the one-dimensional spectral representations of one-particle Green functions (§§ 53 and 54), four- (and five-) dimensional spectral representations for the scattering amplitude (§ 55), and, finally, the dispersion relations for the forward-scattering amplitudes (§§ 56 and 57) relating observable quantities.

**52.4. Optical Theorem.** The case of forward scattering is theoretically the simplest (Secs. 56.2 and 56.4) but, at the same time, particularly important in practice because the so-called optical theorem tells us that the imaginary part of the forward scattering amplitude is proportional to the total transverse cross section for the process.

To show this, consider the matrix element of  $SS^\dagger = 1$  between the state amplitudes  $\langle p's', q'\rho' |$  and  $|ps, q\rangle$  containing a nucleon with momentum  $p'(p)$  and discrete index  $s'(s)$ , and a meson with momentum  $q'(q)$  and discrete index  $\rho(\rho')$ :

$$\langle p's', q'\rho' | S \hat{S} | ps, q\rangle = \langle p's', q'\rho' | | ps, q\rangle = \delta(p-p') \delta_{ss'} \delta(q-q') \delta_{\rho\rho'}.$$

By writing the left-hand side of this expression as a sum over the complete system of state amplitudes  $|n\rangle$ , we obtain

$$\langle p's', q'\rho' | S \hat{S} | ps, q\rangle = \sum_n \langle p's', q'\rho' | S | n \rangle \langle n | \hat{S} | ps, q\rangle. \quad (52.18)$$

The summation over  $n$  in this expression includes both summation over the discrete characteristics of the state  $|n\rangle$  and integration over the continuous characteristics. By restricting the summation on the right-hand side of (52.18) to states containing one nucleon ( $p''$ ,  $s''$ ) and one meson ( $q''$ ,  $\rho''$ ), and by going over to the scattering amplitude with the aid of (25.21), we obtain

$$\begin{aligned} \langle p's', q'\rho' | S \hat{S} | ps, q\rangle &= \delta(p-p') \delta_{ss'} \delta(q-q') \delta_{\rho\rho'} + \\ &+ \frac{i\delta(p'+q'-p-q)}{2\pi \sqrt{q^0 q'^0}} (f_{s'\rho', sp}(p', q', p, q) - \hat{f}_{s'\rho', sp}^*(p', q', p, q)) + \\ &+ \frac{\delta(p'+q'-p-q)}{(2\pi)^2 \sqrt{q^0 q'^0}} \sum_{s'', \rho''} \int \frac{d p'' d q''}{q''^0} \hat{f}_{s'\rho', s''\rho''}(p', q', p'', q'') \hat{f}_{s''\rho'', sp}^*(p'', q'', p, q) \times \\ &\times \delta(p''+q''-p-q). \end{aligned}$$

If we now recall (52.18) and use the above result, we obtain the following expression for forward scattering when  $p' = p$ ,  $q' = q$ ,  $s' = s$ ,  $\rho' = \rho$ :

$$\begin{aligned} i(f_{s\rho, s\rho}(p, q, p, q) - f_{s\rho, s\rho}^*(p, q, p, q)) &= \\ &= - \sum_{s'', \rho''} \int \frac{d\rho'' dq''}{2\pi q''^0} \delta(p'' + q'' - p - q) |f_{s\rho, s''\rho''}(p, q, p'', q'')|^2 = \\ &= - \int \frac{d\rho'' dq''}{2\pi q''^0} \delta(p'' + q'' - p - q) \sum_{s'', \rho''} |f_{s\rho, s''\rho''}(p, q, p'', q'')|^2 \Big|_{p''=p+q-q''}. \end{aligned}$$

Comparison of this expression with the formulas of §25.4 gives

$$\text{Im} f = \frac{|q|_{\text{LAB}}}{4\pi} \sigma,$$

where  $\sigma$  is the total cross section for elastic pion-nucleon scattering. However, one can readily show that inclusion of the other terms in (52.18) has the result that  $\sigma$  must be interpreted as the total cross section for pion-nucleon scattering:

$$\text{Im} f(E, \theta=0) = \frac{|q|_{\text{LAB}}}{4\pi} \sigma_{\text{tot}}(E). \quad (52.19)$$

Finally, let us write the optical theorem (52.19) in terms of invariant variables. In order to do this, we introduce a new invariant variable equal to the square of the total energy of the pion-nucleon system in the center of mass frame:

$$s = W^2 = (p + q)^2 = M^2 + \mu^2 + 2M\sqrt{\mu^2 + q^2} = M^2 + \mu^2 + 2ME.$$

We then obtain

$$\text{Im} f(s) = \frac{\{[s - (M + \mu)^2][s - (M - \mu)^2]\}^{1/2}}{8\pi M} \sigma_{\text{tot}}(s). \quad (52.20)$$

### §53. Spectral Representation of the Pion Green's Function

In this and in the following sections, we shall obtain the so-called spectral representations of the pion and nucleon Green's functions. Representations of this type were first obtained by Källen (1952) for quantum electrodynamics and by Lehmann (1954) for the meson theory. However, the method by which the investigation of these functions was carried out in the papers just cited cannot be considered as satisfactory, since it involved formal manipulation of infinitely large renormalization constants, and so on.

Below we shall obtain the Källen-Lehmann spectral representations for Green's functions in the pseudoscalar meson theory, starting from the general properties of a local field theory, formulated in §52, and using the method of analytic continuation into the complex plane.

*53.1. Radiation Operators of the First and Second Order.* Bearing in mind subsequent applications to meson-nucleon scattering processes, we confine our attention to the

eight-component spinor nucleon field and the three-component meson field interacting with one another in a charge-symmetric fashion. We shall not take into account the presence of the electromagnetic field or the weak interactions with leptons.

In addition to isospin invariance, i.e., invariance under rotations in three-dimensional isospin space, we shall use invariance under gauge transformations of the first kind

$$\psi(x) \rightarrow e^{i\alpha} \psi(x), \quad \bar{\psi}(x) \rightarrow e^{-i\alpha} \bar{\psi}(x), \quad (53.1)$$

i.e., we shall consider that both of these transformations are included in condition (52.2B).

We now turn to the vacuum expectation values of the radiation operators of the first and second kind within the framework of the theory formulated above.

It is not difficult to show that considerations of covariance under rotations in ordinary and in isospin space ensure that the vacuum expectation values of first-order radiation operators are zero, and so are the vacuum expectation values of second-order operators in which one differentiation is carried out with respect to the boson field and the other with respect to the fermion field. Invariance under the gauge transformation (53.1) ensures that the vacuum expectation values of  $\delta^2 S / \delta \psi \delta \psi$  and  $\delta^2 S / \delta \bar{\psi} \delta \bar{\psi}$  also vanish. Thus, only the vacuum expectation values of the radiation operators  $\delta^2 S / \delta \varphi(x) \delta \varphi(y)$  and  $\delta^2 S / \delta \bar{\psi} \delta \psi(y)$  turn out to be different from zero. Invariance under translation then ensures that these expectation values can depend only on the difference  $x - y$ ; we shall therefore write them in the form

$$\left\langle 0 \left| \frac{\delta^2 S}{\delta \varphi_\sigma(x) \delta \varphi_\rho(y)} \hat{S}^\dagger \right| 0 \right\rangle = i Q_{\sigma\rho}(x - y), \quad (53.2)$$

$$i \left\langle 0 \left| \frac{\delta^2 S}{\delta \bar{\psi}_\alpha(x) \delta \psi_\beta(y)} \hat{S}^\dagger \right| 0 \right\rangle = R_{\alpha\beta}(x - y). \quad (53.3)$$

We note that, by virtue of property 52.2G,  $\hat{S}^\dagger |0\rangle = |0\rangle$  so that the factor  $\hat{S}^\dagger$  may be omitted from (53.2) and (53.3).

In order to interpret (53.2) and (53.3) within the framework of the usual theory, we note that they may be very simply related to the complete Green's functions by means of the following argument based on the "generalized Wick's theorem" formulated in §38.1.

By applying this theorem to the complete meson Green's function

$$\Delta_{\sigma\rho}(x - y) = i \frac{\langle T(\varphi_\sigma(x) \varphi_\rho(y) S) \rangle_0}{S_0}, \quad (53.4)$$

we obtain

$$\Delta_{\sigma\rho}(x - y) = i \overline{\varphi_\sigma(x) \varphi_\rho(y)} + \frac{i}{S_0} \sum_\alpha \int \left\langle \overline{\varphi_\sigma(x) \varphi_\rho(y) \varphi_\alpha(z)} \frac{\delta S}{\delta \varphi_\alpha(z)} \right\rangle_0 dz. \quad (53.5)$$

Applying this theorem once again to the second term, we obtain

$$\Delta_{\sigma\rho}(x-y) = \Delta_{\sigma\rho}^0(x-y) + \sum_{\alpha,\beta} \int \Delta_{\alpha\alpha}^0(x-z) Q_{\alpha\beta}(z-t) \Delta_{\beta\rho}^0(t-y) dz dt, \quad (53.6)$$

where

$$\Delta_{\sigma\rho}^0(x-y) = i \overline{\varphi_{\sigma}(x)} \varphi_{\rho}(y).$$

Transforming in (53.6) with the aid of

$$Q_{\alpha\beta}(z-t) = \frac{1}{(2\pi)^4} \int e^{-ip(z-t)} q_{\alpha\beta}(p) dp \quad (53.7)$$

to the momentum representation, we obtain

$$\Delta_{\sigma Q}(p) = \frac{1}{\mu^2 - p^2} + \frac{1}{\mu^2 - p^2} q_{\sigma Q}(p) \frac{1}{\mu^2 - p^2}. \quad (53.8)$$

In a similar way, we may obtain the following expression for the nucleon Green's function  $G$ :

$$G(p) = (M - \hat{p})^{-1} - (M - \hat{p})^{-1} r(p) (M - \hat{p})^{-1} \quad (53.9)$$

We shall now examine  $Q(x-y)$  in greater detail. In order to do this, we introduce the radiation Bose-operator of the first order

$$j_{\sigma}(x) = i \frac{\delta S}{\delta \varphi_{\sigma}(x)} \hat{S}, \quad (53.10)$$

which we call the current operator (in order to establish a correspondence with the usual pseudoscalar meson theory). The operator  $j$  is Hermitian:

$$\hat{j}_{\sigma}^{\dagger}(x) = j_{\sigma}(x),$$

which is a consequence of the fact that  $\varphi$  is real and  $S$  is unitary; this may be shown by an elementary calculation (similar to the one used in §21.2 to prove that the generalized Hamiltonian is Hermitian).

We may now express the radiation operator in (53.2) in terms of  $j$  and of its variational derivative. By varying (53.10) and by taking into account the fact that the  $S$ -matrix is unitary and the current (48.11) is Hermitian, we obtain

$$\begin{aligned} \frac{\delta j_{\sigma}(x)}{\delta \varphi_{\rho}(y)} &= i \frac{\delta^2 S}{\delta \varphi_{\sigma}(x) \delta \varphi_{\rho}(y)} \hat{S} + i \frac{\delta S}{\delta \varphi_{\sigma}(x)} \frac{\delta \hat{S}}{\delta \varphi_{\rho}(y)} = \\ &= i \frac{\delta^2 S}{\delta \varphi_{\sigma}(x) \delta \varphi_{\rho}(y)} \hat{S} + i j_{\sigma}(x) j_{\rho}(y). \end{aligned} \quad (53.11)$$

Now, taking into account the fact that in accordance with condition 52.3B,

$$\frac{\delta j_\sigma(x)}{\delta \varphi_Q(y)} = 0 \quad \text{for} \quad y \lesssim x,$$

we obtain

$$i \frac{\delta^2 S}{\delta \varphi_\sigma(x) \delta \varphi_Q(y)} \dot{S} = -i j_\sigma(x) j_Q(y) \quad \text{for} \quad y \lesssim x. \quad (53.12)$$

Since the left-hand side of this relation is symmetric with respect to an interchange of  $(x, \sigma)$  and  $(y, \rho)$ , we also have

$$i \frac{\delta^2 S}{\delta \varphi_\sigma(x) \delta \varphi_Q(y)} \dot{S} = -i j_Q(y) j_\sigma(x) \quad \text{for} \quad x \lesssim y. \quad (53.13)$$

We thus obtain

$$i \frac{\delta^2 S}{\delta \varphi_\sigma(x) \delta \varphi_Q(y)} \dot{S} = -iT(j_\sigma(x) j_Q(y)). \quad (53.14)$$

We note that the right-hand side of this formula is not, strictly speaking, defined for  $x = y$  [this remark also applies to (53.16) and (53.18) below].

It also follows from (53.12) and (53.13) that

$$[j_\sigma(x), j_Q(y)]_- = 0 \quad \text{for} \quad x \sim y. \quad (53.15)$$

Finally, we note that the following operator relations follow from (53.14):

$$\left( i \frac{\delta^2 S}{\delta \varphi(x) \delta \varphi(y)} \dot{S} \right) + \left( i \frac{\delta^2 S}{\delta \varphi(x) \delta \varphi(y)} \dot{S} \right)^\dagger = -i\varepsilon(x^0 - y^0) [j(x), j(y)]_-, \quad (53.16)$$

$$\left( i \frac{\delta^2 S}{\delta \varphi(x) \delta \varphi(y)} \dot{S} \right) - \left( i \frac{\delta^2 S}{\delta \varphi(x) \delta \varphi(y)} \dot{S} \right)^\dagger = -i \{j(x) j(y) + j(y) j(x)\}. \quad (53.17)$$

For  $Q(x - y)$ , we obtain from (53.14):

$$Q_{\sigma Q}(x - y) = i \langle T(j_\sigma(x) j_Q(y)) \rangle_0 = \begin{cases} i \langle j_\sigma(x) j_Q(y) \rangle_0 & \text{for } x \gtrsim y, \\ i \langle j_Q(y) j_\sigma(x) \rangle_0 & \text{for } y \gtrsim x. \end{cases} \quad (53.18)$$

**53.2. Vacuum Expectation Value of the Product and the Commutator of Two Currents.**  
We have expressed  $Q$  in terms of the vacuum expectation value of the product of two currents,  $\langle j_\sigma(x) j_Q(y) \rangle_0$ .

We now consider in greater detail the structure of this expression, using condition 52.2D

for this purpose. Since, as shown in §53.1, the vacuum expectation value of the current is zero, we may now write

$$\langle j_\sigma(x) j_Q(y) \rangle_0 = \sum_n \int d\mathbf{k} \langle 0 | j_\sigma(x) | n, \mathbf{k} \rangle \langle n, \mathbf{k} | j_Q(y) | 0 \rangle. \quad (53.19)$$

Next, using properties 52.2B and 52.2C, we have

$$\langle n, \mathbf{k} | j_Q(y) | 0 \rangle = \langle n, \mathbf{k} | U_{L_y}^\dagger j_Q(0) U_{L_y} | 0 \rangle = \langle n, \mathbf{k} | j_Q(0) | 0 \rangle e^{i[E_n(\mathbf{k})y^0 - \mathbf{k} \cdot \mathbf{y}]}. \quad (53.20)$$

Writing  $\langle 0 | j_\sigma(x) | n, \mathbf{k} \rangle$  in an analogous way, we obtain the following result in place of (53.17):

$$\begin{aligned} \langle j_\sigma(x) j_Q(y) \rangle_0 &= \\ &= \sum_n \int d\mathbf{k} e^{-i[E_n(\mathbf{k})(x^0 - y^0) - \mathbf{k} \cdot (\mathbf{x} - \mathbf{y})]} \langle 0 | j_\sigma(0) | n, \mathbf{k} \rangle \langle n, \mathbf{k} | j_Q(0) | 0 \rangle. \end{aligned} \quad (53.21)$$

We now suppose that the lowest energy states in (53.19) (after the vacuum states) are states with one, two, and three mesons (this corresponds to the assumption of the absence of bound complexes of mesons and nucleons with mass less than  $3\mu$ , which is not inconsistent with experiment). We shall show that for the one-meson states  $|n, \mathbf{k}\rangle$ , the matrix elements

$$\langle n, \mathbf{k} | j_\sigma(0) | 0 \rangle \quad (53.22)$$

are zero. To do this, we introduce the meson annihilation operators  $a_Q^-(\mathbf{p})$  with the usual commutation relations [cf. (11.7) and (3.36)]:

$$[a_Q^-(\mathbf{p}), \varphi_\sigma(x)] = \pm \frac{\delta_{Q\sigma}}{(2\pi)^{3/2} \sqrt{2p^0}} e^{\pm i\mathbf{p} \cdot \mathbf{x}} \quad (p^0 = \sqrt{\mu^2 + \mathbf{p}^2} > 0)$$

and use condition 53.2C. This gives

$$a_\sigma^-(\mathbf{p}) S - S a_\sigma^-(\mathbf{p}) = \pm \frac{1}{(2\pi)^{3/2} \sqrt{2p^0}} \int e^{i\mathbf{p} \cdot \mathbf{x}} \frac{\delta S}{\delta \varphi_\sigma(x)} dx. \quad (53.23)$$

Now consider the Fourier transform of the matrix element of the current  $\langle n, \mathbf{k} | j(x) | 0 \rangle$ . From (53.20), we obtain

$$\frac{1}{(2\pi)^4} \int e^{-i\mathbf{p} \cdot \mathbf{x}} \langle n, \mathbf{k} | j(x) | 0 \rangle dx = \delta(\mathbf{k} - \mathbf{p}) \langle n, \mathbf{k} | j(0) | 0 \rangle. \quad (53.24)$$

This matrix element will differ from zero only for values of the four-momentum  $p$  equal to the four-momentum  $k$  of one of the states  $n$ .

Consider the single-meson state  $\langle 1, \mathbf{k} |$  when  $p^2 = k^2 = \mu^2$ . If we use (52.14) and the conditions for the stability of vacuum and the one-particle state (52.11), we have, in this case,

$$\begin{aligned} \int e^{-ipx} \langle n, \mathbf{k} | j(x) | 0 \rangle dx &= i \int e^{-ipx} \langle n, \mathbf{k} | \frac{\delta S}{\delta \psi(x)} \dot{S} | 0 \rangle dx = \\ &= i (2\pi)^{3/2} \sqrt{2p^0} \{ \langle n, \mathbf{k} | S a^+(p) S | 0 \rangle - \langle n, \mathbf{k} | a^+(p) | 0 \rangle \} = 0. \end{aligned}$$

Comparison with (53.24) then shows that the matrix element (53.22) is zero for single-meson states. It also follows that

$$\int e^{-ipx} \langle n, \mathbf{k} | j(x) | 0 \rangle dx = 0 \quad \text{for} \quad p^2 = \mu^2 \quad (53.25)$$

for arbitrary states  $\langle n, \mathbf{k} |$ . The fact that (53.22) is zero for the vacuum and two-meson states follows directly from the fact that the mesons are pseudoscalar. Thus, in the expansion given by (53.19), the states  $|n, \mathbf{k}\rangle$  include at least three mesons, and for them

$$E_n^2(\mathbf{k}) - k^2 \geq (3\mu)^2. \quad (53.26)$$

We now take into account the fact that because of translational and isospin invariance,

$$\langle j_\sigma(x) j_\sigma(y) \rangle_0 = \delta_{\sigma Q} u(x-y).$$

The function  $u$  may be represented in the form

$$u(x) = \frac{1}{(2\pi)^3} \int e^{-ikx} v(k) dk,$$

where, in accordance with (53.19), the function  $v(k)$  has the form

$$v(k) = (2\pi)^3 \sum_n |\langle 0 | j_\sigma(0) | n, \mathbf{k} \rangle|^2 \delta(k^0 - E_n(\mathbf{k})). \quad (53.27)$$

On the basis of (53.26) and (53.27), and taking into account Lorentz covariance, we conclude that

$$v(k) = \theta(k^0) J(k^2),$$

where the function  $J$  has the properties

$$J(k^2) = 0 \quad \left. \begin{array}{l} J(k^2) \geq 0 \\ \text{for } k^2 < (3\mu)^2. \end{array} \right\} \quad (53.28)$$

We have thus obtained

$$\langle j_{\sigma}(x) j_{\rho}(y) \rangle_0 = \frac{\delta_{\sigma\rho}}{(2\pi)^3} \int e^{-ik(x-y)} \theta(k^0) J(k^2) dk. \quad (53.29)$$

This formula in fact provides the spectral representation for  $\langle j_{\sigma}(x) j_{\rho}(y) \rangle_0$ .

From (53.18) and (53.29) it follows that the function  $Q(x-y)$  has the "property of causality" similar to the causal Green's functions; for  $x^0 > y^0$  it has only negative frequencies with respect to the argument  $x^0 - y^0$ , and for  $x^0 < y^0$  it has only positive frequencies. We shall therefore denote this function by  $Q^c$ :

$$Q_{\sigma\rho}(x-y) = \delta_{\sigma\rho} Q^c(x-y).$$

We note finally that (53.26) leads to the following expression for the spectral representation of the current commutator:

$$\langle [j_{\sigma}(x), j_{\rho}(y)] \rangle_0 = \frac{\delta_{\sigma\rho}}{(2\pi)^3} \int e^{-ik(x-y)} \varepsilon(k^0) J(k^2) dk. \quad (53.30)$$

**53.3. Analytic Properties of  $Q^r$  and  $Q^a$ .** Relations (53.18) and (53.29) are, however, still insufficient to allow us to draw conclusions with respect to the analytic properties of the function  $Q^c$ . The point is that (53.18) defines  $Q^c$  only for  $x \neq y$ . In order to obtain the information that is lacking, we shall introduce two more functions:

$$\delta_{\sigma\rho} Q^r(x-y) = - \left\langle \frac{\delta j_{\rho}(y)}{\delta \varphi_{\sigma}(x)} \right\rangle_0, \quad \delta_{\sigma\rho} Q^a(x-y) = - \left\langle \frac{\delta j_{\sigma}(x)}{\delta \varphi_{\rho}(y)} \right\rangle_0. \quad (53.31)$$

In writing the left-hand sides of these relations, we have made use of translational and isospin invariance. The indices  $r$  and  $a$  correspond to the properties

$$Q^r(x-y) = Q^a(y-x) = 0 \quad \text{for} \quad x \lesssim y.$$

It is also clear that, in general,

$$Q^r(x-y) = Q^a(y-x). \quad (53.32)$$

By going over to the momentum representation

$$Q(x) = \frac{1}{(2\pi)^4} \int e^{-ikx} q(k) dk,$$

we obtain on the basis of (53.11) and (53.29)

$$q^c(k) = i2\pi\theta(k^0) J(k^2) + q^a(k) = i2\pi\theta(-k^0) J(k^2) + q^r(k) \quad (53.33)$$

and, in particular,



$$q^c(k) = q^a(k) = q^r(k) \quad \text{for} \quad k^2 < (3\mu)^2. \quad (53.34)$$

We shall now consider the analytic properties of the function

$$q^r(k) = \int Q^r(x) e^{ikx} dx, \quad (53.35)$$

taking into account the fact that in consequence of the condition of causality,

$$Q^r(x) = 0 \quad \text{for} \quad x \leq 0. \quad (53.36)$$

The relation given by (53.35) defines  $q^r(k)$  for real values of the components of  $k$ . From (53.36) it follows that  $q^r(k)$  may be continued in a definite way into the region of complex  $k$ . We assume that  $k$  has a nonzero imaginary part, i.e.,

$$\left. \begin{aligned} k &= p + i\Gamma, \\ p &= \text{Re } k, \quad \Gamma = \text{Im } k, \quad \Gamma^2 > 0, \quad \Gamma^0 > 0. \end{aligned} \right\} \quad (53.37)$$

We then obtain

$$q^r(p + i\Gamma) = \int Q^r(x) e^{ipx} e^{-\Gamma x} dx. \quad (53.38)$$

The factor  $\exp(-\Gamma x)$  plays the role of the cutoff factor that guarantees the convergence of the integral. Indeed, in accordance with (53.36) the integral (53.38) is actually taken over the region for which

$$x^0 \geq 0, \quad x^2 \leq (x^0)^2,$$

and, in this region, very rough estimates show that

$$\Gamma x = \Gamma^0 x^0 - \Gamma x > (\Gamma^0 - |\Gamma|) x^0 > \frac{1}{2} (\Gamma^0 - |\Gamma|) (|x^0| + |x|),$$

so that

$$e^{-\Gamma x} \leq e^{-\alpha(|x^0| + |x|)}, \quad (53.39)$$

where, in accordance with (53.37),  $\alpha = (\frac{1}{2})(\Gamma^0 - |\Gamma|) > 0$ .

At the same time, by virtue of condition 52.3A, the function  $Q^r(x)$  is integrable, so that the integral

$$\int Q^r(x) h(x) dx,$$

which is a linear functional in the space of the functions  $h(x)$  in the class  $C(q, n, 1)$ , exists and is bounded in absolute magnitude by a linear combination of the  $h_{mn}$ , where

$$h_{mn} = \sup \left\{ |x|^m \frac{\partial^n h(x)}{(\partial x^0)^{\alpha_0} \dots (\partial x^3)^{\alpha_3}} \right\} \quad \begin{array}{l} m = 0, 1, \dots, r; \\ n = \alpha_0 + \dots + \alpha_3 = 0, 1, \dots, q, \end{array}$$

are bounded.

By virtue of (53.39), the function  $\exp \{ipx - \Gamma x\}$  belongs in any case to the class  $C(q, r, 1)$  with arbitrary finite values of  $q, r$  so that the integral (53.38) and its derivatives with respect to  $k$

$$\int Q^r(x) (x^0)^{\beta_0} \dots (x^3)^{\beta_3} e^{ikx} dx$$

will converge. Thus,  $q^r(k)$  is an analytic function of  $k$  in the region given by (53.37). Since the derivatives of  $\exp ikx$  with respect to the components of  $x$  are proportional to powers of  $k$ , the function  $q^r(k)$  bounded by a combination of the quantities  $h_{mn}$  will increase at infinity not faster than a polynomial in  $k$  of degree not higher than  $n$  (of course, we are dealing with values of  $k$  for which the inequality (53.39) is not relaxed).

The transition to real  $k$  is accomplished by means of the improper limiting transition

$$q^r(p + i\Gamma) \rightarrow q^r(p); \quad \Gamma^2 \rightarrow +0, \quad \Gamma^0 \rightarrow +0. \quad (53.40)$$

It may be shown similarly that the function  $q_a(k)$  may be continued into the complex region

$$k \rightarrow p - i\Gamma; \quad \Gamma^2 > 0, \quad \Gamma^0 > 0 \quad (53.41)$$

with the same analytic properties as the function  $q^r(k)$ .

Since by virtue of (53.34), both these functions coincide for real  $k$  in the region in which  $k^2 < (3\mu)^2$ , they should be regarded as a single analytic function  $f$  defined for (53.37) and (53.41). Since these functions then also coincide with  $q^c(k)$ , which by virtue of covariance and invariance of  $Q^c(x)$  under time reversal depends only on  $k^2$ , the function  $f$  also depends only on  $k^2$ .

The function  $f(k^2)$  is consequently analytic in the range of values of its argument that corresponds to complex components of  $k$  such that

$$\Gamma^2 = (\text{Im } k)^2 > 0. \quad (53.42)$$

We now substitute

$$\text{Re } k^2 = \xi, \quad \text{Im } k^2 = \eta, \quad \text{i.e.,} \quad k^2 = \zeta = \xi + i\eta,$$

so that

$$\xi = p^2 - \Gamma^2, \quad \eta = 2(p\Gamma). \quad (53.43)$$

It is now clear that the condition (53.42) reduces to the exclusion from the complex  $\zeta$  plane of the real positive axis\*

$$\eta = 0, \quad 0 < \xi < \infty. \quad (53.44)$$

Thus the function  $f(\zeta)$  is an analytic function with the cut (53.44). It increases at infinity not faster than a polynomial.

We shall now define the boundary values of the function  $f$  along the upper and the lower edges of the cut as the corresponding improper limits:

$$f_{\pm}(p^2) = \lim_{\eta \rightarrow \pm 0} f(k^2). \quad (53.45)$$

It now follows from (53.4), (53.32), (53.37), (53.41), and (53.43) that

$$q^r(k) = q^a(-k) = \begin{cases} f_+(k^2) & \text{for } k^0 > 0, \\ f_-(k^2) & \text{for } k^0 < 0. \end{cases} \quad (53.46)$$

Taking (53.33) into account, we also obtain

$$f_+(k^2) - f_-(k^2) = 2\pi i J(k^2). \quad (53.47)$$

Now, noting that by virtue of (53.34), the difference (53.47) vanishes for  $k^2 < (3\mu)^2$ , we find that the cut is given not by the whole positive real axis, but only by its segment

$$\eta = 0, \quad (3\mu)^2 \leq \xi < \infty. \quad (53.48)$$

On taking (53.43) into account, these limiting relations may also be written in the more compact form

$$q^r(k) = \lim_{\varepsilon \rightarrow +0} f(k^2 + i\varepsilon k^0), \quad (53.49)$$

$$q^a(k) = \lim_{\varepsilon \rightarrow +0} f(k^2 - i\varepsilon k^0). \quad (53.50)$$

By combining these expressions with (53.32), we also obtain

$$q^r(k) = \lim_{\varepsilon \rightarrow +0} f(k^2 + i\varepsilon). \quad (53.51)$$

**53.4. Spectral Representation for  $q^r$ ,  $q^a$ , and  $q^c$ .** The conclusions just reached with respect to the analytic function  $f(\zeta)$  now enable us to obtain spectral representations such as (53.29) for  $q^r$ ,  $q^a$ , and  $q^c$ . For this purpose, we shall now use the Cauchy integral theorem in a manner similar to the way it was used in §52.1. We introduce the auxiliary function

\*Indeed, if  $\xi > 0$ , then  $p^2 > \Gamma^2 + \xi > 0$ . But since also  $\Gamma > 0$ , then  $\eta = 2p\Gamma \neq 0$ .

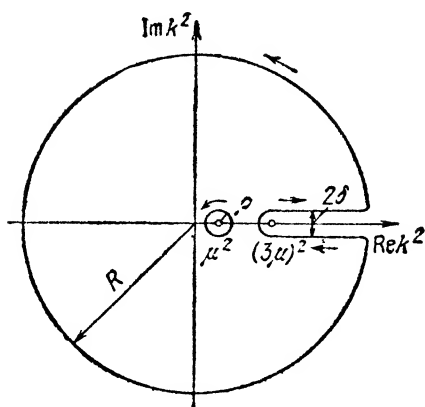


Fig. 66. Contour of integration for spectral representations of Green's functions.

$$h(\zeta) = \frac{f(\zeta)}{(\zeta - \mu^2)^{n+1}}$$

with an appropriate  $n$  and choose the closed contour as shown in Fig. 66. By going to the limit as  $R \rightarrow \infty$ , and subsequently setting  $\rho \rightarrow 0$ ,  $\delta \rightarrow 0$ , we obtain

$$f(z) = \frac{(z - \mu^2)^{n+1}}{2\pi i} \int_{(3\mu)^2}^{\infty} \frac{f_+(z') - f_-(z')}{(z' - \mu^2)^{n+1}(z' - z)} dz' - \frac{(z - \mu^2)^{n+1}}{2\pi i} \oint_Q \frac{dz'}{(z' - \mu^2)^{n+1}} \frac{f(z')}{z' - z}.$$

By evaluating the integral along the small circle of radius  $\rho$  about the point  $\mu^2$ , and by utilizing (53.47), we find that

$$f(k^2) = (k^2 - \mu^2)^{n+1} \int_{(3\mu)^2}^{\infty} \frac{J(z) dz}{(z - \mu^2)^{n+1}(z - k^2)} + \sum_{0 \leq m \leq n} \frac{(k^2 - \mu^2)^m}{m!} f^{(m)}(\mu^2), \quad (53.52)$$

$$f^{(m)}(\mu^2) = \left[ \frac{\partial^m f(z)}{\partial z^m} \right]_{z=\mu^2}.$$

Relations (53.49)–(53.51) enable us at once to go over in (53.52) to any one of the functions  $q^a$ ,  $q^r$ ,  $q^c$ . Thus, for example, we obtain

$$q^c(k^2) = (k^2 - \mu^2)^{n+1} \int_{(3\mu)^2}^{\infty} \frac{J(z) dz}{(z - \mu^2)^{n+1}(z - k^2 - i\epsilon)} + \sum_{0 \leq m \leq n} \frac{(k^2 - \mu^2)^m}{m!} f^{(m)}(\mu^2). \quad (53.53)$$

Similar relations, which differ from (53.53) by the replacement of  $i\epsilon$  by  $\pm i\epsilon k^0$ , may be written for  $q^a$  and  $q^r$ .

We note that the spectral representation (53.53) could have been obtained formally directly from (53.18) and (53.19) with the aid of the "subtraction formalism." Indeed, from the relations referred to above, we have

$$Q(x) = \frac{i}{(2\pi)^3} \int e^{-ikx\theta(k^0)} J(k^2) dk = \int_{(3\mu)^2}^{\infty} dz' J(z') \frac{i}{(2\pi)^3} \int e^{-ikx\theta(k^0)} \delta(k^2 - z') dk = \int_{(3\mu)^2}^{\infty} dz' J(z') D_{z'}^-(x) \quad \text{for } x \geq 0. \quad (53.54)$$

Here  $D_{z'}^-$  is the negative-frequency part of the Pauli-Villars function corresponding to the mass  $\sqrt{z'}$ :

$$D_{z'}^-(x) = \frac{i}{(2\pi)^3} \int e^{-ikx\theta(k^0)} \delta(k^2 - z') dk.$$

Combining (53.54) with the corresponding representation for  $Q(x)$  in the region  $x \leq 0$ , we obtain

$$Q(x) = \int_{(3\mu)^2}^{\infty} J(z') D_{z'}^c(x) dz' \quad \text{for } x \neq 0,$$

from which it follows that

$$q^c(k) = \int_{(3\mu)^2}^{\infty} \frac{J(z) dz}{z - k^2 - i\epsilon} + P(k^2), \quad (53.55)$$

where  $P(k^2)$  is a polynomial in  $k^2$ .

If the function  $J(z)$  does not fall off sufficiently rapidly at infinity, the integral in (53.55) will diverge. However, it may be made to converge by means of a subtraction procedure and in this way a definite meaning may be given to formula (53.55). We shall utilize the formula

$$\frac{1}{z - k^2} = \frac{1}{z - \mu^2 - (k^2 - \mu^2)} = \frac{1}{z - \mu^2} \left\{ 1 + \frac{k^2 - \mu^2}{z - \mu^2} + \dots + \left( \frac{k^2 - \mu^2}{z - \mu^2} \right)^n \right\} + \frac{(k^2 - \mu^2)^{n+1}}{(z - k^2)(z - \mu^2)^{n+1}},$$

with the aid of which we can represent (53.55) in the form

$$q^c(k) = (k^2 - \mu^2)^{n+1} \int_{(3\mu)^2}^{\infty} \frac{J(z) dz}{(z - k^2 - i\epsilon)(z - \mu^2)^{n+1}} + \sum_{0 \leq m \leq n} (k^2 - \mu^2)^m \int_{(3\mu)^2}^{\infty} \frac{J(z) dz}{(z - \mu^2)^{m+1}} + P(k^2). \quad (53.56)$$

We choose  $n$  to be sufficiently large, so that the first integral in (53.56) turns out to be convergent. The divergent terms in the sum involving the powers of  $(k^2 - \mu^2)$  may be

compensated by means of the polynomial  $P(k^2)$ . This "compensation of divergences" is typical for the usual subtraction procedure and leads us to (53.53).

We shall now show that by virtue of our conditions, the zero term is absent from the sum in (53.53), i.e.,

$$f(\mu^2) = 0. \quad (53.57)$$

We note that by virtue of property 47.3C, the matrix element of  $S$  between two single-meson states  $\langle p'\sigma |$  and  $|p\rho\rangle$  may be represented with the aid of (53.23) in the form

$$\begin{aligned} \langle p'\sigma | S | p\rho \rangle &= \langle 0 | S a_{\sigma}^{-}(\mathbf{p}') a_{\rho}^{+}(\mathbf{p}) | 0 \rangle + \\ &+ \frac{1}{(2\pi)^{3/2} \sqrt{2p^0}} \int \left\langle 0 \left| \frac{\delta S}{\delta \varphi_{\sigma}(x)} a_{\rho}^{+}(\mathbf{p}) \right| 0 \right\rangle e^{ip'x} dx \quad \text{for} \quad p^2 = p'^2 = \mu^2. \end{aligned}$$

By commuting  $a^{(-)}$  with  $\delta S/\delta \varphi$  under the integral on the right-hand side, and taking into account 47.2G and (53.2), we obtain

$$\begin{aligned} \langle p'\sigma | S | p\rho \rangle &= \langle a_{\sigma}^{-}(\mathbf{p}') a_{\rho}^{+}(\mathbf{p}) \rangle_0 + \frac{i}{(2\pi)^3 2 \sqrt{p^0 p'^0}} \int Q_{\sigma\rho}(x, y) e^{i(p'x - py)} dx dy = \\ &= \langle p'\sigma | p\rho \rangle + \delta_{\sigma\rho} \frac{i\pi}{p^0} \delta(p - p') q^c(p) \quad \text{for} \quad p^2 = \mu^2. \end{aligned}$$

On the other hand, by virtue of the same condition, we may directly write

$$\langle p'\sigma | S | p\rho \rangle = \langle p'\sigma | p\rho \rangle,$$

from which (53.57) follows after (53.34) and (53.57) have been taken into account.

We shall also show that the constants  $f^m(\mu^2)$  are purely imaginary. Indeed, by virtue of the definition (53.4) and the fact that  $\varphi$  is real, the function  $O$  is Hermitian, so that, taking into account the fact that  $q^c(k)$  is even, we obtain

$$q^{c*}(k^2) = q^c(k^2).$$

We have thus proved that the functions  $q^r$ ,  $q^a$ , and  $q^c$  possess spectral representations of the type of (53.52) in which

$$f^{(m)*}(\mu^2) = f^{(m)}(\mu^2).$$

and

$$f^{(0)}(\mu^2) = f(\mu^2) = 0 \quad (53.58)$$

$$f^{(m)*}(\mu^2) = f^{(m)}(\mu^2). \quad (53.59)$$

The well-known result of Källen and Lehmann for the boson Green's function may be obtained from formulas (53.53), (53.58), and (53.59), with the aid of (53.8) if the additional assumption is made that the "rate of increase"  $n$  is equal to unity. Substituting (53.53) into (53.8), we then obtain

$$\Delta(p) = \frac{1 + f^{(1)}(\mu^2)}{\mu^2 - p^2 - i\varepsilon} + \int_{(3\mu)^2}^{\infty} \frac{I(z) dz}{z - p^2 - i\varepsilon}, \quad I(z) = \frac{J(z)}{(z - \mu^2)^2}. \quad (53.60)$$

## §54. Spectral Representation of the Fermion Green's Function

*54.1. Radiation Fermi Operators.* We now turn to the development of the spectral distribution for the vacuum expectation of the second-order radiation Fermi operator

$$R_{\alpha\beta}^c(x-y) = i \left\langle \frac{\delta^2 S}{\delta \psi_\alpha(x) \delta \bar{\psi}_\beta(y)} \dot{S} \right\rangle_0, \quad (54.1)$$

which, as shown in §53.1, is the only radiation Fermi operator of the second order containing the second variational derivative and having a nonzero vacuum expectation value. Since the discussion given below differs from the material in §53 only by the difference between the transformation properties of  $R$  and  $Q$ , the next few paragraphs are given in a more abbreviated form.

It will be useful to introduce the radiation Fermi operators

$$\vartheta(x) = iS \frac{\delta \dot{S}}{\delta \psi(x)} = -i \frac{\delta S}{\delta \bar{\psi}(x)} \dot{S} \quad (54.2)$$

and

$$\bar{\vartheta}(x) = i \frac{\delta S}{\delta \bar{\psi}(x)} \dot{S} = -iS \frac{\delta \dot{S}}{\delta \psi(x)}, \quad (54.3)$$

that are related by

$$\bar{\vartheta}(x) = [\vartheta(x)]^\dagger \gamma^0.$$

The definitions given by (54.2) and (54.3) are chosen so that the "Fermi currents" introduced above correspond in lowest-order perturbation theory to the Schwinger sources in the interaction Lagrangian

$$\mathcal{L}(x) \sim \bar{\psi}(x) \vartheta_0(x) + \vartheta_0(x) \psi(x).$$

Expressing the variational derivatives of  $\vartheta$ ,  $\bar{\vartheta}$  with respect to the Fermion arguments in terms of the second variational derivatives and the products of currents, we obtain

$$\frac{\delta \vartheta(x)}{\delta \psi(y)} = i \frac{\delta^2 S}{\delta \psi(x) \delta \bar{\psi}(y)} \dot{S} - i \vartheta(x) \bar{\vartheta}(y), \quad (54.4)$$

$$\frac{\delta \vartheta(y)}{\delta \psi(x)} = i \frac{\delta^2 S}{\delta \psi(x) \delta \bar{\psi}(y)} \dot{S} + i \bar{\vartheta}(y) \vartheta(x). \quad (54.5)$$

Using the causality condition, we obtain from this [see the remark in connection with (53.14)]:

$$\frac{\delta^2 S}{\delta \bar{\psi}(x) \delta \psi(y)} \dot{S} = T(\vartheta(x) \bar{\vartheta}(y)), \quad (54.6)$$

$$[\vartheta(x), \bar{\vartheta}(y)]_+ = 0 \quad \text{for} \quad (x-y)^2 < 0. \quad (54.7)$$

We now introduce the vacuum expectation values

$$\left. \begin{aligned} R^r(x-y) &= \left\langle \frac{\delta}{\delta \bar{\psi}(x)} \bar{\vartheta}(y) \right\rangle_0, \\ R^a(x-y) &= \left\langle \frac{\delta}{\delta \psi(y)} \vartheta(x) \right\rangle_0, \end{aligned} \right\} \quad (54.8)$$

$$\left. \begin{aligned} R^{(-)}(x-y) &= i \langle \vartheta(x) \bar{\vartheta}(y) \rangle_0, \\ R^{(+)}(x-y) &= i \langle \bar{\vartheta}(y) \vartheta(x) \rangle_0, \end{aligned} \right\} \quad (54.9)$$

which, because of (54.4) and (54.5), are related to  $R^c$  by the following:

$$\left. \begin{aligned} R^a(x) &= R^c(x) + R^{(-)}(x), \\ R^r(x) &= R^c(x) + R^{(+)}(x). \end{aligned} \right\} \quad (54.10)$$

By virtue of the causality condition, we have

$$\left. \begin{aligned} R^r(x) &= 0 & \text{for} & \quad x \lesssim 0, \\ R^a(x) &= 0 & \text{for} & \quad x \gtrsim 0. \end{aligned} \right\} \quad (54.11)$$

Because of isospin and Lorentz invariance, all these functions have the following structure:

$$R(x) = \delta_{st} (i \hat{\partial} u_1(x) + u_2(x)),$$

where  $u_1$  and  $u_2$  are invariant scalar functions,  $s$  and  $t$  are the isospin (proton and neutron) indices, and the components  $\hat{\partial}$  contain the usual Dirac matrices of rank four.

If we now take into account the invariance properties of (54.8)–(54.9) under charge conjugation (cf. §13.4), we find that the functions  $u(x)$  are related by the following equations:



$$\left. \begin{aligned} u_1^{(+)}(x) &= -u_1^{(-)}(-x), & u_2^{(+)}(x) &= u_2^{(-)}(-x), \\ u_1^r(x) &= -u_1^a(-x), & u_2^r(x) &= u_2^a(-x). \end{aligned} \right\} \quad (54.12)$$

54.2. *Derivation of the Spectral Representation.* We now examine the function  $R^{(-)}$ . In accordance with condition 52.2C, we may write it in the form

$$R^{(-)}(x-y) = \frac{i}{(2\pi)^3} \sum_n \int dk \langle 0 | \bar{\psi}(0) | n, k \rangle \langle n, k | \bar{\psi}(0) | 0 \rangle e^{-ik(x-y)}. \quad (54.13)$$

As was also the case in §53.2, some of the initial terms in this sum are zero. Thus, covariance ensures that

$$\langle 0 | \bar{\psi}(0) | n, k \rangle = 0 \quad (54.14)$$

for the states  $|n, k\rangle$  containing no nucleons. Moreover, using arguments similar to those in §53.2, we see that (54.14) also holds for a single-nucleon state; therefore, the integral (54.13) is in fact taken over the region in which  $k^2 \geq (M + \mu)^2$ . It is now clear that  $R^{(-)}$  may be written in the form

$$\begin{aligned} R^{(-)}(x) &= \frac{\delta_{\sigma\sigma}}{(2\pi)^3 i} \int e^{-ikx} (k^0) [\hat{k} Q_1(k^2) + Q_2(k^2)] dk, \\ Q_{1,2}(k^2) &= 0 \quad \text{for} \quad k^2 < (M + \mu)^2. \end{aligned} \quad (54.15)$$

By introducing similar momentum representations for  $R^a$ ,  $R^r$ , and  $R^c$  in terms of the scalar functions  $\varphi_1$  and  $\varphi_2$ :

$$R(x) = \frac{\delta_{\sigma\sigma}}{(2\pi)^4} \int e^{-ikx} [\hat{k} \varphi_1(k) + \varphi_2(k)] dk, \quad (54.16)$$

we obtain, with the aid of (54.10) and (54.12),

$$\varphi_i^c(k) = \varphi_i^r(k) - 2\pi i \theta(k^0) Q_i(k^2) = \varphi_i^a(k) + 2\pi i \theta(-k^0) Q_i(k^2) \quad (i = 1, 2). \quad (54.17)$$

By using (54.15) in the above, we also obtain

$$\varphi_i^c(k) = \varphi_i^r(k) = \varphi_i^a(k) \quad \text{for} \quad k^2 < (M + \mu)^2. \quad (54.18)$$

Since (54.17) and (54.18) are completely analogous to (53.33) and (53.35), we may now repeat word for word for the functions  $u_i$  the arguments used in §53.3 for the functions  $q$ .

In this way we obtain

$$\left. \begin{aligned} \varphi_i^r(k) &= \lim_{\varepsilon \rightarrow +0} f_i(k^2 + i\varepsilon k^0), \\ \varphi_i^a(k) &= \lim_{\varepsilon \rightarrow +0} f_i(k^2 - i\varepsilon k^0), \\ \varphi_i^c(k) &= \lim_{\varepsilon \rightarrow +0} f_i(k^2 + i\varepsilon), \end{aligned} \right\} \quad (54.19)$$

where the functions  $f_i$  are analytic over the whole complex plane of their argument with the exception of the cut

$$\operatorname{Im} k^2 = 0, \quad \operatorname{Re} k^2 > (M + \mu)^2,$$

and increase at infinity not faster than a polynomial of the  $n$ th degree, so that they may be represented in a form similar to (53.52):

$$f_i(k^2) = (k^2 - M^2)^{n+1} \int_{(M+\mu)^2}^{\infty} \frac{\varrho_i(z) dz}{(z - M^2)^{n+1} (z - k^2)} + \sum_{0 \leq m \leq n} \frac{(k^2 - M^2)^m}{m!} f_i^{(m)}(M^2). \quad (54.20)$$

We shall now establish certain inequalities which must be satisfied by the functions  $\rho_i$ . In order to do this, we note that (54.13) and (54.15) allow us to write

$$\hat{k}_{\alpha\beta Q_1}(k^2) + \delta_{\alpha\beta Q_2}(k^2) = \sum_{n, \omega} \langle 0 | \vartheta_{\alpha}(0) | n, k \rangle \langle n, k | \bar{\vartheta}_{\omega}(0) | 0 \rangle \gamma_{\omega\beta}^c, \quad (54.21)$$

which is valid for all  $k^2 = M_n^2$ ,  $k^0 > 0$ . In it we set\*

$$k = 0, \quad k^0 = v > 0, \quad k^2 = v^2, \quad \alpha = \beta$$

and make use of the fact\*\* that in the usual representation,

$$\gamma^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

We thus obtain

\*I.e., we choose  $k$  to be purely time-like and denote by  $v$  the set of  $M_n$  for which the right-hand side of (54.21) is different from zero.

\*\*It is not essential to introduce a special representation of the Dirac matrices at this point. We have introduced it only in order to simplify the presentation.

$$\left. \begin{aligned} \nu \varrho_1(\nu^2) + \varrho_2(\nu^2) &= \sum_n |\langle 0 | \vartheta_a(0) | n, 0 \rangle|^2 > 0, \\ -\nu \varrho_1(\nu^2) + \varrho_2(\nu^2) &= -\sum_n |\langle 0 | \vartheta_a(0) | n, 0 \rangle|^2 < 0. \end{aligned} \right\} \quad (54.22)$$

We now introduce the functions

$$J_1(\nu) = \frac{\varrho_1(\nu^2) - \frac{\varrho_2(\nu^2)}{\nu}}{2}, \quad J_2(\nu) = \frac{\varrho_1(\nu^2) + \frac{\varrho_2(\nu^2)}{\nu}}{2},$$

which, by virtue of (54.22) are nonnegative:

$$J_1(\nu) \geq 0, \quad J_2(\nu) \geq 0, \quad (54.23)$$

and we note that

$$\hat{k} \varrho_1(\nu^2) + \varrho_2(\nu^2) = (\hat{k} - \nu) J_1(\nu^2) + (\hat{k} + \nu) J_2(\nu^2). \quad (54.24)$$

From (54.20) and (54.24) we can now conclude that the combinations

$$R(\hat{k}) = \hat{k} f_1(k^2) + f_2(k^2) \quad (54.25)$$

have a spectral representation of the form

$$\begin{aligned} R(\hat{k}) &= (k^2 - M^2)^{n+1} \int_{\nu > M+\mu}^{\infty} \frac{(\hat{k} - \nu) J_1(\nu) + (\hat{k} + \nu) J_2(\nu)}{(\nu^2 - M^2)^{n+1}} \frac{d\nu^2}{\nu^2 - k^2} + \\ &+ \sum_{0 \leq m \leq n} \frac{(k^2 - M^2)^m}{m!} (\hat{k} f_1^{(m)}(M^2) + f_2^{(m)}(M^2)). \end{aligned} \quad (54.26)$$

It is clear that because of (54.16) and (54.19), we can take for  $R(\hat{k})$  in (54.26) the momentum representation of  $R^r$ ,  $E^a$ , or  $R^c$ , choosing in accordance with (54.19) the appropriate method of going around the pole  $\nu^2 = k^2$ .

The spectral representation (54.26) may be brought into a somewhat different form. In order to do this, we note that the differences

$$\left( \frac{k^2 - M^2}{\nu^2 - M^2} \right)^{n+1} \frac{1}{\nu \pm \hat{k}} - \left( \frac{\hat{k} - M}{\nu \pm M} \right)^{2n+2} \frac{1}{\nu \pm \hat{k}}$$

are polynomials of degree  $2n + 1$  with respect to  $\hat{k}$ . In view of this, we may rewrite (54.26) in the form

$$R(\hat{k}) = (\hat{k} - M)^{2n+2} \int_{(v \geq M + \mu)} \left( -\frac{I_1(v)}{v + \hat{k}} + \frac{I_2(v)}{v - \hat{k}} \right) dv + \sum_{0 \leq m \leq 2n+1} B_m (\hat{k} - M)^m, \quad (54.27)$$

where

$$\left. \begin{aligned} I_1(v) &= \frac{2vJ_1(v)}{(v+M)^{2n+2}} \geq 0, \\ I_2(v) &= \frac{2vJ_2(v)}{(v-M)^{2n+2}} \geq 0, \end{aligned} \right\} \quad (54.28)$$

and  $B_m$  are scalar constants not containing the Dirac matrices.

Just as in the case of the boson function  $q$ , it may be shown that  $B_0 = 0$ , since, by virtue of (53.9), only in this case does the corresponding Green's function have a pole of the first order at the point  $\hat{k} = M$ . The corresponding proof may be carried out by investigating the matrix element of  $S$  between two single-nucleon states, just as in the boson case.

Finally, by making use of the rule for taking conjugates

$$\overline{R^r}(x-y) \equiv (R^r(x-y))^* \gamma^0 = R^a(y-x),$$

which follows from the definitions (54.5) and (54.7), and from which it also follows that

$$\gamma^0 \overline{R^r}(k) \gamma^0 = R^a(k),$$

we find that all the  $B_m$  are real.

The Källén-Lehmann representation for the fermion Green's function may be obtained from the foregoing with the aid of (53.9) and with the additional assumption that  $n = 0$ .

We are thus again (as in §53.5) confronted by the interesting fact that under our system of conditions (§§52.2, 3), a specification of the "rate of increase" is equivalent to specifying the form of the Lagrangian.

**54.3. Proximity to a Contradiction.** In conclusion, we shall say a few words about condition 52.3A which demands that the vacuum expectation values of the radiation operators must be "integrable functions" in the generalized sense.

Let us consider what would happen if we were to impose on these vacuum expectation values  $h(x_1, \dots, x_n)$  the more restrictive condition that they should be ordinary functions for which integrals of the form

$$\int |h(x_1, \dots, x_n)| \varrho(x_1, \dots, x_n) dx_1 \dots dx_n \quad (54.29)$$

exist with weighting functions  $\rho$  falling off at infinity as

$$\left( \frac{1}{|x_1| + \dots + |x_n|} \right)^m \quad (|x| = |x^0| + |x|)$$

with a sufficiently high value of  $m$ .

Since we are not relaxing our system of conditions, but, on the contrary, are making it stricter, all the theorems which were proved above continue to hold. In particular, (53.52), (54.26), and (54.27) remain valid.

Let us go over to the  $x$ -representation, for example, in (53.53). We obtain:

$$Q_{\sigma Q}(x-y) = \delta_{Q\sigma} (\square_x - \mu^2)^{n+1} \int_{(3\mu)^2}^{\infty} I(z) D_z^c(x-y) dz + \\ + \sum_{1 \leq m \leq n} c_m (\mu^2 - \square_x)^m \delta(x-y), \quad (54.30)$$

where

$$\square_x = \Delta_x - \frac{\partial^2}{(\partial x^0)^2}, \quad I(z) = \frac{J(z)}{(z - \mu^2)^{n+1}} \geq 0,$$

and  $D_z^c$  is a causal function for a field of mass  $\sqrt{z}$ .

However, it is well known (see §16.2) that the function  $D_z^c(x-y)$  has a singularity of the type  $1/(x-y)^2$  on the light cone. Because the weighting function  $I$  in the integral (44.30) is positive, *all these singularities add instead of compensating each other*. The operator  $(\square - \mu^2)^m$  intensifies the singularities on the light cone still further. As a result of this, integrals such as (54.29) diverge for functions  $Q$  belonging to the class of functions under investigation—the class of vacuum expectation values of radiation operators.

It thus turns out that it is sufficient to relax somewhat one of the conditions in §52.2 and 52.3 to produce an internal *contradiction*. Of course, this does not in itself allow us to draw any conclusions about the compatibility of the “undistorted” set of conditions in §52.2 and 52.3. We merely draw attention to the fact that the internal consistency of the entire local theory is to some extent an open question.

## §55. The Jost-Lehmann-Dyson Representation

*55.1. Formulation of the Problem.* When important problems of quantum field theory (for example, the analytic properties of elastic scattering amplitudes, some special properties of the so-called deep inelastic processes, and so on) are examined, it is exceedingly useful to have the integral representation of the Fourier transform of the matrix element of the current commutator

$$I(x) = \langle p', \alpha \left[ j_1\left(\frac{x}{2}\right), j_2\left(-\frac{x}{2}\right) \right]_- | p, \beta \rangle \quad (55.1)$$

The integral representation of the Fourier transform of (55.1)

$$\left[ j_1\left(\frac{x}{2}\right), j_2\left(-\frac{x}{2}\right) \right]_- = 0 \quad \text{for} \quad x^2 < 0.$$

has a number of simple properties that follow from the spectral condition. To exhibit these properties in detail, we substitute (55.1) in (55.2), we then use the general property of translational invariance (52.2C) in the form given by (52.19) and, finally, we integrate over the configuration four-space. We thus obtain

$$f(q) = f_1(q) - f_2(q),$$

$$f_1(q) = i \sum_{n_1} \delta\left(\frac{p+p'}{2} - q(n_1) + q\right) \langle p', \alpha | j_1(0) | n_1 \rangle \langle n_1 | j_2(0) | p, \beta \rangle, \quad (55.3)$$

$$f_2(q) = i \sum_{n_2} \delta\left(\frac{p+p'}{2} - q(n_2) - q\right) \langle p', \alpha | j_2(0) | n_2 \rangle \langle n_2 | j_1(0) | p, \beta \rangle. \quad (55.4)$$

Transforming, for simplicity, to the reference frame in which

$$p' + p = 0 \quad (55.5)$$

and substituting

$$\frac{p'_0 + p_0}{2} = a,$$

we conclude that

$$f(q) = 0 \quad \text{for} \quad q \in R, \quad (55.6)$$

where the region  $R$  of four-space  $q$  is defined by the conditions

$$a - \sqrt{q^2 + m_2^2} < q^0 < \sqrt{q^2 + m_1^2} - a, \quad (55.7)$$

where  $m_1$  and  $m_2$  are the minimum masses of the intermediate states  $n_1$  and  $n_2$  on the right-hand side of (55.3) and (55.4). When  $m_1 = m_2 = m$ , the region  $R$  is symmetric in  $q^0$ :

$$|q^0| < \sqrt{q^2 + m^2} - a. \quad (55.8)$$

This case will be referred to as *symmetric*.

The integral representation of  $f(q)$  was obtained by Jost and Lehmann (1957) for the symmetric case. It has the form of the quadruple integral

$$f(q) = \int du \int_0^\infty d\lambda^2 e(q^0) \delta[(q^0)^2 - (q - u)^2 - \lambda^2] (\Phi_1(u, \lambda^2) + q^0 \Phi_2(u, \lambda^2)). \quad (55.9)$$

In the more general, nonsymmetric case, the corresponding generalization of (55.9) was established by Dyson (1958) in the form of the quintuple integral

$$f(q) = \int d^4u \, d\lambda^2 \varepsilon(q^0 - u^0) \delta[(q - u)^2 - \lambda^2] \Psi(u, \lambda^2). \quad (55.10)$$

The value of (55.9) and (55.10) in physically important cases is connected with the fact that the region of integration with respect to  $u$  turns out to be finite, whereas the integral with respect to  $\lambda^2$  is evaluated only over part of the real positive axis (see below, §55.4).

The general form of (55.9) and (55.10) will be shown later to be based on the important fact that according to (55.2) and (55.1),  $f(q)$  is the Fourier transform of the function that vanishes outside the light cone in configuration space:

$$\tilde{f}(x) = 0 \quad \text{for} \quad x^2 < 0. \quad (55.11)$$

Such functions  $f(q)$  will be called "causal functions."

Moreover, the restrictions on the range of integration in (55.9) and (55.10) are reflections of the spectral properties (55.6) of the function  $f(q)$  in momentum  $q$ -space.

We begin by considering the general form of the spectral representation.

**55.2. General Form of Representation.** Let us try to construct the integral representation of all the "causal" functions  $f(q)$ .

To do this, we extend the four-dimensional  $x$  and  $q$  spaces to the six-dimensional  $X$  and  $Q$  spaces

$$\{x\} \rightarrow \{X\} = \{x, y\}, \quad \{q\} \rightarrow \{Q\} = \{q, \lambda\},$$

where

$$y = \{y_1, y_2\}, \quad \lambda = \{\lambda_1, \lambda_2\}.$$

Thus, for example,

$$(x^0, x^1, x^2, x^3) \rightarrow (X^0 = x^0, X^1 = x^1, X^2 = x^2, X^3 = x^3, X^4 = y_1, X^5 = y_2).$$

The additional components  $y_1, y_2, \lambda_1, \lambda_2$  will be assumed to be space-like in six-dimensional space so that

$$X^2 = x^2 - y^2 = (x^0)^2 - (x)^2 - y_1^2 - y_2^2, \quad Q^2 = q^2 - \lambda^2, \quad XQ = xq - y\lambda.$$

We now introduce the function  $\tilde{F}$  in the six-dimensional configurational representation:

$$\tilde{F}(X) = 4\pi \tilde{f}(x) \delta(x^2 - y^2) = 4\pi \tilde{f}(x) \delta(X^2). \quad (55.12)$$

By virtue of (55.11), the operation of multiplication of  $\tilde{f}$  by  $\delta(x^2)$  turns out to be defined and the function  $\tilde{F}$  unambiguously defines the function  $\tilde{f}$ . In point of fact,

$$\frac{1}{4\pi} \int_0^\infty \tilde{F}(X) dy^2 = \frac{1}{4\pi^2} \int \tilde{F}(X) d^2y = \begin{cases} \tilde{f}(x) & \text{for } x^2 \geq 0, \\ 0 & \text{for } x^2 < 0. \end{cases} \quad (55.13)$$

Evaluating the four-dimensional Fourier transform of (55.12), we obtain

$$F(Q) = \frac{1}{(2\pi)^6} \int e^{iQX} \tilde{F}(X) d^6X = \frac{1}{\pi} \int f(q') D^1(Q - \tilde{q}') d^4q'. \quad (55.14)$$

Here, the symbol  $\tilde{q}$  represents the special six-vector  $(q^0, q^1, q^2, q^3, 0, 0)$  and  $D^1$  is the even invariant function

$$D^1(Q) = \frac{1}{(2\pi)^5} \int e^{iQX} \delta(X^2) d^6X = \frac{1}{\pi^2 (Q^2)^2}, \quad (55.15)$$

satisfying the six-dimensional wave equation

$$\square_6 D^1(Q) = 0, \\ \square_6 = \frac{\partial^2}{\partial Q_0^2} - \frac{\partial^2}{\partial Q_1^2} - \dots - \frac{\partial^2}{\partial Q_5^2}.$$

It follows from (55.14) [see also (55.12)] that  $F(Q)$  will also satisfy the six-dimensional wave equation

$$\square_6 F(Q) = 0.$$

Finally, we note that according to (55.2), (55.13), and (55.14),

$$f(q) = F(\tilde{q}), \quad (55.16)$$

where  $F(Q)$  is the extension of the function  $f(q)$  to a larger number of dimensions.

We now recall the well-known property of the solution of the wave equation, according to which, this solution can be expressed in terms of its (initial) values and the values of its derivatives on a given space-like surface (this is the generalization of the Kirchhoff integral). This formula contains the odd solution of the homogeneous wave equation, and is the six-dimensional analog of the Pauli-Jordan function (10.18). In the case which we are considering, this function is

$$D(Q) = \frac{1}{(2\pi)^5 i} \int e^{iQX} \varepsilon(X^0) \delta(X^2) d^6X = \frac{\varepsilon(Q^0) \delta'(Q^2)}{2\pi^2}. \quad (55.17)$$

The generalized Kirchhoff integral can now be written in the form [see §31 in the book by Vladimirov (1964)]

$$F(Q) = \int_{\Sigma} d\Sigma' \left[ D(Q - Q') \frac{\partial F(Q')}{\partial \Gamma'} + F(Q') \frac{\partial D(Q' - Q)}{\partial \Gamma'} \right], \quad (55.18)$$



where  $\Sigma$  is an arbitrary space-like surface in six-dimensional space, which can be defined by

$$Q^0 = \omega(Q^1, \dots, Q^5), \quad (55.19)$$

$d\Sigma$  is an element of area on  $\Sigma$ , and the operator  $\partial/\partial\Gamma$  is the normal derivative on  $\Sigma$ , defined by

$$\frac{\partial}{\partial\Gamma} d\Sigma = \frac{\partial}{\partial Q^0} dQ^1 \dots dQ^5 + \dots + \frac{\partial}{\partial Q^5} dQ^0 \dots dQ^4. \quad (55.20)$$

The right-hand side of (55.18) can be interpreted as the convolution of the generalized functions  $D$ ,  $F$ , and their derivatives. If we now introduce the generalized functions (layers)

$$\delta(\Sigma) \frac{\partial F(Q)}{\partial\Gamma} \quad \text{and} \quad \frac{\partial}{\partial\Gamma} (\delta(\Sigma) F(Q)),$$

related to the space-like surface  $\Sigma$  and satisfying the rule

$$\begin{aligned} \int d^6Q \delta(\Sigma) \frac{\partial F(Q)}{\partial\Gamma} \varphi(Q) &= \int_{\Sigma} d\Sigma \frac{\partial F(Q)}{\partial\Gamma} \varphi(Q), \\ \int d^6Q \frac{\partial}{\partial\Gamma} [\delta(\Sigma) F(Q)] \varphi(Q) &= - \int_{\Sigma} d\Sigma F(Q) \frac{\partial \varphi(Q)}{\partial\Gamma}, \end{aligned}$$

then (55.18) can be written in the form

$$F(Q) = \int d^6Q' D(Q - Q') \left[ \delta(\Sigma') \frac{\partial F(Q')}{\partial\Gamma'} + \frac{\partial}{\partial\Gamma'} (\delta(\Sigma') F(Q')) \right]. \quad (55.21)$$

If we use (55.16), we find that (55.18) and (55.21) with  $Q = \tilde{q}$  provide the required integral representation of  $f(q)$ . We now choose the space-like surface  $\Sigma$  to have a special form, so that its equation (55.19) is independent of the coordinates  $Q^4 = \lambda_1$ ,  $Q^5 = \lambda_2$ . The corresponding derivatives are then no longer present in (55.20), so that the integration can be factorized:

$$\frac{\partial}{\partial\Gamma} d\Sigma = \frac{\partial}{\partial\gamma} d\sigma dQ^4 dQ^5 = \pi \frac{\partial}{\partial\gamma} d\sigma d\lambda^2, \quad \lambda^2 = \lambda_1^2 + \lambda_2^2,$$

where  $\partial/\partial\gamma$  is the conormal derivative to the surface  $\sigma$  in the four-dimensional space:

$$\frac{\partial}{\partial\gamma} d\sigma = \frac{\partial}{\partial Q^0} dQ^1 \dots dQ^3 + \dots + \frac{\partial}{\partial Q^3} dQ^0 \dots dQ^2,$$

and we have

$$\left. \begin{aligned} f(q) &= \int d^4u \int_0^\infty d\lambda^2 \varepsilon(q^0 - u^0) \delta'((q-u)^2 - \lambda^2) \psi_1(u, \lambda^2), \\ \psi_1(u, \lambda^2) &= \frac{1}{2\pi^2} \left[ \delta(\sigma') \frac{\partial F(u, \lambda^2)}{\partial \gamma'} + \frac{\partial}{\partial \gamma'} (\delta(\sigma') F(u, \lambda^2)) \right]. \end{aligned} \right\} \quad (55.22)$$

In transforming from (55.21) to (55.22), we use the explicit form (55.17) of the function  $D$ , and change the variables of integration as follows:

$$(Q'_0, \dots, Q'_3) \rightarrow (u^0, u^1, u^2, u^3, \lambda_1, \lambda_2).$$

We note that according to (55.14) and (55.15), the function  $F(u, \lambda^2)$  can be formally represented by the following "integral":

$$F(u, \lambda^2) = \frac{1}{\pi^3} \int \frac{f(q) d^4q}{[(u-q)^2 - \lambda^2]^2}. \quad (55.23)$$

If we look upon the integrals on the right-hand side of (55.22) as convolutions of generalized functions, we can carry out integration by parts with respect to  $\lambda^2$ , and reduce (55.22) to the form given by (55.10). We then have

$$\Psi(u, \lambda^2) = \frac{\partial \psi_1(u, \lambda^2)}{\partial \lambda^2}.$$

If the equation of the surface is specialized still further in the form

$$u^0 = 0,$$

then, instead of (55.22), we have

$$\begin{aligned} f(q) &= \frac{1}{2\pi} \int d\mathbf{u} du^0 \int_0^\infty d\lambda^2 \varepsilon(q^0 - u^0) \delta'((q^0 - u^0)^2 - (q - \mathbf{u})^2 - \lambda^2) \times \\ &\quad \times \left[ \delta(u^0) \frac{\partial F(u, \lambda^2)}{\partial u^0} + \frac{\partial}{\partial u^0} (\delta(u^0) F(u, \lambda^2)) \right]. \end{aligned} \quad (55.24)$$

By evaluating the integrals with respect to  $\lambda^2$  and  $u^0$  by parts, we obtain the representation given by (55.9), where

$$\begin{aligned} \Phi_1(u, \lambda^2) &= -\frac{1}{2\pi} \frac{d}{d\lambda^2} \left[ \mathfrak{F}(\lambda^2) \frac{\partial F(u, \lambda^2)}{\partial u^0} \right]_{u^0=0}, \\ \Phi_2(u, \lambda^2) &= \frac{1}{\pi} \frac{d^2}{d(\lambda^2)^2} [\mathfrak{F}(\lambda^2) F(u, \lambda^2)]_{u^0=0}. \end{aligned} \quad (55.25)$$

This completes the derivation of the general form of the integral representation. We note

that the representations given by (55.10) do not have the property of uniqueness (if only because of the different choice of the surface  $\Sigma$ ). However, the representation given by (55.9) does have this property, i.e., the relationships between the causal function  $f(q)$  and the weighting functions  $\Phi_i(u, \lambda^2)$  are mutually unique. Readers interested in the proof of this are referred to the original paper by Jost and Lehmann (1957).

**55.3. Region of Integration.** We now consider the restrictions on the region of integration with respect to  $u$  and  $\lambda^2$  that follow from the spectral property (55.6). The conditions defining  $R$  can be written in the general form that includes the physically important case (55.7):

$$g_-(q) < q^0 < g_+(q). \quad (55.26)$$

These conditions determine two space-like sufficiently smooth surfaces  $\sigma_{\pm} [q^0 = g_{\pm}(q)]$  in four-dimensional  $q$  space.

It follows from (55.10) that the relation between the spaces  $q$  and  $u, \lambda^2$  can be established with the aid of the condition

$$(q - u)^2 = \lambda^2 \geq 0. \quad (55.27)$$

This defines a hyperboloid of two sheets,  $h(u, \lambda^2)$ , in the four-dimensional  $q$  space, which depends parametrically on the variables  $u, \lambda^2$ . We shall refer to this hyperboloid as the allowed hyperboloid if its upper sheet  $h_+$  has no points lying below  $\sigma_+$ , whereas the lower sheet  $h_-$  has no points lying above  $\sigma_-$  (see Fig. 67), i.e.,

$$\left. \begin{aligned} u^0 + \sqrt{(q - u)^2 + \lambda^2} &\geq g_+(q), \\ u^0 - \sqrt{(q - u)^2 + \lambda^2} &\leq g_-(q). \end{aligned} \right\} \quad (55.28)$$

The set of allowed hyperboloids corresponds to the set of points  $S$  in the five-dimensional  $U$ -space

$$\{U\} = \{u^0, u^1, u^2, u^3, u^4 = \lambda^2\}, \quad u^4 \geq 0.$$

It is now clear that if the integrals in (55.10) are evaluated only over the set  $S$ , then  $f(q)$  satisfies the spectral condition (55.6). The converse proposition is also true, i.e., any causal function  $f(q)$  satisfying the spectral condition can be represented by an integral such as (55.10), where the region of integration is contained in  $S$ . The proof of this converse proposition is based on the uniqueness of the Jost-Lehmann (1957) representation. Another proof that takes into account the singular character of the generalized functions in a more rigorous fashion is given in §32 of the monograph by Vladimirov (1964).

We must now determine the explicit form of the region  $S$ . Using (55.7) for  $g_+$  and  $g_-$ , we have from (55.28)

$$m(u, \lambda) \leq u^0 \leq M(u, \lambda), \quad (55.29)$$

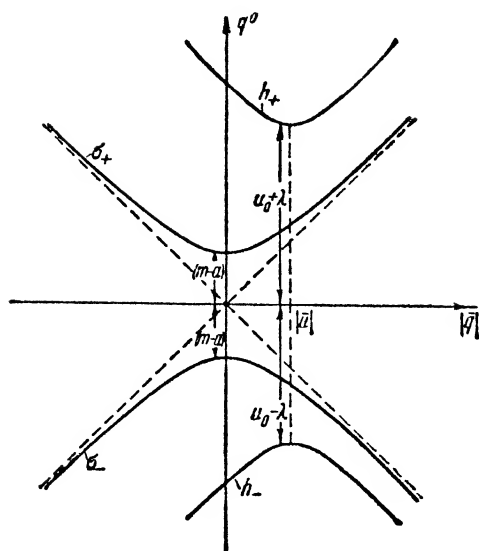


Fig. 67. Example of allowed hyperboloid in the symmetric case.

where

$$M(u, \lambda) = \min_q \{g_-(q) + \sqrt{(q-u)^2 + \lambda^2}\} = \begin{cases} a - \sqrt{u^2 + (m_2 - \lambda)^2} & \text{for } \lambda \leq m_2, \\ a - \sqrt{u^2} & \text{for } \lambda \geq m_2 \end{cases} \quad (55.30)$$

and

$$m(u, \lambda) = \max_q \{g_+(q) - \sqrt{(q-u)^2 - \lambda^2}\} = \begin{cases} \sqrt{u^2 + (m_1 - \lambda)^2} - a & \text{for } \lambda \leq m_1, \\ \sqrt{u^2} - a & \text{for } \lambda \geq m_1. \end{cases} \quad (55.31)$$

For the symmetric case,  $m_1 = m_2 = m$ , this yields

$$|u^0| + |u| \leq a, \quad \lambda^2 \geq (m - \sqrt{(a - |u^0|)^2 - u^2})^2. \quad (55.32)$$

If we choose the hyperplane of integration in the form  $u^0 = 0$ , we obtain the region of integration  $S_0$  for the Jost-Lehmann representation (see Fig. 67)

$$|u| \leq a, \quad \lambda^2 \geq (m - \sqrt{a^2 - u^2})^2. \quad (55.33)$$

In the nonsymmetric case, the region  $S$  turns out to be nonsymmetric under the substitution  $u^0 \rightarrow -u^0$ , and the surface of integration cannot be chosen in the form  $u^0 = 0$ .

When this is so, we replace (55.32) by a more general restriction which can be written in the form

$$\begin{aligned} |u^0| + |u| &\leq a, \\ \lambda^2 &\geq \max \left\{ (m_2 - \sqrt{(a - u^0)^2 - |u|^2})^2, \right. \\ &\quad \left. (m_1 - \sqrt{(a + u^0)^2 - |u|^2})^2 \right\}. \end{aligned} \quad (55.34)$$

*55.4. Some Consequences.* From representations such as (55.10) for the Fourier transform of the matrix elements of the current commutator, we can obtain the corresponding representation for the Fourier transforms of the matrix elements of retarded, advanced, and so on, current products.

Such representations are useful for many applications. Consider the matrix elements of the "retarded" commutator

$$\tilde{f}^{\text{ret}}(x) = i\theta(x^0) \langle p' \alpha | \left[ j_1\left(\frac{x}{2}\right), j_2\left(-\frac{x}{2}\right) \right] | p \beta \rangle. \quad (55.35)$$

The operation of multiplication of the matrix element (55.1) by  $\theta(x^0)$  is not, of course, always unique. It can, however, be defined to within a finite linear combination of  $\delta^4(x)$  and derivatives (see §§ 53 and 54). In momentum representation [formula (55.36) and thereafter], this corresponds to a polynomial ambiguity. The corresponding coefficients may depend on  $p, p', \alpha, \beta$ . If we use (55.1), (55.2), and (55.10) to evaluate the Fourier transform

$$f^{\text{ret}}(q) = \frac{1}{(2\pi)^4} \int dx e^{iqx} \tilde{f}^{\text{ret}}(x),$$

explicitly, we obtain

$$f^{\text{ret}}(q) = \frac{1}{2\pi i} \int \frac{d^4u \, d\lambda^2 \Psi(u, \lambda^2)}{(q-u)^2 - \lambda^2 - i\epsilon(q^0 - u^0)}. \quad (55.36)$$

The Fourier transform of the matrix element of the "advanced commutator" can be evaluated similarly. This can be obtained from (55.36) by substituting  $-i\epsilon \rightarrow +i\epsilon$ . In the symmetric case, (55.36) becomes

$$f^{\text{ret}}(q) = \frac{1}{2\pi i} \int_{S_0} \frac{du \, d\lambda^2}{q_0^2 - (q-u)^2 - \lambda^2 - i\epsilon q^0} \{ \Phi_1(u, \lambda^2) + q^0 \Phi_2(u, \lambda^2) \}, \quad (55.37)$$

where the region of integration  $S_0$  coincides with the region of integration in the Jost-Lehmann representation (55.9) and is defined by the inequalities given by (55.33).

The integral representations for the matrix element of the ordinary product of currents

$$f_1(q) = \frac{i}{(2\pi)^4} \int dx e^{iqx} \langle p', \alpha | j_1\left(\frac{x}{2}\right) j_2\left(-\frac{x}{2}\right) | p\beta \rangle$$

and the "causal" matrix element

$$\begin{aligned} f^c(q) &= \frac{i}{(2\pi)^4} \int dx e^{iqx} \langle p', \alpha | T\left(j_1\left(\frac{x}{2}\right) j_2\left(-\frac{x}{2}\right)\right) | p\beta \rangle = \\ &= \frac{1}{(2\pi)^4 i} \int dx e^{iqx} \langle p', \alpha | \frac{\delta^2 S}{\delta \varphi_1(x/2) \delta \varphi_2(-x/2)} \hat{S}^\dagger | p\beta \rangle, \end{aligned} \quad (55.38)$$

$$f^c(q) = f^{\text{ret}}(q) + f_1(-q) \quad (55.39)$$

can be obtained in a simple fashion only under certain conditions.

Thus, if, in the symmetric case,

$$m > a, \quad (55.40)$$

then  $f_1(q)$  contains only positive frequencies, i.e.,

$$f_1(q) = 0 \quad \text{for} \quad q^0 < 0.$$

Hence, if we write

$$f_1(q) = \theta(q^0) f(q)$$

and use (55.9), we obtain

$$f_1(q) = \theta(q^0) \int du \int d\lambda^2 \delta(q_0^2 - (q-u)^2 - \lambda^2) [\Phi_1(u, \lambda^2) + q^0 \Phi_2(u, \lambda^2)]. \quad (55.41)$$

From (55.39), (55.41), and (55.37), we find that

$$f^c(q) = \frac{1}{2\pi i} \int \frac{du d\lambda^2}{q_0^2 - (q-u)^2 - \lambda^2 + i\varepsilon} [\Phi_1(u, \lambda^2) + q^0 \Phi_2(u, \lambda^2)]. \quad (55.42)$$

We note that the above integral representations are direct generalizations of the integral representations for the vacuum matrix element considered in §§53 and 54, and become identical with them in the corresponding limiting cases. Thus, for example, substituting

$$p = p' = 0, \quad j_1(x) = j_2(x) = i \frac{\delta S}{\delta \varphi(x)} \hat{S}^\dagger,$$

in (55.1), we obtain the vacuum expectation value of the commutator of pseudoscalar currents (symmetric case). Here  $a = 0$  and, in accordance with (55.33), the region of integration in the Jost-Lehmann representation is

$$u = 0, \quad \lambda^2 > m^2.$$

Substituting

$$\Phi_1(u, \lambda^2) = \frac{1}{(2\pi)^3} \delta(u) J(\lambda^2), \quad \Phi_2 = 0,$$

we obtain from (55.9)

$$f(q) = \frac{1}{(2\pi)^3} \varepsilon(q^0) J(q^2),$$

and, if we recall (55.2), we find that this corresponds exactly to the spectral representation (53.30) for the vacuum expectation value of the current commutator.

## § 56. Derivation of Dispersion Relations for Pion-Nucleon Scattering

*56.1. Connection between Scattering Amplitude and the Retarded and Advanced Matrix Elements.* We suppose that prior to scattering, the nucleon is a state with momentum  $\mathbf{p}$  and discrete spin and isospin indices which are together denoted by the letter  $s$ , and the meson (pion) is in the state with momentum  $\mathbf{q}$  and discrete indices  $\rho$ . The corresponding quantities after scattering will be denoted by primed letters ( $\mathbf{p}'$ ,  $s'$  for the nucleon,  $\mathbf{q}'$ ,  $\rho'$  for the pion). We shall find it convenient to separate out the pion momenta  $\mathbf{q}$  and  $\mathbf{q}'$ , and to denote the remaining quantities characterizing the initial and the final states by a single symbol:

$$\alpha = (\mathbf{p}, s, \varrho), \quad \omega = (\mathbf{p}', s', \varrho').$$

The scattering amplitude for the above process will be expressed in terms of the matrix element

$$\langle \mathbf{p}' s', \mathbf{q}' \varrho' | S | \mathbf{p} s, \mathbf{q} \varrho \rangle.$$

After picking out the pion operators  $a_{\rho}^{(-)}(q')$  and  $a_{\rho}^{(+)}(q)$  in the amplitudes of the initial and final states, we commute  $a^{(+)}$  with the  $S$ -matrix:

$$\begin{aligned} \langle \mathbf{p}' s', \mathbf{q} \varrho' | S | \mathbf{p} s, \mathbf{q} \varrho \rangle &= \langle \mathbf{p}' s' | a_{\varrho'}^-(q') S a_{\varrho}^+(q) | \mathbf{p} s \rangle = \\ &= \langle \mathbf{p}' s' | a_{\varrho'}^-(q') a_{\varrho}^+(q) S | \mathbf{p} s \rangle + \frac{1}{(2\pi)^{3/2} \sqrt{2q^0}} \int dy e^{-iqy} \langle \mathbf{p}' s' | a_{\varrho'}^-(q') \frac{\delta S}{\delta \varphi^{\varrho}(y)} | \mathbf{p} s \rangle. \end{aligned} \quad (56.1)$$

To evaluate the first term, we use the condition of stability of the single-nucleon state. Then, by commuting the operators, we find that this term is equal to a product of  $\delta$ -functions. In the second term, it is also necessary to commute  $a^{(-)}$  with  $\delta S / \delta \varphi$ . By taking (25.21) into account, we obtain from this the following expression for the scattering amplitude:

$$\begin{aligned}
\delta(p+q-q'-p') f(\alpha q, \omega q') &= \\
&= \frac{\pi}{(2\pi)^3 i} \int dx dy e^{i(q'x - qy)} \langle p's' | \frac{\delta^2 S}{\delta \varphi_{Q'}(x) \delta \varphi_Q(y)} \dot{S} | ps \rangle, \quad (56.2) \\
p'^0 &= \sqrt{p^2 + M^2}, \dots, q^0 = \sqrt{q^2 + \mu^2}.
\end{aligned}$$

Using the condition of translational invariance (52.9) in the integrand

$$\langle p's' | \frac{\delta^2 S}{\delta \varphi_{Q'}(x) \delta \varphi_Q(y)} \dot{S} | ps \rangle = e^{i(p'-p)\frac{x+y}{2}} \langle p's' | \frac{\delta^2 S}{\delta \varphi\left(\frac{x-y}{2}\right) \delta \varphi\left(\frac{y-x}{2}\right)} \dot{S} | ps \rangle$$

and performing one explicit integration, we obtain from (56.2)

$$f(\alpha q, \omega q') = \frac{2\pi^2}{i} \int dx e^{ix\frac{q'+q}{2}} \langle p's' | \frac{\delta^2 S}{\delta \varphi\left(\frac{x}{2}\right) \delta \varphi\left(-\frac{x}{2}\right)} \dot{S} | ps \rangle. \quad (56.3)$$

By analogy with the vacuum expectation values introduced in §§53 and 54, and the single-particle matrix elements introduced in §55 [cf. (55.38)], the matrix element under the integral sign is conveniently referred to as the causal matrix element

$$F^c(x) = \frac{2\pi^2}{i} \langle p's' | \frac{\delta^2 S}{\delta \varphi\left(\frac{x}{2}\right) \delta \varphi\left(-\frac{x}{2}\right)} \dot{S} | ps \rangle. \quad (56.4)$$

Henceforth, it will be convenient to use the retarded and advanced matrix elements

$$F_{Q'Q}^{\text{ret}}(x) = \frac{2\pi^2}{i} \langle p's' | \frac{\delta}{\delta \varphi_{Q'}\left(\frac{x}{2}\right)} \left( \frac{\delta S}{\delta \varphi_Q\left(-\frac{x}{2}\right)} \dot{S} \right) | ps \rangle, \quad (56.5)$$

$$F_{Q'Q'}^{\text{adv}}(x) = \frac{2\pi^2}{i} \langle p's' | \frac{\delta}{\delta \varphi_Q\left(-\frac{x}{2}\right)} \left( \frac{\delta S}{\delta \varphi_{Q'}\left(\frac{x}{2}\right)} \dot{S} \right) | ps \rangle = F_{Q'Q}^{\text{ret}}(-x), \quad (56.6)$$

with the properties

$$F^{\text{ret}}(x) = 0 \quad \text{for} \quad x \lesssim 0, \quad (56.7)$$

$$F^{\text{adv}}(x) = 0 \quad \text{for} \quad x \gtrsim 0, \quad (56.8)$$

and the "frequency" matrix elements



$$F_{\bar{Q}'Q}^-(x) = \frac{2\pi^2}{i} \langle p's' | j_{Q'}\left(\frac{x}{2}\right) j_Q\left(-\frac{x}{2}\right) | ps \rangle, \quad (56.9)$$

$$F_{Q\bar{Q}'}^+(x) = \frac{2\pi^2}{i} \langle p's' | j_Q\left(-\frac{x}{2}\right) j_{Q'}\left(\frac{x}{2}\right) | ps \rangle = F_{\bar{Q}'Q}^-(x), \quad (56.10)$$

related to  $F^{\text{ret}}$ ,  $F^{\text{adv}}$ , and  $F^c$  by

$$F^c = F^{\text{ret}} - F^+ = F^{\text{adv}} - F^-, \quad (56.11)$$

It will also be convenient to introduce the Fourier representations for the functions  $F(x)$

$$T_{\alpha\omega}(k) = \int dx e^{ikx} F(x). \quad (56.12)$$

From the definitions (56.5)–(56.6) the following relations follow for  $T$  ( $P_{\rho\rho'}$  denotes an interchange of indices  $\rho$  and  $\rho'$ ):

$$\left. \begin{aligned} T_{\alpha\omega}^{\text{ret}}(k) &= P_{Q\bar{Q}'} T_{\alpha\omega}^{\text{adv}}(-k), \\ T_{\alpha\omega}(k) &= P_{Q\bar{Q}'} T_{\alpha\omega}^+(k), \end{aligned} \right\} \quad (56.13)$$

from which after taking (56.10) into account we also obtain:

$$T_{\alpha\omega}(k) = T_{\alpha\omega}^{\text{ret}}(k) - T_{\alpha\omega}^+(k) = T_{\alpha\omega}^{\text{adv}}(k) - T_{\alpha\omega}^-(k) \quad (56.14)$$

and

$$T_{\alpha\omega}^{\text{ret}}(k) - T_{\alpha\omega}^{\text{adv}}(k) = T_{\alpha\omega}^+(k) - P_{Q\bar{Q}'} T_{\alpha\omega}^+(-k). \quad (56.15)$$

We note that the scattering amplitude is related to  $T^c$  by the equation

$$f(\alpha q, \omega q') = T_{\alpha\omega}^c\left(\frac{q+q'}{2}\right). \quad (56.16)$$

As was done previously, we now reduce the causal function  $T^c$  (or  $F^c$ ) to the retarded or the advanced one, remembering that by virtue of (56.7) and (56.8), the functions  $F^{\text{ret}}$  and  $F^{\text{adv}}$  are more convenient for analytic continuation. With this in view, we examine in greater detail the quantity  $T^{(+)}$ :

$$T_{\alpha\omega}^+\left(\frac{q+q'}{2}\right) = \frac{2\pi^2}{i} \int dx e^{i\frac{q+q'}{2}x} \langle p's' | j_Q\left(\frac{-x}{2}\right) j_{Q'}\left(\frac{x}{2}\right) | ps \rangle.$$

We use the fact that the set of functions (52.21) is complete. By making replacements such as

$$j(z) = U_{L_z}^\dagger j(0) U_{L_z}$$

and using (52.9), we obtain after integrating over  $x$ :

$$\begin{aligned} T_{\alpha\omega}\left(\frac{q+q'}{2}\right) &= \frac{2\pi^2}{i} \int dx e^{i\frac{q+q'}{2}x} \sum_n \int dk \langle \mathbf{p}'s' | j_Q\left(-\frac{x}{2}\right) | \mathbf{kn} \rangle \langle \mathbf{kn} | j_{Q'}\left(\frac{x}{2}\right) | \mathbf{ps} \rangle = \\ &= \frac{(2\pi)^3 \pi}{i} \sum_n \delta\left(\sqrt{\mathbf{k}^2 + M_n^2} + \frac{q^0 + q'^0 - p^0 - p'^0}{2}\right) \langle \mathbf{p}'s' | j_Q(0) | \mathbf{kn} \rangle \langle \mathbf{kn} | j_{Q'}(0) | \mathbf{ps} \rangle; \\ &\quad \mathbf{k} = \frac{\mathbf{p} + \mathbf{p}' - \mathbf{q} - \mathbf{q}'}{2}. \end{aligned} \quad (56.17)$$

Since by virtue of (56.16) we are interested in the region in which

$$p + q - p' - q' = 0,$$

we see that the argument of the  $\delta$ -function in (56.17) may be represented in the form

$$\sqrt{(\mathbf{p} - \mathbf{q}')^2 + M_n^2} + q'^0 - p^0 = \sqrt{(\mathbf{p} - \mathbf{q}')^2 + M_n^2} + \sqrt{\mu^2 + \mathbf{q}'^2} - \sqrt{M^2 + \mathbf{p}^2}. \quad (56.18)$$

Since (56.18) appears inside the  $\delta$ -function we must have

$$\sqrt{M_n^2 + (\mathbf{p} - \mathbf{q}')^2} + \sqrt{\mu^2 + \mathbf{q}'^2} = \sqrt{M^2 + \mathbf{p}^2}.$$

This relation corresponds to the transformation of a particle of mass  $M$  into two particles of masses  $\mu$  and  $M_n$ . But since in any case

$$M_n \geq M,$$

the foregoing is impossible because of conservation of energy and momentum. From this it follows that the expression given by (56.18) is essentially positive, as a result of which  $T^{(+)}$  vanishes.

Thus, if the matrix elements are evaluated between the states of real particles with positive energies,

$$p^0 = \sqrt{\mathbf{p}^2 + M^2} > 0, \quad \text{and so on,}$$

and the four-momentum conservation law is satisfied, the causal matrix element is equal to the retarded matrix element:

$$T_{\alpha\omega}^c(k) = T_{\alpha\omega}^{\text{ret}}(k) \quad \text{for } k^0 > \mu. \quad (56.19)$$

If we recall (56.16), we obtain from this

$$f(\alpha q, \omega q') = T_{\alpha\omega}^{\text{ret}}\left(\frac{q+q'}{2}\right). \quad (56.20)$$

We note that (56.20) can also be obtained directly by using the condition of stability for the one-nucleon state  $|ps\rangle$  before commuting  $a^-$  in (56.1), and then commute  $a^-$  not with  $\delta S/\delta\varphi(x)$  but with  $i[\delta S/\delta\varphi(x)]S^\dagger = j(x)$ .

A completely analogous procedure will establish that

$$T_{\alpha\omega}^c(k) = T_{\alpha\omega}^{\text{adv}}(k) \quad \text{for } k^0 < -\mu. \quad (56.21)$$

*56.2. Transition to a Fixed Frame of Reference. Difficulties of Analytic Continuation.* As was pointed out in §52, to obtain the dispersion relations, it is necessary to establish the existence of analytic properties in the complex plane of the energy variable. Such properties for the retarded function of a single argument in the example of §52 were directly obtained from a property analogous to (56.7).

In the present case, we are dealing with functions of a large number of independent arguments (momenta of different particles), where in contrast to the case discussed in §§53 and 54, the arguments are not reducible to a single scalar (of the type of the square of the four-momentum,  $k^2$ ). This makes the problem considerably more complicated.

To enable us to write individual independent energy and momentum variables in explicit form, we must fix our frame of reference. It is most convenient to employ the generally accepted frame of reference in which the sum of the nucleon momenta before and after scattering is equal to zero:

$$\mathbf{p} + \mathbf{p}' = 0 \quad (56.22)$$

(this system reduces to the laboratory system in the case of forward scattering for which  $\mathbf{p} = \mathbf{p}' = 0$ ). In this system  $\mathbf{p}^2 = \mathbf{p}'^2$ , and also, by virtue of the law of conservation of energy,  $q^2 = q'^2$ . The law of conservation of momentum in turn leads to

$$\mathbf{p} = \frac{\mathbf{q}' - \mathbf{q}}{2} \text{ and } (\mathbf{q}' + \mathbf{q}) \cdot \mathbf{p} = 0, \quad (56.23)$$

so that we may write

$$\frac{\mathbf{q}' + \mathbf{q}}{2} = \lambda \mathbf{e}, \quad (56.24)$$

where  $\mathbf{e}$  is a unit vector orthogonal to  $\mathbf{p}$ :

$$\mathbf{e}^2 = 1, \quad \mathbf{e} \cdot \mathbf{p} = 0.$$

It follows from (56.23) and (56.24) that

$$q = -p + \lambda e, \quad q' = p + \lambda e$$

and, consequently, that

$$q^2 = q'^2 = p^2 + \lambda^2, \quad q^0 = q'^0 = \sqrt{\mu^2 + p^2 + \lambda^2},$$

i.e., the four-vector argument  $(q + q')/2$  may be replaced by the quantities  $\lambda$  and  $e$ . In doing this, it is convenient in addition to  $\lambda$  to make use of the meson energy

$$\frac{q^0 + q'^0}{2} = q^0 = E = \sqrt{\mu^2 + p^2 + \lambda^2}. \quad (56.25)$$

For a fixed  $p$ , the quantities  $E$  and  $\lambda$  are related to each other in an unambiguous way.

Thus, in the frame of reference (56.22), the quantities  $T^{\text{ret}}$  may be represented with the aid of (56.12) in the following way:

$$T_{\alpha\omega}^{\text{ret}}(E, e) = \int dx e^{i(Ex^0 - ex \sqrt{E^2 - \mu^2 - p^2})} F_{\alpha\omega}^{\text{ret}}(x). \quad (56.26)$$

In accordance with (56.7), the integral in (56.26) is actually taken over the upper light cone  $x^0 > |x|$ .

Let us now investigate the possibility of analytic continuation of (56.26) into the upper half of the complex plane of the variable  $E$ . We note, first of all, that for real positive  $a$  we always have\*

$$\text{Im} \sqrt{E^2 - a} > \text{Im} E. \quad (56.27)$$

It follows that if a purely imaginary quantity  $i\Gamma$  is added to the real  $E$ :

$$E \rightarrow E + i\Gamma \quad (\Gamma > 0),$$

then for each value of  $x^0$  there will always exist a region of  $x$  for which

$$x^0 \Gamma + ex \text{Im} \sqrt{E^2 - \mu^2 - p^2} > 0,$$

as a result of which (56.26) will not be analytic in the upper half-plane of the variable  $E$ .

We see that this difficulty completely precludes the possibility of a direct analytic continuation of (56.26) into the domain of complex values of the variable  $E$ . Moreover, (56.26) is meaningful for real  $E$  only if  $E^2 > \mu^2 + p^2$ , since after going through the branch point  $E_0 = \pm \sqrt{\mu^2 + p^2}$ , the integrand acquires the increasing factor  $\exp \{ex \sqrt{\mu^2 + p^2 - E^2}\}$ , and is no longer meaningful.

\*A direct algebraic proof of (56.27) is rather awkward. Its validity may be verified in a simpler manner by graphical constructions in the complex plane.

Thus (56.26) represents the function  $T$  only along the two segments of the real axis

$$-\infty < E < -\sqrt{\mu^2 + p^2}, \quad \sqrt{\mu^2 + p^2} < E < \infty$$

and cannot be directly continued beyond the limits of these segments.

It is also clear that analogous difficulties will arise if we attempt the analytic continuation of the advanced function  $T^{\text{adv}}(E, e)$ . It is therefore necessary to adopt complicated artificial devices for the analytic continuation of the expressions  $T^{\text{ret}}$  and  $T^{\text{adv}}$ . The analytic continuation in the case of forward scattering may be regarded as a relatively simpler procedure, and we now proceed to investigate it.

56.3. *Outline of the Method of Obtaining Dispersion Relations for Forward Scattering.* In the case of forward scattering, when  $p = 0$ ,

$$T(E, e) = \int dx e^{i(Ex - ex\sqrt{E^2 - \mu^2})} F(x), \quad (56.28)$$

and also

$$\begin{aligned} T^{(\pm)}\left(\frac{q+q'}{2}\right) &= T^{(\pm)}(q) = \frac{(2\pi)^5 \pi}{i} \delta(\sqrt{M^2 + q^2} - M \pm E) \sum_{s''} \langle 0s' | j_Q(0) | qs'' \rangle \times \\ &\times \langle qs'' | j_{Q'}(0) | 0s \rangle + \frac{(2\pi)^5 \pi}{i} \sum_{(M_n > M+\mu)} \delta(\sqrt{M_n^2 + q^2} - M \pm E) \langle 0s' | j_Q(0) | qn \rangle \times \\ &\times \langle qn | j_{Q'}(0) | 0s \rangle. \end{aligned} \quad (56.29)$$

Equation (56.28) can be used for the analytic continuation of the amplitude  $T$  to complex values of the energy variable  $E$ . We shall also consider the corresponding continuation of (56.29), assuming that

$$E^2 \neq \mu^2 + q^2$$

and regarding  $E$  as a variable independent of  $q$  (i.e., we shall leave the pion mass surface).

The first term in (56.29) provides a nonzero contribution for

$$E = \pm (\sqrt{M^2 + q^2} - M),$$

or its equivalent

$$E = \pm \frac{E^2 - q^2}{2M} = \pm \frac{q^2}{2M}. \quad (56.30)$$

The contribution of the second term is nonzero for

$$|E| = \sqrt{M_n^2 + q^2} - M > \sqrt{(M+\mu)^2 + q^2} - M > (M+\mu) - M = \mu. \quad (56.31)$$

Thus, along the segment of the real axis  $-\mu < E < \mu$ , a contribution to  $T$  is provided only

by the intermediate single-nucleon state for two values of the energy, which are related to the square of the four-momentum by (56.30). Therefore, if instead of the function  $F$  we consider the expression

$$\tilde{F}(x) = - \left[ \left( \frac{\partial}{\partial x^0} \right)^2 + \left( \frac{\square}{2M} \right)^2 \right] F(x), \quad (56.32)$$

which in the momentum representation is equivalent to multiplication by the polynomial

$$\tilde{T}(q) = \left[ (q^0)^2 - \left( \frac{q^2}{2M} \right)^2 \right] T(q), \quad (56.33)$$

then for  $-\mu < q^0 < \mu$  the functions  $\tilde{T}^{(+)}$  and  $\tilde{T}^{(-)}$  will vanish,\*\* as a result of which\*\*

$$\tilde{T}(q) = \tilde{T}^{\text{ret}}(q) = \tilde{T}^{\text{adv}}(q) \quad \text{for} \quad -\mu < q^0 < \mu. \quad (56.34)$$

It follows from (56.28) that the points  $E = \pm\mu$  are, generally speaking, branch points of the functions  $\tilde{T}$ . In order to eliminate the double sign in front of the square root  $\pm \sqrt{E^2 - \mu^2}$ , it is simplest of all to consider in place of the functions  $T(E, e)$  their symmetrized and antisymmetrized combinations

$$\left. \begin{aligned} S_+ T(E, e) &= \frac{T(E, e) + T(E, -e)}{2}, \\ S_- T(E, e) &= \frac{T(E, e) - T(E, -e)}{2\lambda}, \end{aligned} \right\} \quad (56.35)$$

which we shall in future collectively denote by  $ST$ . The functions  $ST^{\text{ret}}$  and  $ST^{\text{adv}}$  may be continued into the domain of complex values of  $E$  in the following manner. We introduce the expressions

$$S \int \tilde{F}^{\text{ret}}(x) e^{i(Ex^0 - \epsilon x \sqrt{E^2 - \mu^2}) - \epsilon x^2} dx^0 dx = \Phi^r(E, e; \epsilon), \quad (56.36)$$

$$S \int \tilde{F}^{\text{adv}}(x) e^{i(Ex^0 - \epsilon x \sqrt{E^2 - \mu^2}) - \epsilon x^2} dx = \Phi^a(E, e; \epsilon), \quad (56.37)$$

where in accordance with (56.35)

$$S_+ e^{-i\epsilon x \sqrt{E^2 - \mu^2}} = \cos(\lambda \epsilon x), \quad S_- e^{-i\epsilon x \lambda} = \frac{\sin(\lambda \epsilon x)}{i\lambda}.$$

It is clear, first of all, that the presence of the factor  $\exp(-\epsilon x^2)$  ensures that the function  $\Phi^r(E, e; \epsilon)$  will be analytic in the upper half of the complex plane of  $E$ , while  $\Phi^a(E, e; \epsilon)$  will be analytic in the lower half. Second, by virtue of (56.34)

\*By virtue of relations of the type  $x\delta(x) = 0$ .

\*\*By virtue of the assumption  $p = 0$ , since (56.31) ceases to hold for  $p \neq 0$ .

$$\Phi^r(E, e; \epsilon) - \Phi^a(E, e; \epsilon) = 0 \quad \text{for} \quad -\mu \leq \operatorname{Re} E < \mu, \quad \operatorname{Im} E = 0.$$

Thus the set of functions  $\Phi^r$  and  $\Phi^a$  represents the function

$$\Phi(E, e; \epsilon) = \begin{cases} \Phi^r(E, e; \epsilon) & \text{for } \operatorname{Im} E > 0, \\ \Phi^a(E, e; \epsilon) & \text{for } \operatorname{Im} E < 0, \end{cases} \quad (56.38)$$

which is analytic in the whole complex plane of its argument with the exception of the cuts

$$\text{and} \quad \left. \begin{array}{l} -\infty < \operatorname{Re} E < -\mu, \quad \operatorname{Im} E = 0 \\ \mu < \operatorname{Re} E < \infty, \quad \operatorname{Im} E = 0. \end{array} \right\} \quad (56.39)$$

The values of  $\Phi$  along the upper edges of the cuts are equal to the values of  $\Phi^r$ , while along the lower edges of the cuts they are equal to the values of  $\Phi^a$ .

If we denote by  $n$  the rate of increase of the function  $\Phi$  for large values of  $E$ , we may apply Cauchy's integral theorem to the expression

$$\frac{\Phi(E, e; \epsilon)}{(E - E_0)^{n+2}},$$

where  $E_0$  is a real parameter lying in the interval  $(-\mu, +\mu)$ .

We choose the contour of integration in the form of a circle of small radius  $\delta$  around the point  $E_0$ , two semicircles of large radius  $R$ , and two contours connecting the aforementioned semicircles along the edges of the cuts and separated from the cuts by a distance  $\delta$  (Fig. 68). By letting  $R$  tend to infinity we ensure that the integrals along the large semicircles vanish. Then, by going to the limit as  $\delta \rightarrow 0$ , we obtain

$$\begin{aligned} \Phi(E, e; \epsilon) = & \frac{(E - E_0)^{n+2}}{2\pi i} \int_{-\infty}^{-\mu} \frac{\Phi(E' + i0, e; \epsilon) - \Phi(E' - i0, e; \epsilon)}{(E' - E)(E' - E_0)^{n+2}} dE' + \\ & + \frac{(E - E_0)^{n+2}}{2\pi i} \int_{\mu}^{\infty} \frac{\Phi(E' + i0, e; \epsilon) - \Phi(E' - i0, e; \epsilon)}{(E' - E)(E' - E_0)^{n+2}} dE' + \tilde{P}_{n+1}(E), \end{aligned} \quad (56.40)$$

where  $\tilde{P}_{n+1}(E)$  is a polynomial of the  $(n + 1)$ -th degree in  $E$ . We now go to the limit as  $\epsilon \rightarrow 0$  in the integrals in (56.40). Since the integration is taken over the observable region  $E'^2 > \mu^2$ , the numerators in the integrand will, in accordance with (56.36) and (56.37), assume the form

$$\Phi^r(E', e; 0) - \Phi^a(E', e; 0) = S\hat{T}^{\text{ret}}(E', e) - S\hat{T}^{\text{adv}}(E', e).$$

The integrals themselves, multiplied by  $(E - E_0)^{n+1}$ , will for  $\epsilon = 0$  represent functions that are analytic over the whole complex plane of the variable  $E$  with the exception of the cuts (56.39). It therefore follows from (56.40) that for  $\epsilon = 0$ , the right-hand side defines a function  $\Phi(E, e; 0)$  that is analytic over the whole plane of  $E$  with the exception of the two

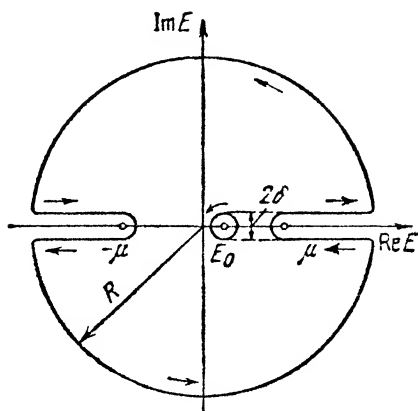


Fig. 68.

cuts referred to above. However, it follows from (56.33), (56.36), (56.37), and (56.40), that  $\Phi(E, e; 0)$  differs from the function

$$ST(E, e) = \begin{cases} ST^{\text{ret}}(E, e) & \text{for } \text{Im } E > 0, \\ ST^{\text{adv}}(E, e) & \text{for } \text{Im } E < 0 \end{cases} \quad (56.41)$$

only by the factor

$$E^2 - \left( \frac{E^2 - q^2}{2M} \right)^2,$$

which assumes the following form on the mass surface (for  $E^2 = \mu^2 + q^2$ ):

$$E^2 - \left( \frac{\mu^2}{2M} \right)^2 = \left( E - \frac{\mu^2}{2M} \right) \left( E + \frac{\mu^2}{2M} \right). \quad (56.42)$$

Thus, returning to the mass surface, we obtain

$$\Phi(E, e, 0) = \tilde{ST}(E, e) = [E^2 - (\mu^2/2M)^2] ST(E, e).$$

It is therefore clear that the function  $ST(E, e)$  will also be analytic over the entire complex domain of the variable  $E$  with the exception of the two cuts (56.39) and the points at which the factor (56.42) is zero. At such points lying outside the cuts, the function  $ST(E, e)$  will have first-order poles. The function  $ST$  will then increase at infinity not faster than a polynomial of degree  $n - 1$ .

Consequently, the Cauchy integral formula may be applied to the function

$$ST(E, e)/(E - E_0)^n$$



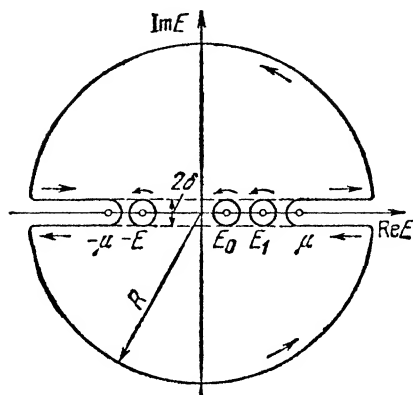


Fig. 69. Contour of integration in the case of the dispersion relation for forward scattering.

where the contour of integration must be chosen so as to take into account the presence of the additional poles connected with the factor (56.42):

$$E_1 = \frac{\mu^2}{2M} < \mu, \quad E_2 = -E_1 = -\frac{\mu^2}{2M},$$

i.e., these poles must be surrounded by two further circles (Fig. 69).

If we then go to the limit as  $R \rightarrow \infty$ ,  $\delta \rightarrow 0$ , we reduce the integral term to integrals over the observable region  $(-\infty < E < -\mu)$ ,  $(\mu < E < \infty)$  [by analogy with (56.40)] and to residues at the poles  $\pm E_1$ .

We obtain

$$ST(E, e) = \frac{S\tilde{T}(E_1, e)}{2E_1(E - E_1)} + \frac{S\tilde{T}(E_2, e)}{2E_2(E - E_2)} + \\ + \frac{(E - E_0)^n}{2\pi i} \left( \int_{\mu}^{\infty} + \int_{-\infty}^{-\mu} \right) \frac{ST^{\text{ret}}(E', e) - ST^{\text{adv}}(E', e)}{(E' - E)(E' - E_0)^n} dE' + P_{n-1}(E). \quad (56.43)$$

Moreover, according to (56.41), (56.20), and (56.16), the physical scattering amplitude  $f$  is expressed in terms of the limiting values of the function  $ST$  on the upper edge of the first cut

$$ST(E + i0, e) = ST^{\text{ret}}(E, e), \quad \text{Im } E = 0, \quad E > \mu. \quad (56.44)$$

Equation (56.43) is the spectral representation of the amplitude  $ST$  continued into the complex plane of the variable  $E$ . If we use (56.44) to carry out in (56.43) the limiting transition to real physical values of  $E$  lying on the upper edge of the right (physical) cut, we obtain the dispersion relation for the pion-nucleon scattering amplitude whose structure is

analogous to that of (52.4) and (52.5). This will be done in detail in §57, where the so-called physical dispersion relation for forward pion-nucleon scattering will be derived.

56.4. *The Case of Scattering with  $p \neq 0$ .* We shall now briefly consider the more general case of scattering, i.e., other than just forward scattering. We shall examine the properties of the scattering amplitude as a function of the energy variable  $E$  for a fixed value of the transferred momentum  $p \neq 0$  (but not, say, the scattering angle  $\theta$ ). The scattering amplitude was expressed in §56.2 in terms of the integral

$$T^c(E, e) = \int e^{i(Ex^0 - \lambda e x)} F^c(x) dx, \quad (56.45)$$

where

$$\lambda^2 = E^2 - p^2 - \mu^2.$$

It was shown there that the difficulties with the analytic continuation into the domain of complex values of  $E$  were connected with the fact that

$$\lambda^2 < E^2. \quad (56.46)$$

It is possible to weaken the inequality (56.46) by formally transforming from  $\mu^2$  to a new variable  $\tau$  defined by

$$\lambda^2 = E^2 - p^2 - \tau, \quad (56.47)$$

and subject to the condition

$$\tau < -p^2, \quad (56.48)$$

i.e., instead of  $T(E, e)$ , we consider the function  $T(E, e, \tau)$ . This operation is equivalent to leaving the pion mass surface. Examination of the analytic properties of the amplitude  $T(E, e, \tau)$  in the complex plane of the variable  $E$  will not now present any essential difficulty. It turns out that it is possible to construct dispersion relations in  $E$  for  $T(E, e, \tau)$ . However, in this program, the reverse transition to the pion mass surface turns out to be quite difficult. The corresponding procedure involves some quite subtle mathematical discussions, using the technique of analytic continuation of generalized functions, first developed by Bogolyubov in the middle 1950's. The most general and classic result is the theorem stating that it is possible to combine the advanced and retarded functions into a single analytic function [Bogolyubov, Medvedev, and Polivanov (1958), Appendix A, Theorem 1]. This was subsequently called the "edge of the wedge theorem." The theorem can be used as a basis of derivations of dispersion relations for different cases. This method has been used to establish the dispersion relation for pion-nucleon scattering with fixed  $t = -(\Delta p)^2$  in the range

$$t_{\min} < t < 0; \quad t_{\min} = -8\mu^2.$$

These methods have been further extended by Bogolyubov and Vladimirov (1958), Bremerman, Oehme, and Taylor (1958), Lehmann (1959), Todorov (1960), and others. In particular, Lehmann (1959) used the Jost-Lehmann-Dyson representation to extend  $|t_{\min}|$  to

$$|t_{\min}| \simeq -\frac{32}{3} \mu^2.$$

### §57. Dispersion Relations for Pion-Nucleon Forward Scattering

*57.1. Transition to Real Quantities.* It was established in §56.3 that (56.43) was valid for any  $E$  not lying on the real axis ( $\text{Im } E \neq 0$ ), where  $ST(E, e)$  was the continuation of the pion-nucleon forward scattering amplitude into the complex domain of the energy variable  $E$ . According to (56.44), physical values of the scattering amplitude correspond to values of the function  $ST(E, e)$  on the upper edge of the right cut. The integral on the right-hand side of (56.43) extends into the physical region ( $\mu < E < \infty$ ) and to the left cut ( $-\infty < E < -\mu$ ). We shall begin by showing that the numerator in the integrand in the integral over the physical region can be expressed in terms of the physical scattering amplitude.

With this aim in view, we consider the properties of the Hermitian functions  $ST^{\text{ret}}$  and  $ST^{\text{adv}}$ . We note, first of all, that it follows directly from the definition (56.5) that

$$\tilde{F}_{\alpha\omega}^{\text{ret}}(x) = P_{\text{qq}} F_{\omega\alpha}^{\text{ret}}(x). \quad (57.1)$$

From this in turn follows the relation expressing the Hermitian property in the momentum representation

$$\tilde{T}_{\alpha\omega}^{\text{ret}}(k) = P_{\text{qq}} T_{\omega\alpha}^{\text{ret}}(-k). \quad (57.2)$$

On the other hand, we have by virtue of (56.13),

$$T_{\alpha\omega}^{\text{adv}}(k) = P_{\text{qq}} T_{\omega\alpha}^{\text{ret}}(-k). \quad (57.3)$$

By combining (57.2) and (57.3), we obtain

$$\tilde{T}_{\alpha\omega}^{\text{adv}}(k) = T_{\omega\alpha}^{\text{ret}}(k). \quad (57.4)$$

where

$$\tilde{T}_{\alpha\omega}(k) \equiv \tilde{T}_{\omega\alpha}(k). \quad (57.5)$$

By going over to the arguments  $E$  and  $e$  in (52.14), we also obtain

$$\tilde{ST}_{\alpha\omega}^{\text{adv}}(E) = ST_{\omega\alpha}^{\text{ret}}(E). \quad (57.6)$$

This relation is very important since it shows directly that the linear combinations

$$\frac{ST_{\alpha\omega}^{\text{ret}}(E) + ST_{\alpha\omega}^{\text{adv}}(E)}{2} = D_{\alpha\omega}(E) \quad (57.7)$$

and

$$\frac{ST_{\alpha\omega}^{\text{ret}}(E) - ST_{\alpha\omega}^{\text{adv}}(E)}{2i} = A_{\alpha\omega}(E) \quad (57.8)$$

are Hermitian:

$$\dot{D}_{\alpha\omega}(E) = D_{\alpha\omega}(E), \quad \dot{A}_{\alpha\omega}(E) = A_{\alpha\omega}(E), \quad (57.9)$$

and represent the Hermitian and the anti-Hermitian parts of the function  $ST^{\text{ret}}$ , i.e.,

$$ST^{\text{ret}}(E) = D(E) + iA(E). \quad (57.10)$$

Thus, according to (57.8), the numerator of the integrand in the physical integral on the right-hand side of (56.43) is the anti-Hermitian part of the physical amplitude  $ST^{\text{ret}}$ .

We now choose some value of the complex argument  $E$  above the physical cut  $\text{Re } E > \mu$ ,  $\text{Im } E > 0$  and perform the limiting transition to the real axis, i.e.,

$$E \rightarrow E + i\varepsilon, \quad E > \mu, \quad \varepsilon \rightarrow +0.$$

Using (56.44) for the left-hand side of (56.43) and (52.3) for the right-hand side, we obtain

$$\begin{aligned} D_{\alpha\omega}(E) = & \frac{\tilde{D}_{\alpha\omega}(E_1)}{2E_1(E-E_1)} + \frac{\tilde{D}_{\alpha\omega}(E_2)}{2E_2(E-E_2)} + \frac{(E-E_0)^n}{\pi} \left( \oint_{\mu}^{\infty} + \int_{-\infty}^{-\mu} \right) \frac{A_{\alpha\omega}(E') dE'}{(E'-E)(E'-E_0)^n} + P_{n-1}(E), \end{aligned} \quad (57.11)$$

where the integral over the physical region must be interpreted as the principal value.

The pole terms on the right-hand side of (57.11) can be formally included in the integral by assuming that the anti-Hermitian part  $A_{\alpha\omega}$  contains the single-particle subthreshold contribution

$$[A_{\alpha\omega}(E)]_1 = -\frac{\pi\tilde{D}_{\alpha\omega}(E_1)}{2E_1} \delta(E-E_1) - \frac{\pi\tilde{D}_{\alpha\omega}(E_2)}{2E_2} \delta(E-E_2). \quad (57.12)$$

This formula corresponds to the relations given by (57.8) and (56.15) and gives the contribution of the single-nucleon term in (56.29) to the anti-Hermitian part of the amplitude.

This term contains products of the single-nucleon matrix elements of the current  $j_\rho(0)$ , and isospin and Lorentz covariance ensure that they can be written in the form

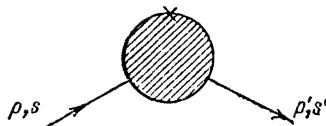


Fig. 70.

$$\langle p's' | j_Q(0) | ps \rangle = \frac{ig((p-p')^2)}{(2\pi)^3} \bar{v}^{s'} + (p') \gamma^5 \tau_Q v^s - (p), \quad (57.13)$$

where  $g$  is a function of the square of the 4-vector  $p - p'$ .

In the case in which we are interested, matrix elements such as (57.13) are evaluated between states for which  $(p - p')^2 = \mu^2$ . The real constant

$$g = g(\mu^2) \quad (57.14)$$

will be called the pion-nucleon interaction constant. We note that this definition of the pion-nucleon interaction constant is completely natural. If we use the correspondence with the Lagrangian for the pion-nucleon interaction in the form given by (8.8)

$$\mathcal{L}(x) = ig_0 : \bar{\Psi}(x) \gamma^5 \tau \Psi(x) \varphi(x) :$$

and evaluate the left-hand side for the “free pion current,”

$$(j_Q(0))_0 = -ig_0 : \bar{\Psi}(0) \gamma^5 \tau_Q \Psi(0) :,$$

we find that, instead of  $g[(p - p')^2]$  on the right-hand side of (57.13), we obtain  $g_0$ . On the other hand, the matrix element  $\langle p's' | j_Q(0) | ps \rangle$  does not, in fact, contain all the higher corrections and corresponds to the diagram shown in Fig. 70. The shaded circle in this diagram describes the  $\pi NN$  vertex function and the cross (X) represents symbolically the pion current  $j_\rho(0)$ .

The matrix element (57.13) thus describes the vertex function shown in Fig. 52b which, in general, is a function of the squares of all three four-momenta  $p^2$ ,  $p'^2$ ,  $(p - p')^2$  contained by it. In the case of (57.13), two of these squares are fixed,  $p'^2 = p^2 = M^2$ , and we have a function of the single variable,  $g[(p - p')^2]$ . This function is known as the *pion-nucleon formfactor*. When  $(p - p')^2 < (3\mu)^2$ , this function turns out to be real.

It is important to note that for real four-vectors  $p$  and  $p'$  lying on the mass surface ( $p^2 = p'^2 = M^2$ ), the quantity  $q^2 = (p - p')^2$  can assume values  $q^2 < 0$  and  $q^2 > 4M^2$ . The point  $q^2 = \mu^2$  is not, therefore, a physical point. The choice of this point in the determination of the renormalized  $\pi N$  interaction constant is the result of a convention, and is dictated by considerations of formal convenience, since it is precisely the quantity given by (57.14) that is present in the single-nucleon term in the dispersion relations.

It is well known that the procedure for introducing the mesonic charge is not unique even in the usual theory. The point is that because the mesonic mass  $\mu$  is not zero, the range of nuclear forces is finite. This means that in contrast to electrodynamics, it is impossible to construct a classical macroscopic mesodynamics and to relate the procedure for determining the mesonic charge to macroscopic experiments such as the Millikan experiment or experiments on the deflection of pith balls. Accordingly, the mesonic charge  $g$  must somehow be introduced in terms of the basic variables of the theory, and its numerical value must be subsequently determined by comparison between the theory and macroscopic experiments. It is clear that the particular way in which macroscopic charge is introduced is not in itself important provided only that the method by which the numerical value of  $g$  is determined is specified.

In accordance with (57.13), the formula given by (57.14) corresponds to the following definition from the point of view of the usual theory:

$$g\gamma^5\tau_Q = g_0\Gamma_Q^5(M^2, M^2; \mu^2), \quad (57.15)$$

where  $g$  is the bare charge in the interaction Lagrangian (8.8) and  $\Gamma_Q^5$  is the vertex operator. If we define chronological products so that

$$\Gamma_Q^5(M^2, M^2; \mu^2) = \gamma^5\tau_Q, \quad (57.16)$$

we can identify  $g_0$  with  $g$ .

Finally, we note that the function  $\Gamma_Q^5(M^2, M^2; \mu^2)$  is not directly related to real processes because the emission (or absorption) of a real meson ( $q^2 = \mu^2$ ) by a real nucleon ( $p^2 = M^2$ ,  $(p \pm q)^2 = M^2$ ) is forbidden by the energy-momentum conservation law.

Substituting (57.13) into the first term on the right-hand side of (56.17), we obtain the single-nucleon contributions in the form

$$(T^+(q))_1 = \frac{g^2}{2i} \left(1 - \frac{\mu^2}{2M^2}\right) \delta\left(E - \frac{\mu^2}{2M}\right) \{\dots\}_{QQ'}^{s's'}, \quad (57.17)$$

$$(T^-(q))_1 = \frac{g^2}{2i} \left(1 - \frac{\mu^2}{2M^2}\right) \delta\left(E + \frac{\mu^2}{2M}\right) \{\dots\}_{QQ'}^{s's'}, \quad (57.18)$$

where

$$\{\dots\}_{QQ'}^{s's'} = \bar{v}^{s'} + (p')\tau_Q\tau_Q \frac{\hat{k} - M}{2k^0} v^s - (p); \quad k^0 = \sqrt{k^2 + M^2} = M - \frac{\mu^2}{2M}, \quad k = -\lambda e. \quad (57.19)$$

Using (56.15) and (57.8) together with (57.17) and (57.18), we can now express the coefficients in the single-pole terms in (57.12) in terms of  $g^2$ . It is convenient to perform this procedure by explicitly taking into account the spin and isospin structure of the pion-nucleon scattering amplitude.

57.2. *Isospin and Spin Structure.* Covariance in ordinary three-dimensional space shows that the amplitude  $T^{\text{ret}}$  can be written as the sum of a term independent of the nucleon spin  $\sigma$  (diagonal in the spin indices  $s, s'$ ) and a term that is linear in  $\sigma$ . The linear term must contain the product of  $\sigma$  and any axial vector. The two vectors  $\mathbf{p}$  and  $\mathbf{e}$  can be used to form only one axial vector, namely,  $[\mathbf{p} \times \lambda \mathbf{e}]$ . We therefore have

$$T_{\alpha\omega}^{\text{ret}} = \delta_{s's} T_{(1)} + i (\sigma [\mathbf{p} \times \lambda \mathbf{e}])_{ss'} T_{(2)}. \quad (57.20)$$

The second term in this expression, which describes the nucleon spin flip, is equal to zero for forward scattering (when  $\mathbf{p} = 0$ ). The product  $[\mathbf{p} \times \lambda \mathbf{e}]$  is proportional to  $\sin \theta$ , where  $\theta$  is the scattering angle in the laboratory frame. In order to take into account the main contribution to the amplitude with spin flip, it is usual to consider  $T_{(2)}$  for forward scattering at small scattering angles. To segregate the corresponding structure in expressions such as (57.19), we consider small terms proportional to first powers of  $\mathbf{p} = -\mathbf{p}'$ :

$$2k^0 \{ \dots \}_{Q'Q}^{s's} = \tau_{Q'} \tau_Q \left( \delta_{s's} E_1 + i \sigma_{s's} [\mathbf{p} \times \mathbf{e}] \frac{\lambda}{2M} \right). \quad (57.21)$$

Finally, if we take into account the isospin structure of the functions  $T_{(1,2)}$  (see Appendix I),

$$T_{(i)} = \delta_{i'i} \delta_{Q'Q} T_{(i)}^{(1)} + \frac{[\tau_{Q'}, \tau_Q]_{i'i}}{2} T_{(i)}^{(2)}, \quad (57.22)$$

we arrive at the four structure functions

$$T_{(i)}^{(k)} \quad (k, i = 1, 2) \quad (57.23)$$

of the pion-nucleon scattering amplitude.

The operation  $S$  then segregates the amplitude with spin flip  $T_{(2)}$ , from the amplitude  $T_{(1)}$ :

$$S_+ T = \delta_{s's} T_{(1)}, \quad S_- T = i \sigma [\mathbf{p} \times \mathbf{e}] T_{(2)}, \quad (57.24)$$

and the operations of symmetrization and antisymmetrization in the isospin index determine the amplitudes  $T^{(1)}$  and  $T^{(2)}$ :

$$\frac{1+P_{Q'Q}}{2} T = T^{(1)}, \quad \frac{1-P_{Q'Q}}{2} T = \frac{[\tau_{Q'}, \tau_Q]}{2} T^{(2)}. \quad (57.25)$$

For the single-nucleon contributions, we use (57.17), (57.18), and (57.21) to obtain, instead of (57.12),

$$\begin{aligned}
(A_{(1)}^{(1)})_1 &= \frac{1}{2i} S_+ \frac{1+P_{q'q}}{2} \{(T^+)_1 - (T^-)_1\} = \frac{g^2 E_1}{4M} [\delta(E - E_1) - \delta(E + E_1)], \quad E_1 = \frac{\mu^2}{2M}, \\
(A_{(2)}^{(1)})_1 &= -\frac{g^2 E_1}{4\mu^2 M} [\delta(E - E_1) + \delta(E + E_1)], \\
(A_{(1)}^{(2)})_1 &= \frac{g^2 E_1}{4M} (\delta(E - E_1) + \delta(E + E_1)), \\
(A_{(2)}^{(2)})_1 &= -\frac{g^2 E_1}{4\mu^2 M} (\delta(E - E_1) - \delta(E + E_1)).
\end{aligned} \tag{57.26}$$

57.3. *Symmetry in Energy.* We must now take into account the symmetry properties of the amplitudes (57.23) with respect to  $E$ . Combining (57.3), (57.4), (57.7), and (57.8), we have

$$(1 \pm P_{q'q}) D_{\alpha\omega}(q) = (P_{q'q} \pm 1) D_{\alpha\omega}(-q), \tag{57.27}$$

$$(1 \pm P_{q'q}) A_{\alpha\omega}(q) = -(P_{q'q} \pm 1) A_{\alpha\omega}(-q). \tag{57.28}$$

Using the explicit dependence on the vector  $q = \lambda e$  given by (57.20), and the relations in (57.24), we obtain the parity properties of the real and imaginary parts of the structure functions (57.23)

$$D_{\text{even}}(E) = \{D_{(1)}^{(1)}(E), D_{(2)}^{(2)}(E)\}, \quad D_{\text{odd}}(E) = \{D_{(2)}^{(1)}(E), D_{(1)}^{(2)}(E)\}, \tag{57.29}$$

$$A_{\text{even}}(E) = \{A_{(1)}^{(2)}(E), A_{(2)}^{(1)}(E)\}, \quad A_{\text{odd}}(E) = \{A_{(1)}^{(1)}(E), A_{(2)}^{(2)}(E)\}. \tag{57.30}$$

We note that the single-nucleon contribution (57.26) to the absorptive parts  $[A_{(k)}^{(i)}]_1$  have the properties defined by (57.30).

The parity properties of  $A_{(k)}^{(i)}$  can now be used to express the integrals over the domain of negative values  $E' < -\mu$  in (57.11) in terms of integrals over the domain of positive physical values of the energy  $E' > \mu$ . To obtain the dispersion relations (57.11) in the final form, we must also fix the rate of growth  $n + 1$ .

Let us suppose that  $n = 2$ , i.e., we assume that the rate of growth of the amplitudes  $T_{(k)}^{(i)}$  corresponds to the situation where the asymptotic behavior of the scattering amplitude at high energies is such that

$$\frac{T_{(k)}^{(i)}(E)}{E^2} \rightarrow 0, \tag{57.31}$$

where the approach to zero takes place in a power-law fashion. Condition (57.31) can be obtained [see Jin and Martin (1964); Logunov, Nguyen Van Hieu, and Khrustalev (1969)] if we assume that the dispersion relations are valid for finite  $n$  and fixed  $t$  in some range  $0 > t > t_{\min}$ . The corresponding experimental data on total cross sections show that these



cross sections vary only slightly at high energies and, apparently, tend to constant limiting values, which is not inconsistent with (57.31).

When  $n = 2$ , the dispersion relations given by (57.11) require two subtractions. These subtractions are usually performed at  $E = \mu$ , i.e., on the physical threshold. The subtraction is convenient because the subtraction constant, which is then introduced, can be expressed in terms of the threshold values of the scattering amplitude which, in turn, can be reduced to the  $s$ - and  $p$ -wave scattering lengths.

Substituting  $n = 2$  and  $E_0 = \mu$  in (57.11), and using (57.29) and (57.26), we obtain

$$D_{\text{even}}(E) - D_{\text{even}}(\mu) = \frac{E^2 - \mu^2}{\pi} \int_0^\infty \frac{A_{\text{odd}}(E') dE'^2}{(E'^2 - E^2)(E'^2 - \mu^2)}, \quad (57.32)$$

$$D_{\text{odd}}(E) - \frac{E}{\mu} D_{\text{odd}}(\mu) = 2E \frac{E^2 - \mu^2}{\pi} \int_0^\infty \frac{A_{\text{even}}(E') dE'}{(E'^2 - E^2)(E'^2 - \mu^2)}. \quad (57.33)$$

*57.4. Physical Dispersion Relations.* We now proceed to particular functions  $T_{(k)}^{(i)}$ , and take into account the explicit form of the one-nucleon contributions (57.26). Using (57.22) and (57.33), we obtain the following expressions for the scattering amplitudes without spin flip:

$$D_{(1)}^{(1)}(E) - D_{(1)}^{(1)}(\mu) = \frac{g^2}{2\pi} \frac{E_1^2 (M - E_1) q^2}{M^2 (E^2 - E_1^2) (\mu^2 - E_1^2)} + \frac{q^2}{\pi} \int_\mu^\infty \frac{A_{(1)}^{(1)}(E') dE'^2}{(E'^2 - E^2) q'^2}, \quad (57.34)$$

$$D_{(1)}^{(2)}(E) - \frac{E}{\mu} D_{(1)}^{(2)}(\mu) = -\frac{g^2}{2\pi} \frac{E_1 (M - E_1) q^2 E}{M^2 (E^2 - E_1^2) (\mu^2 - E_1^2)} + \frac{2Eq^2}{\pi} \int_\mu^\infty \frac{A_{(1)}^{(2)}(E') dE'}{(E'^2 - E^2) q'^2}. \quad (57.35)$$

We now express  $T_{(1)}^{(i)}$  in terms of the amplitudes for the scattering of charged pions by protons:

$$T_{\pm} \equiv T(\pi^\pm p \rightarrow \pi^\pm p)$$

and use the relations (see Appendix I, equation A1C.11)

$$T_{(1)}^{(1)} = \frac{T_+ + T_-}{2}, \quad T_{(1)}^{(2)} = \frac{T_- - T_+}{2} \quad (57.36)$$

and the optical theorem (52.19)

$$\text{Im } T_{\pm}(E) = \frac{q}{4\pi} \sigma_{\pm}(E), \quad q = \sqrt{E^2 - \mu^2}.$$

The result of this is that we obtain, instead of (57.34) and (57.35), the physical dispersion relations for the scattering of charged pions by protons:

$$D_{\pm}(E) = \frac{1}{2} \left(1 + \frac{E}{\mu}\right) D_{\pm}(\mu) + \frac{1}{2} \left(1 - \frac{E}{\mu}\right) D_{\mp}(\mu) + \\ + \frac{2f^2}{\mu^2} \frac{q^2}{E_{\mp} \frac{\mu^2}{2M}} + \frac{q^2}{4\pi^2} \int_{\mu}^{\infty} \frac{dE'}{q'} \left[ \frac{\sigma_{\pm}(E')}{E' - E} + \frac{\sigma_{\mp}(E')}{E' + E} \right]. \quad (57.37)$$

In the single-nucleon term, we have transformed to the new coupling constant

$$f^2 = \frac{1}{4\pi} \left( \frac{g\mu}{2M} \right)^2 \quad (57.38)$$

and have neglected small terms of order  $\mu^2/M^2 \simeq 0.02$ . The constant  $f$  is known as the vector pion-nucleon interaction constant. The dispersion relations for amplitudes with spin flip can be written out in an analogous fashion. The dispersion relations given by (57.37) are, however, of particular interest. They contain quantities that are directly observable, namely, the total cross sections for the scattering of charged pions by protons and the real parts of the elastic amplitudes for the zero-angle scattering of charged pions by protons. The total cross sections  $\sigma_{\pm}(E)$  can be observed directly and, to determine  $D_{\pm}(E)$ , one must use data on elastic differential forward-scattering cross sections:

$$\left. \frac{d\sigma_{\pm}^{el}}{d\Omega} \right|_{\theta=0} = |T_{\pm}(E)|^2 = D_{\pm}^2(E) + A_{\pm}^2(E) = D_{\pm}^2(E) + \frac{q^2}{16\pi^2} \sigma_{\pm}^2(E). \quad (57.39)$$

The sign of  $D_{\pm}$  is determined by interference with Coulomb scattering. The threshold values of the amplitudes  $D_{\pm}(\mu)$  can be expressed in terms of the  $s$ -wave pion-nucleon scattering lengths. The only free parameter in (57.37) is, therefore,  $f^2$ . The verification of the dispersion relations (57.37) therefore consists in establishing whether there exists a value of  $f^2$  for which the left- and right-hand sides of both relations in (57.37) are equal. This procedure will also yield the numerical value of  $f^2$  (and  $g^2$ ).

The results of verifications of this kind [see, for example, Shirkov, Serebryakov, and Meshcheryakov (1967), §4] show that these dispersion relations are in good agreement with experimental data in a broad energy range for  $f^2 = 0.08$  ( $g^2/4\pi = 14.7$ ). The general principles of the local quantum field theory, which lie at the basis of the derivation of the dispersion relations, have thus received direct experimental confirmation.

**57.5. Further Development of the Method.** We conclude our presentation with a brief review of further developments in the method of dispersion relations and its more important applications.

Soon after the middle 1950's, the method of dispersion relations assumed a very important position in the theory of interaction between particles and, especially, in the theory of strong interactions. We shall now list some of the more important rigorous results obtained by this method.

Soon after the rigorous derivation of the dispersion relations for pion-nucleon scattering [Bogolyubov (1956), Symanzik (1957), Bogolyubov, Medvedev, and Polivanov (1957)], they were proved for Compton scattering by nucleons [Bogolyubov and Shirkov (1957),

for the photoproduction of pions or nucleons [Logunov and Stepanov (1956), Logunov, Tavkhelidze, and Sovol'yev (1957), Chew, Goldberger, Low and Nambu (1957)], for several virtual scattering processes, i.e., when one or two of the four particles were not on the mass surface [Vladimirov and Logunov (1959), Oehme and Taylor (1959), Logunov and Sovol'yev (1958, 1959)], and for some other inelastic processes of the form 2 particles  $\rightarrow$  3 particles [Logunov and Tavkhelidze (1958), Logunov, Bilen'kiy, and Tafkhelidze (1958)].

The next important group of results involves the establishment of rigorous limitations on the asymptotic behavior of the scattering amplitudes at high energies. The first important step in this direction was made by Froissart (1961) who, starting with hypothetical dispersion relations in the square of transferred momentum  $T$ , showed that the total cross section could not increase faster than  $\ln^2 E$  with increasing energy. Greenberg and Low (1961) showed that, to obtain this result, it is sufficient to use the analyticity of the amplitude in  $\cos \theta$  within an ellipse, the semimajor axis of which is determined by the nearest singularity in the amplitude at  $t = 4\mu^2$ . The appropriate analyticity properties were then established for the amplitude by Martin (1963), who essentially used the unitarity condition. Froissart's result was subsequently improved somewhat, and its modern form [Sing and Roy (1970)] is

$$\sigma_{\text{tot}}(s) \leq \frac{\pi}{\mu^2} \ln \frac{s}{\sigma_{\text{el}}(s)}, \quad (57.40)$$

where  $\mu$  is the mass of the particle transporting the interaction in the cross  $t$ -channel (the mass of the pion). This result is physically important because it explicitly reflects the influence of the properties of the cross channel on the asymptotic behavior of the direct channel, and provides an estimate which is relatively close to experimental data.

Dispersion relations were at the basis of the well-known result obtained by Pomeranchuk (1958), according to which the total cross sections for the scattering of particles and antiparticles by the same target are equal (this is the *Pomeranchuk theorem*):

$$\sigma(\infty) = \tilde{\sigma}(\infty). \quad (57.41)$$

Some improvements and simple derivations of the Pomeranchuk theorem, based on the Lindelöf theorem in the theory of analytic functions, were published by Sugawara and Kanazawa (1961) and Meiman (1962). The Pomeranchuk theorem (57.41) was then generalized by Logunov, Nguyen Van Hieu, Todorov, and Khrustalev (1963, 1965) and Van Hove (1963) [see also Logunov, Nguyen Van Hieu and Todorov (1966)] to the case of elastic differential cross sections. The following asymptotic result was established:

$$\lim_{s \rightarrow \infty} \frac{\frac{d\sigma}{d\Omega}(s, t)}{\frac{d\tilde{\sigma}}{d\Omega}(s, t)} = 1 \quad (57.42)$$

for a fixed square of transferred momentum  $t$ .

The set of axioms (§52.2) used in the proof of the dispersion relations has turned out to be of broader significance. Medvedev, Pavlov, Polivanov, and Sukhanov (1972) and others [see also the review by Medvedev and Polivanov (1964)] have shown that it is very convenient for the systematic development of quantum field theory.

Finally, we note some of the more important results obtained with the aid of the method of dispersion relations in theoretical schemes of semiphenomenologic character.

The application of the dispersion method to weak vertices has led to the well-known Goldberger-Treiman (1958) relation. It has also been used to relate hyperon decay probabilities with the pion-nucleon scattering phases [Okubo, Marshak, and Sudarshan (1959), Todorov and Khrustalev (1959)].

By combining the dispersion relations for fixed momentum transfer with the hypothesis of the Regge-type asymptotic behavior, Logunov, Solov'yev, and Tavkhelidze (1967) and Igi and Matsuda (1967) succeeded in obtaining the so-called *finite-energy sum rules*, explicitly reflecting the connection between the scattering properties of direct and cross channels. These sum rules subsequently led to the idea of *duality*. The dispersion sum rules have been used widely in the formulation of the results obtained on the basis of the hypothesis of current algebra [see, for example, the book by Collins and Squires (1968)] and in the soft-pion approximation (the Adler sum rule) [see, for example, the book by Adler and Dashen (1968)].

# APPENDIX I

## ELEMENTS OF ISOSPIN FORMALISM

The concept of isospin was first introduced by Heisenberg in 1932 in order to describe the neutron and the proton as two different (charge) states of the same particle, i.e., the nucleon. The nucleon wave function is then assigned the properties of a two-component spinor under rotational transformations in a certain fictitious three-dimensional space, namely, the isospin space. Analogous properties are assigned to the wave functions of the other charge multiplets. For example, the pion triplet  $\pi^+$ ,  $\pi^0$ ,  $\pi^-$  is described by three functions that are the components of a vector in the isospin space, and so on.

Invariance under rotations in isospin space correspond physically to the equivalence of particles within each multiplet. It is clear that this equivalence is violated by the electromagnetic interaction. However, if we ignore the latter, the property of isospin invariance is found to be valid for strong (nuclear) interactions.

We shall now give a very brief review of the basic formulas of the isospin formalism for nucleons, pions, and the pion-nucleon scattering amplitude. A more detailed discussion can be found in Chapter 3 of the monograph by Nishijima (1964).

*A. The Nucleon Doublet.* Consider, to begin with, the nucleon doublet. In accordance with the foregoing (§7.1), a spinor in three-dimensional isospace has two components corresponding to the proton  $\Psi_p$  and neutron  $\Psi_n$  wave functions:

$$\Psi = \begin{pmatrix} \psi_p \\ \psi_n \end{pmatrix}, \quad \Psi^* = (\psi_p^*, \psi_n^*). \quad (\text{A1A.1})$$

Operators corresponding to rotations in isospin space are constructed from matrices

of rank two. They are identical with the Pauli matrices (7.10) but are represented by different symbols:

$$\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (\text{A1A.2})$$

The matrices (A1A.2) anticommute with one another:

$$\tau_\alpha \tau_\beta + \tau_\beta \tau_\alpha = 0, \quad \alpha \neq \beta; \quad \tau_\alpha^2 = 1 \quad (\text{A1A.3})$$

and satisfy the relations

$$\tau_1 \tau_2 = i \tau_3 \quad (\text{and cyclic transposition of } 1, 2, 3). \quad (\text{A1A.4})$$

The operators for isospin rotations of nucleons are related to the matrices  $\tau$ , as follows:

$$I_\alpha = \frac{\tau_\alpha}{2}. \quad (\text{A1A.5})$$

By virtue of (A1A.3) and (A1A.4), they satisfy the commutation relations for the rotation operators

$$[I_1, I_2] = i I_3 \quad (\text{and cyclic transposition of } 1, 2, 3), \quad (\text{A1A.6})$$

and have the property

$$I^2 = I_1^2 + I_2^2 + I_3^2 = 3/4, \quad (\text{A1A.7})$$

so that the eigenvalues of the resultant isospin  $I$  given by the quantum-mechanical relation

$$I^2 = I(I+1), \quad (\text{A1A.8})$$

turn out to be  $I = 1/2$ . The linear combinations of the matrices  $\tau_1$  and  $\tau_2$

$$\tau_+ = \tau_1 + i\tau_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \tau_- = \tau_1 - i\tau_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad (\text{A1A.9})$$

transform a neutron state into a proton state, and vice versa:

$$\tau_+ \begin{pmatrix} 0 \\ \psi \end{pmatrix} = \begin{pmatrix} \psi \\ 0 \end{pmatrix}, \quad \tau_- \begin{pmatrix} \psi \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \psi \end{pmatrix}, \quad (\text{A1A.10})$$

so that  $\tau_+$  and  $\tau_-$  are occasionally referred to as charge increasing and reducing operators. The transformation formulas for the spinor  $\Psi$  under rotation in isospin space around the  $z_\alpha$  axis through an angle  $\varphi$  have the form analogous to (6.31):

$$\Psi' = \Lambda_a(\varphi) \Psi, \quad \bar{\Psi}' = \Lambda_a^{-1}(\varphi) \bar{\Psi}, \quad (\text{A1A.11})$$

$$\Lambda_a(\varphi) = e^{-i\tau_a \frac{\varphi}{2}} = \cos \frac{\varphi}{2} - i\tau_a \sin \frac{\varphi}{2}. \quad (\text{A1A.12})$$

The bilinear forms  $\bar{\Psi}\tau_\alpha\Psi$  ( $\alpha = 1, 2, 3$ ) form the spatial density of the isospin vector

$$I = \int \bar{\Psi}(x) \frac{\tau}{2} \Psi(x) dx, \quad (\text{A1A.13})$$

the third component of which is related by means of (2.21) to the electric charge and the hypercharge:

$$Q = I_3 = \frac{B+S}{2} = I_3 + \frac{Y}{2}. \quad (\text{A1A.14})$$

*B. The Pion Triplet.* The triple of pseudoscalar pions  $\pi^+$ ,  $\pi^0$ ,  $\pi^-$  forms an isospin triplet that can be described by three pseudoscalar field functions forming an isospin vector. Two representations of these functions are commonly used. In the first, the field functions are real:

$$\pi(x) = \{\pi_1(x), \pi_2(x), \pi_3(x)\}, \quad \pi_\alpha^*(x) = \pi_\alpha(x). \quad (\text{A1B.1})$$

In the second, the two real functions  $\pi_1, \pi_2$  are used to form the complex combinations

$$\varphi_1 = \frac{-\pi_1 + i\pi_2}{\sqrt{2}}, \quad \varphi_2 = \pi_3, \quad \varphi_3 = \frac{\pi_1 + i\pi_2}{\sqrt{2}}. \quad (\text{A1B.2})$$

The transformation (A1B.2) of the representation

$$\Psi = \{\varphi_1, \varphi_2, \varphi_3\} \quad (\text{A1B.3})$$

can be written in the matrix form

$$\varphi = O\pi; \quad O = \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 \end{pmatrix}, \quad (\text{A1B.4})$$

where  $O$  is the unitary matrix

$${}^{\dagger}OO = 1. \quad (\text{A1B.5})$$

The transformation formulas for the isospin vector (A1B.1) under rotations of three-dimensional isospin space have the form

$$\pi \rightarrow \pi' = \Lambda(\alpha) \pi, \quad \Lambda(\alpha) = e^{-i\omega_v \alpha_v}, \quad (\text{A1B.6})$$

where  $\alpha_v$  ( $v = 1, 2, 3$ ) are the angles of rotation and  $\omega_v$  are the isospin matrices in the representation (A1B.1)

$$\omega_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \omega_2 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, \quad \omega_3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (\text{A1B.7})$$

The square of the matrix vector  $\omega$  is given by

$$\omega(\omega + 1) = \omega_1^2 + \omega_2^2 + \omega_3^2 = 2, \quad (\text{A1B.8})$$

so that  $\omega$  is the angular momentum operator with eigenvalues 1, 0, -1.

The operator corresponding to rotation through an angle  $\alpha$  about the  $z_3$  axis will be written in the form

$$\Lambda(\alpha) = e^{-i\omega_3 \alpha} = 1 - \omega_3^2 + \omega_3^2 \cos \alpha - i\omega_3 \sin \alpha = \begin{pmatrix} \cos \alpha & -\sin \alpha & 0 \\ \sin \alpha & \cos \alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (\text{A1B.9})$$

The components of the matrices (A1B.7) can be expressed in terms of a completely antisymmetric unit tensor of rank three

$$(\omega_\alpha)_{\beta\gamma} = -ie_{\alpha\beta\gamma}. \quad (\text{A1B.10})$$

The matrices  $\omega_\alpha$  satisfy the commutation relations

$$[\omega_1, \omega_2] = i\omega_3 \quad (\text{and cyclic transposition of } 1, 2, 3). \quad (\text{A1B.11})$$

The transition to the representation given by (A1B.3) is achieved by the unitary transformation

$$T_v = O\omega_v \dot{O}. \quad (\text{A1B.12})$$

The matrices  $T_v$  have the form

$$T_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad T_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad T_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad (\text{A1B.13})$$



and satisfy the commutation relations given by (A1B.11). An important point is that the matrix  $T_3$  is diagonal. This property reflects the fact that, in the representation given by (A1B.3), the components  $\varphi_\nu$  correspond to definite values of the electric charge. On the basis of (A1A.14), we make the following identifications:

$$\varphi_1 \sim \pi^+, \quad \varphi_2 \sim \pi^0, \quad \varphi_3 \sim \pi^-. \quad (\text{A1B.14})$$

*C. Pion-Nucleon Scattering Amplitude.* Invariance under isospin rotations shows that the pion-nucleon scattering amplitude has the following isotopic structure (see §§56 and 57 for notation):

$$T_{\alpha\omega} = \delta_{t't}\delta_{\rho'\rho} T^{(1)} + \frac{1}{2} \{ \tau_{\rho'} \tau_{\rho} - \tau_{\rho} \tau_{\rho'} \}_{t't} T^{(2)}, \quad (\text{A1C.1})$$

where  $\tau_\rho$  are the nucleon isospin matrices (A1A.2),  $t', t$  are the isospin indices of the nucleons, and  $\rho, \rho'$  are the isospin indices of the pions.

(A1C.1) can also be written in the form

$$T_{\alpha\omega} = \delta_{t't}\delta_{\rho'\rho} T^{(1)} - \omega_{\rho'\rho} \tau_{t't} T^{(2)}, \quad (\text{A1C.2})$$

where

$$(\omega_\alpha)_{\rho\rho'} = -ie_{\alpha\rho\rho'}$$

is the isospin operator for pions introduced in (A1B.10).

To determine the possible values of the scalar product  $\omega \cdot \tau$ , we consider the resultant isospin operator of the pion-nucleon system

$$\Omega = \omega + \frac{\tau}{2}. \quad (\text{A1C.3})$$

The square of this operator can be expressed in terms of the resultant isospin  $T$ :

$$\Omega^2 = T(T+1). \quad (\text{A1C.4})$$

From (A1C.3) and (A1C.4), and using (A1A.7) and (A1A.8), we obtain

$$\omega\tau = \Omega^2 - \omega^2 - \frac{\tau^2}{4} = T(T+1) - 2 - \frac{3}{4} = T(T+1) - \frac{11}{4}. \quad (\text{A1C.5})$$

Hence, it follows that the eigenvalue of the operator  $\omega \cdot \tau$  in the state with resultant isospin of the pion-nucleon system  $T = 2/3$  is

$$\frac{3}{2} \left( \frac{3}{2} + 1 \right) - \frac{11}{4} = 1,$$

and, in the state with  $T = \frac{1}{2}$ , the eigenvalue is

$$\frac{1}{2} \left( \frac{1}{2} + 1 \right) - \frac{11}{4} = -2.$$

We now introduce the amplitudes  $T_{3/2}$  and  $T_{1/2}$ , which refer to states with isospin  $3/2$  and  $1/2$ , and have from (A1C.2)

$$T_{3/2} = T^{(1)} - T^{(2)}, \quad T_{1/2} = T^{(1)} + 2T^{(2)}. \quad (\text{A1C.6})$$

The inverse relations are

$$T^{(1)} = \frac{T_{1/2} + 2T_{3/2}}{3}, \quad T^{(2)} = \frac{T_{1/2} - T_{3/2}}{3}. \quad (\text{A1C.7})$$

The scattering amplitudes for charged pions on protons

$$T(\pi^{\pm} + p \rightarrow \pi^{\pm} + p) \equiv T_{\pm} \quad (\text{A1C.8})$$

are related to the amplitudes (A1C.6) by

$$T_{+} = T_{3/2}, \quad T_{-} = \frac{T_{3/2} + 2T_{1/2}}{3} \quad (\text{A1C.9})$$

so that

$$T_{+} = T^{(1)} - T^{(2)}, \quad T_{-} = T^{(1)} + T^{(2)}, \quad (\text{A1C.10})$$

and

$$T^{(1)} = \frac{T_{+} + T_{-}}{2}, \quad T^{(2)} = \frac{T_{-} - T_{+}}{2}. \quad (\text{A1C.11})$$

# APPENDIX II

## LIST OF SINGULAR FUNCTIONS

*A. Auxiliary Singular Functions. The one-dimensional Dirac  $\delta$ -function:*

$$\delta(\alpha) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{i\alpha\tau} d\tau. \quad (\text{A2A.1})$$

*The four-dimensional Dirac  $\delta$ -function:*

$$\delta(x) = \frac{1}{(2\pi)^4} \int_{-\infty}^{+\infty} e^{ikx} dk \quad (\text{A2A.2})$$

$$(kx \equiv k^0 x^0 - \mathbf{k} \cdot \mathbf{x}, \quad dk = dk^0 d\mathbf{k}),$$

$$\delta(x) = \delta(x^0) \delta(\mathbf{x}) = \delta(x^0) \delta(x^1) \delta(x^2) \delta(x^3).$$

*The step functions  $\theta(\alpha)$  and  $\varepsilon(\alpha)$ :*

$$\theta(\alpha) = \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{e^{i\alpha\tau}}{\tau - i\varepsilon} d\tau = \begin{cases} 1 & \text{for } \alpha > 0, \\ 0 & \text{for } \alpha < 0, \end{cases} \quad (\text{A2A.3})$$

$$\varepsilon(\alpha) = \frac{1}{\pi i} \mathcal{P} \int_{-\infty}^{+\infty} e^{i\alpha\tau} \frac{d\tau}{\tau} = \theta(\alpha) - \theta(-\alpha) = \begin{cases} 1 & \text{for } \alpha > 0, \\ -1 & \text{for } \alpha < 0 \end{cases} \quad (\text{A2A.4})$$

( $\mathcal{P}$  — denotes the principal value).

The positive- and negative-frequency parts of the  $\delta$ -function:

$$\delta_{\pm}(\alpha) = \frac{1}{2\pi} \int_0^{\infty} e^{\pm i\alpha\tau} d\tau = \frac{1}{2} \left( \delta(\alpha) \pm \frac{i}{\pi} \mathcal{P} \frac{1}{\alpha} \right). \quad (\text{A2A.5})$$

Some useful relations:

$$\frac{1}{\alpha + i\varepsilon} = \frac{2\pi}{i} \delta_+(\alpha) = \frac{\pi}{i} \delta(\alpha) + \mathcal{P} \frac{1}{\alpha} = -i \int_0^{\infty} e^{i\alpha t} dt, \quad (\text{A2A.6})$$

$$\frac{1}{\alpha - i\varepsilon} = 2\pi i \delta_-(\alpha) = \pi i \delta(\alpha) + \mathcal{P} \frac{1}{\alpha} = i \int_0^{\infty} e^{-i\alpha t} dt, \quad (\text{A2A.7})$$

$$\mathcal{P} \frac{1}{\tau} = \frac{1}{2i} \int_{-\infty}^{+\infty} \varepsilon(\alpha) e^{i\alpha\tau} d\alpha. \quad (\text{A2A.8})$$

B. Scalar Field Functions. The Pauli-Jordan commutation function  $D(x)$ :

$$[\varphi(x), \varphi(y)]_- = \frac{1}{i} D(x - y); \quad (\text{11.3})$$

$$(\square - m^2) D(x) = 0;$$

$$D(x) = \frac{i}{(2\pi)^3} \int e^{-ikx} \varepsilon(k^0) \delta(k^2 - m^2) dk = \frac{1}{(2\pi)^3} \int \frac{dk}{\sqrt{k^2 + m^2}} e^{ikx} \sin(x^0 \sqrt{k^2 + m^2}), \quad (\text{10.18})$$

$$\left. \frac{\partial D(x^0, \mathbf{x})}{\partial x^0} \right|_{x^0=0} = \delta(\mathbf{x}). \quad (\text{A2B.1})$$

$$D(x) = \frac{1}{2\pi} \varepsilon(x^0) \delta(\lambda) - \frac{m}{4\pi \sqrt{\lambda}} \varepsilon(x^0) \theta(\lambda) J_1(m \sqrt{\lambda}) \quad (\text{16.12})$$

$$(\lambda = x^2 \equiv (x^0)^2 - \mathbf{x}^2).$$

In the neighborhood of the light cone  $D(x)$  has the form

$$D(x) \approx \frac{1}{2\pi} \varepsilon(x^0) \delta(\lambda) - \frac{m^2}{8\pi} \varepsilon(x^0) \theta(\lambda). \quad (\text{16.17})$$

The positive- and negative-frequency parts of the Pauli-Jordan function  $D^{(+)}(x)$  and  $D^{(-)}(x)$ :

$$[\varphi^-(x), \varphi^+(y)]_- = \langle \varphi(x) \varphi(y) \rangle_0 = \frac{1}{i} D^-(x - y),$$

$$[\varphi^+(x), \varphi^-(y)]_- = \frac{1}{i} D^+(x - y) = i D^-(y - x); \quad (\text{11.2})$$

$$(\square - m^2) D^-(x) = 0, \quad (\square - m^2) D^+(x) = 0;$$

$$D^+(x) = \frac{1}{(2\pi)^3 i} \int e^{ikx} \theta(k^0) \delta(k^2 - m^2) dk = \frac{1}{(2\pi)^3 i} \int \frac{dk}{2\sqrt{k^2 + m^2}} e^{ix^0 \sqrt{k^2 + m^2} - i\mathbf{x}\mathbf{k}}, \quad (16.1)$$

$$D^-(x) = \frac{i}{(2\pi)^3} \int e^{ikx} \theta(-k^0) \delta(k^2 - m^2) dk = \\ = \frac{i}{(2\pi)^3} \int \frac{dk}{2\sqrt{k^2 + m^2}} e^{-ix^0 \sqrt{k^2 + m^2} + i\mathbf{x}\mathbf{k}} = -D^+(-x); \quad (16.2)$$

$$D^+(x) = \frac{1}{4\pi} \varepsilon(x^0) \delta(\lambda) - \frac{mi}{8\pi\sqrt{\lambda}} \theta(\lambda) [N_1(m\sqrt{\lambda}) - i\varepsilon(x^0) J_1(m\sqrt{\lambda})] - \\ - \frac{mi}{4\pi^2\sqrt{-\lambda}} \theta(-\lambda) K_1(m\sqrt{-\lambda}), \quad (16.10)$$

$$D^-(x) = \frac{1}{4\pi} \varepsilon(x^0) \delta(\lambda) + \frac{mi}{8\pi\sqrt{\lambda}} \theta(\lambda) [N_1(m\sqrt{\lambda}) + i\varepsilon(x^0) J_1(m\sqrt{\lambda})] + \\ + \frac{mi}{4\pi^2\sqrt{-\lambda}} \theta(-\lambda) K_1(m\sqrt{-\lambda}). \quad (16.11)$$

In the neighborhood of the light cone,  $D^{(+)}$  and  $D^{(-)}$  have the form

$$\left. \begin{aligned} D^+(x) &\approx \frac{1}{4\pi} \varepsilon(x^0) \delta(\lambda) + \frac{i}{4\pi^2\lambda} - \frac{im^2}{8\pi^2} \ln \frac{m|\lambda|^{1/2}}{2} - \frac{m^2}{16\pi} \varepsilon(x^0) \theta(\lambda), \\ D^-(x) &\approx \frac{1}{4\pi} \varepsilon(x^0) \delta(\lambda) - \frac{i}{4\pi^2\lambda} + \frac{im^2}{8\pi^2} \ln \frac{m|\lambda|^{1/2}}{2} - \frac{m^2}{16\pi} \varepsilon(x^0) \theta(\lambda). \end{aligned} \right\} \quad (16.16)$$

Even solution of homogeneous equation

$$D^1(x) = \frac{1}{(2\pi)^3} \int e^{-ikx} \delta(k^2 - m^2) dk, \quad (A2B.2)$$

$$(\square - m^2) D^1(x) = 0,$$

$$D^1(x) = i(D^+(x) - D^-(x)), \quad (A2B.3)$$

$$D^1(x) = \frac{1}{(2\pi)^3} \int \frac{dk}{\sqrt{k^2 + m^2}} e^{ikx} \cos(x^0 \sqrt{k^2 + m^2}), \quad (A2B.4)$$

$$D^1(x) = \frac{m}{4\pi\sqrt{\lambda}} \theta(\lambda) N_1(m\sqrt{\lambda}) + \\ + \frac{m}{2\pi^2\sqrt{-\lambda}} \theta(-\lambda) K_1(m\sqrt{-\lambda}) \approx -\frac{1}{2\pi^2\lambda} + \frac{m^2}{4\pi^2} \ln \frac{m|\lambda|^{1/2}}{2} + \dots \quad (A2B.5)$$

The causal Green's functions  $D^c(x)$ :

$$\langle T(\varphi(x) \varphi(y)) \rangle_0 = \frac{1}{i} D^c(x - y); \quad (15.15)$$

$$(\square - m^2) D^c(x) = -\delta(x); \\ D^c(x) = \frac{1}{(2\pi)^4} \int e^{-ikx} D^c(k) dk, \quad D^c(k) = \frac{1}{m^2 - k^2 - i\varepsilon}; \quad (15.13)$$

$$D^c(x) = \frac{1}{4\pi} \delta(\lambda) - \frac{m}{8\pi\sqrt{\lambda}} \theta(\lambda) [J_1(m\sqrt{\lambda}) - iN_1(m\sqrt{\lambda})] + \\ + \frac{mi}{4\pi^2\sqrt{-\lambda}} \theta(-\lambda) K_1(m\sqrt{-\lambda}).$$

The behavior of  $D^c(x)$  in the neighborhood of the light cone:

$$D^c(x) \approx \frac{1}{4\pi} \delta(\lambda) + \frac{1}{4\pi^2 i \lambda} + \frac{im^2}{8\pi^2} \ln \frac{m|\lambda|^{1/2}}{2} - \frac{m^2}{16\pi} \theta(\lambda). \quad (16.18)$$

Causal function for zero mass

$$D_0^c(x) = D^c(x)|_{m=0} = \frac{1}{4\pi} \left( \delta(\lambda) - \frac{i}{\pi\lambda} \right) = \frac{1}{4\pi} \delta_+(-\lambda) \equiv \frac{1}{4\pi^2} \int_0^\infty e^{ix(-\lambda)} dx. \quad (A2B.6)$$

The retarded and the advanced Green's functions  $D^{\text{ret}}$  and  $D^{\text{adv}}$ :

$$D^{\text{ret}}(x) = 0 \quad \text{for } x^0 < 0; \quad D^{\text{adv}}(x) = 0 \quad \text{for } x^0 > 0; \\ (\square - m^2) D^{\text{ret}}(x) = -\delta(x); \quad (\square - m^2) D^{\text{adv}}(x) = -\delta(x); \\ D^{\text{ret}}(x) = \frac{1}{(2\pi)^4} \int \frac{e^{-ikx}}{m^2 - k^2 - i\epsilon k^0} dk, \quad (15.7)$$

$$D^{\text{adv}}(x) = \frac{1}{(2\pi)^4} \int \frac{e^{-ikx} dk}{m^2 - k^2 + i\epsilon k^0}, \quad (15.9)$$

$$D^{\text{ret}}(x) = \frac{1}{2\pi} \theta(x^0) \left\{ \delta(\lambda) - \theta(\lambda) \frac{m}{2\sqrt{\lambda}} J_1(m\sqrt{\lambda}) \right\}, \quad (16.15)$$

$$D^{\text{adv}}(x) = \frac{1}{2\pi} \theta(-x^0) \left\{ \delta(\lambda) - \theta(\lambda) \frac{m}{2\sqrt{\lambda}} J_1(m\sqrt{\lambda}) \right\}.$$

Even solution of the homogeneous equation

$$D^s(x) = \frac{1}{(2\pi)^4} \mathcal{P} \int \frac{e^{ikx}}{m^2 - k^2} dk, \quad (A2B.7)$$

$$(\square - m^2) D^s(x) = -\delta(x), \\ D^s(x) = \frac{1}{2\pi} \delta(\lambda) - \theta(\lambda) \frac{m}{2\sqrt{\lambda}} J_1(m\sqrt{\lambda}). \quad (A2B.8)$$

Relation between the functions  $D$ ,  $D^+$ ,  $D^-$ ,  $D'$ ,  $D^c$ ,  $D^{\text{adv}}$ ,  $D^{\text{ret}}$ ,  $D^s$ :

$$\begin{aligned} D(x) &= D^+(x) + D^-(x), \\ D^1(x) &= i(D^+(x) - D^-(x)), \end{aligned} \quad (\text{A2B.9})$$

$$D^c(x) = \theta(x^0) D^-(x) - \theta(-x^0) D^+(x), \quad (15.12)$$

$$D^{\text{ret}}(x) = \theta(x^0) D(x) = D^c(x) + D^+(x), \quad (15.8)$$

$$\begin{aligned} D^{\text{adv}}(x) &= -\theta(-x^0) D(x) = D^c(x) - D^-(x), \\ D(x) &= D^{\text{ret}}(x) - D^{\text{adv}}(x), \\ D^s(x) &= \frac{1}{2} [D^{\text{ret}}(x) + D^{\text{adv}}(x)] = \frac{\varepsilon(x^0)}{2} D(x). \end{aligned} \quad (\text{A2B.10})$$

*C. Singular Functions of the Electromagnetic, Vector and Spinor Fields. The electromagnetic field:*

$$[A_m(x), A_n(y)]_- = i g^{mn} D_0(x-y); \quad (12.4)$$

$$\begin{aligned} \langle A_m(x) A_n(y) \rangle_0 &= i g^{mn} D_0^-(x-y); \\ \langle T(A_m(x) A_n(y)) \rangle_0 &= i g^{mn} D_0^c(x-y) \\ (g^{mn} = 0 \quad \text{for } m \neq n; \quad g^{00} = -g^{11} = -g^{22} = -g^{33} = 1). \end{aligned} \quad (24.3)$$

The functions  $D_0$ ,  $D_0^{(-)}$ ,  $D_0^c$ , etc., are obtained from the functions  $D$ ,  $D^{(-)}$ ,  $D^c$  of the scalar field for  $m = 0$ , for example:

$$D_0^c(x) = D^c(x)|_{m=0} = -\frac{1}{(2\pi)^4} \int \frac{e^{-ikx}}{k^2 + i\varepsilon} dk. \quad (15.16)$$

*The vector field:*

$$[U_m(x), U_n(y)]_- = i D_{mn}(x-y); \quad (11.27)$$

$$\begin{aligned} \langle U_m(x) U_n(y) \rangle_0 &= i D_{mn}^-(x-y); \\ \langle T(U_m(x) U_n(y)) \rangle &= i D_{mn}^c(x-y). \end{aligned} \quad (15.19)$$

The functions  $D_{mn}$ ,  $D_{mn}^{(-)}$ ,  $D_{mn}^c$ , etc., may be obtained from the corresponding functions of the scalar field by applying the differential operator

$$g^{mn} + \frac{1}{m^2} \frac{\partial^2}{\partial x^n \partial x^m},$$

for example:

$$D_{nl}(x) = \left( g^{nl} + \frac{1}{m^2} \frac{\partial^2}{\partial x^n \partial x^l} \right) D(x) = \frac{i}{(2\pi)^3} \int \left( g^{nl} - \frac{k_n k_l}{m^2} \right) e^{-ikx} \delta(k^2 - m^2) \varepsilon(k^0) dk,$$

$$D_{nl}^c(x) = \frac{1}{(2\pi)^4} \int \left( g^{nl} - \frac{k_n k_l}{m^2} \right) \frac{e^{-ikx}}{m^2 - k^2 - i\varepsilon} dk. \quad (15.18)$$

The spinor field:

$$[\psi_\alpha(x), \bar{\psi}_\beta(y)]_+ = \frac{1}{i} S_{\alpha\beta}(x-y); \quad (13.4)$$

$$\langle \psi_\alpha(x) \bar{\psi}_\beta(y) \rangle_0 = \frac{1}{i} S_{\alpha\beta}^-(x-y);$$

$$\langle T(\psi_\alpha(x) \bar{\psi}_\beta(y)) \rangle_0 = \frac{1}{i} S_{\alpha\beta}^c(x-y). \quad (15.21)$$

The functions  $S_{\alpha\beta}$ ,  $S_{\alpha\beta}^{(-)}$ ,  $S_{\alpha\beta}^c$ , etc., may be obtained from the corresponding functions for the scalar field by applying the operator

$$(i\hat{\partial} + m)_{\alpha\beta} \equiv i\gamma_{\alpha\beta}^n \frac{\partial}{\partial x^n} + m\mathcal{I}_{\alpha\beta}$$

(the Dirac matrices  $\gamma$  are here defined by the relations  $\gamma^n \gamma^m + \gamma^m \gamma^n = 2g^{mn}$ ), for example:

$$S^-(x) = \left( i\gamma^n \frac{\partial}{\partial x^n} + m \right) D^-(x) = \frac{i}{(2\pi)^3} \int e^{-ikx} \theta(k^0) (m + \hat{k}) \delta(k^2 - m^2) dk =$$

$$= \frac{i}{(2\pi)^3} \int \frac{dk}{2\sqrt{k^2 + m^2}} (m + \gamma^0 \sqrt{k^2 + m^2} - \gamma k) e^{-ix^0 \sqrt{k^2 + m^2} + i\mathbf{x} \cdot \mathbf{k}}, \quad (A2C.1)$$

$$S(x) = \left( i\gamma^n \frac{\partial}{\partial x^n} + m \right) D(x) = \frac{i}{(2\pi)^3} \int e^{-ikx} \varepsilon(k^0) (\hat{k} + m) \delta(k^2 - m^2) dk = \quad (13.5)$$

$$= \frac{1}{(2\pi)^3} \int dk e^{ikx} \left\{ i\gamma^0 \cos(x^0 \sqrt{k^2 + m^2}) + (\gamma k + m) \frac{\sin(x^0 \sqrt{k^2 + m^2})}{\sqrt{k^2 + m^2}} \right\}, \quad (A2C.2)$$

$$S(x)|_{x^0=0} = i\gamma^0 \delta(x), \quad (A2C.3)$$

$$S^c(x) = \frac{1}{(2\pi)^4} \int \frac{(m + \hat{p})}{m^2 - p^2 - i\varepsilon} e^{-ipx} dp. \quad (15.17), (24.4)$$

*D. Connection Between the Notation for Singular Functions Used in This Book and the Notation of Other Authors.*



Our notation	Schwinger (1948, 1949)	Akhiyezer & Berestetskiy (1969)	Schweber (1961)	Bjorken & Drell (1965)
$D$	$-\Delta$	$\Delta_0$	$-\Delta$	$-\Delta$
$D^-$	$-\Delta^{(+)}$	$i\Delta_+$	$-\Delta^{(+)}$	$i\Delta_+$
$D^+$	$-\Delta^{(-)}$	$-i\Delta_-$	$-\Delta^{(-)}$	$-i\Delta_-$
$D^1$	$\Delta^{(1)}$	$\Delta_1$	$\Delta_1$	$\Delta_1$
$D_0$	$D$	$D_0$	$D$	$D$
$S$	$S$	$iS$	$S$	$S$
$D^c$	$\Delta_+$	$i\Delta_c$	$\frac{i}{2} \Delta_F$	$-\Delta_F$
$D_0^c$	$D_+$	$iD_c$	$\frac{1}{2i} D_F$	$D_F$
$S^c$	$S_+$	$iS_c$	$\frac{1}{2i} S_F$	$-S_F$

# APPENDIX III

## LIST OF FORMULAS FOR THE EVALUATION OF MATRIX ELEMENTS

*A. Formulas Involving the Dirac Matrices and Spinors. Definitions:*

$$\gamma^n \gamma^m + \gamma^m \gamma^n = 2g^{mn} \quad (m, n = 0, 1, 2, 3, 5),$$

$$g^{mn} = 0 \quad (m \neq n), \quad g^{00} = -g^{11} = -g^{22} = -g^{33} = g^{55} = 1, \quad (6.12)$$

$$\gamma^5 = -i\gamma^0\gamma^1\gamma^2\gamma^3, \quad \gamma^{5\dagger} = \gamma^5, \quad (\gamma^5)^2 = 1,$$

$$\sigma^{mn} = i \frac{\gamma^m \gamma^n - \gamma^n \gamma^m}{2}. \quad (7.30)$$

The explicit form in the representation in which  $\gamma^0$  is diagonal and the connection between the matrices  $\alpha, \beta, \sigma, \rho$  are discussed in § 6.2.

*Commutation properties ( $\hat{a} \equiv a_n \gamma^n$ ):*

$$\hat{a} \hat{b} = 2(ab) - \hat{b} \hat{a},$$

$$\hat{a} \gamma^n = 2a^n - \gamma^n \hat{a} \quad (n \neq 5)$$

$$\hat{a} \gamma^5 = -\gamma^5 \hat{a}.$$

*Traces:*

$$\text{Tr } \gamma^n = 0, \quad \text{Tr } \hat{a} = 0,$$

$$\text{Tr } \gamma^n \gamma^m = 4g^{mn}, \quad \text{Tr } \hat{a} \hat{b} = 4(ab),$$

$$\text{Tr } \gamma^k \gamma^l \gamma^m \gamma^n = 4(g^{kl}g^{mn} + g^{nk}g^{lm} - g^{km}g^{ln}),$$

$$\begin{aligned}\text{Tr } \hat{a}\hat{b}\hat{c} &= 0, \\ \text{Tr } \hat{a}\hat{b}\hat{c}\hat{d} &= \frac{1}{4} ((ab)(cd) + (ad)(bc) - (ac)(bd)).\end{aligned}$$

*Spinor functions:*

$$\begin{aligned}\psi^\pm(x) &= \frac{1}{(2\pi)^{3/2}} \int dp e^{\pm i p x} \sum_{\nu=1,2} a_\nu^\pm(p) v^{\nu,\pm}(p), \\ \bar{\psi}^\pm(x) &= \frac{1}{(2\pi)^{3/2}} \int dp e^{\pm i p x} \sum_{\mu=1,2} \bar{a}_\mu^\pm(p) \bar{v}^{\mu,\pm}(p), \\ \bar{v}^{\mu,\pm}(p) &= \bar{v}^{\mu,\pm}(p) \gamma_0, \quad \bar{v}^{\mu,\pm}(p) = (v^{\mu,\pm}(p))^*, \\ (m + \hat{p}) v^{\nu,+}(p) &= 0, \quad (m - \hat{p}) v^{\nu,-}(p) = 0, \\ \bar{v}^{\nu,+}(p) (\hat{p} - m) &= 0, \quad \bar{v}^{\nu,-}(p) (\hat{p} + m) = 0.\end{aligned}$$

The spinors  $v^{\nu,-}(p)[\bar{v}^{\nu,+}(p)]$  describe an electron in the initial (final) state and the spinors  $\bar{v}^{\nu,-}(p), v^{\nu,+}(p)$  describe the positron. These functions are frequently denoted in the literature as follows:

$$\begin{aligned}v^{\nu,-}(p) &= u(p, \nu), \quad v^{\nu,+}(p) = v(p, \nu), \\ \bar{v}^{\nu,-}(p) &= \bar{v}(p, \nu), \quad \bar{v}^{\nu,+}(p) = \bar{u}(p, \nu).\end{aligned}$$

*Orthonormalization properties:*

$$\bar{v}^{\nu,\pm}(p) v^{\mu,\mp}(p) = \delta^{\nu\mu}, \quad (7.16)$$

$$\bar{v}^{\nu,\pm}(p) v^{\mu,\mp}(p) = \pm \frac{m}{p^0} \delta^{\nu\mu}, \quad (7.17)$$

$$\bar{v}^{\nu,\pm}(p) v^{\mu,\pm}(-p) = 0. \quad (7.18)$$

*Summation over the spin index:*

$$\sum_{\nu} v^{\nu,+}(p) \bar{v}^{\nu,-}(p) = \frac{\hat{p} - m}{2p^0}, \quad (7.20)$$

$$\sum_{\nu} v^{\nu,-}(p) \bar{v}^{\nu,+}(p) = \frac{\hat{p} + m}{2p^0}. \quad (7.21)$$

*B. Evaluation of Feynman Integrals.* Transformation of the  $\alpha$ -representation:

$$\frac{1}{p^2 - m^2 + i\varepsilon} = \frac{1}{i} \int_0^\infty e^{i\alpha(p^2 - m^2 + i\varepsilon)} d\alpha. \quad (27.12)$$

*Four-dimensional Gaussian quadratures* ( $a$ —positive number,  $b$ —four-vector):

$$\int e^{i(ak^2 + 2bk)} dk = \frac{\pi^2}{ia^2} e^{-ib^2/a}, \quad (27.13)$$

$$\int e^{i(ak^2 + 2bk)} k^n dk = -\frac{b^n}{a} \left( \frac{\pi^2}{ia^2} \right) e^{-ib^2/a}, \quad (27.16)$$

$$\int e^{i(ak^2 + 2bk)} k^n k^m dk = \frac{2b^n b^m + iag^{nm}}{2a^2} \left( \frac{\pi^2}{ia^2} \right) e^{-ib^2/a}, \quad (27.17)$$

$$\int e^{i(ak^2 + 2bk)} k^2 dk = \frac{b^2 + 2ia}{a^2} \left( \frac{\pi^2}{ia^2} \right) e^{-ib^2/a}, \quad (27.18)$$

$$\int e^{i(ak^2 + 2bk)} k^n k^2 dk = -\frac{b^n}{a} \frac{b^2 + 3ia}{a^2} \left( \frac{\pi^2}{ia^2} \right) e^{-ib^2/a}. \quad (A3B.1)$$

*Typical integrals with respect to  $\lambda$ :*

$$\int_0^\infty d\lambda e^{iA\lambda - \varepsilon\lambda} = \frac{i}{A + i\varepsilon}, \quad (A3B.2)$$

$$\int_0^\infty \frac{d\lambda}{\lambda} (e^{iA\lambda} - e^{iB\lambda}) e^{-\varepsilon\lambda} = \ln \frac{B + i\varepsilon}{A + i\varepsilon}, \quad (27.23)$$

$$\begin{aligned} \int_0^\infty \frac{d\lambda}{\lambda^2} (e^{iA\lambda} - e^{iB\lambda}) (e^{iC\lambda} - e^{iD\lambda}) e^{-\varepsilon\lambda} &= \\ &= iA \ln \frac{A + D + i\varepsilon}{A + C + i\varepsilon} + iB \ln \frac{B + C + i\varepsilon}{B + D + i\varepsilon} + iC \ln \frac{C + B + i\varepsilon}{C + A + i\varepsilon} + iD \ln \frac{D + A + i\varepsilon}{D + B + i\varepsilon} = \\ &= i(A + D) \ln(A + D + i\varepsilon) - i(A + C) \ln(A + C + i\varepsilon) + \\ &\quad + i(B + C) \ln(B + C + i\varepsilon) - i(B + D) \ln(B + D + i\varepsilon). \end{aligned} \quad (A3B.3)$$

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